ESAIM: PROCEEDINGS, April 2012, Vol. 36, p. 61-67
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GIBBS-MARKOV-YOUNG STRUCTURES *, **, ***

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Abstract. We discuss the geometric structures defined by Young in [9, 10], which are used to prove the existence of an ergodic absolutely continuous invariant probability measure and to study the decay of correlations in expanding or hyperbolic systems on large parts.

AMS (2000) subject classification. 37A05, 37A25, 37C40, 37D25.
Keywords. GMY structure, Lyapunov exponents.

Résumé. Nous discutons les structures géométriques définies par Young dans [9,10], qui sont utilisées pour prouver l'existence d'une mesure de probabilité ergodique invariante absolument continue et pour étudier la décroissance des corrélations dans systèmes en expansion ou hyperboliques sur de grandes parties.

Mots clefs. Structure GMY, exposants de Lyapunov.

Introduction

One of the most fruitful ways of studying the dynamic properties of a system is to verify the existence of some type of Markovian structure. This generally consists of a partition of a reference set such that the image of an element of the partition is the union of elements of the partition. In the setting of expanding or hyperbolic systems on large parts, Young used a countable partition Markovian and with additional hypotheses about the format of the partition, studied properties such as: existence of equilibria state and speed of convergence of equilibria.

This work is divided into three sections.

1) We define Gibbs-Markov-Young *(GMY)* structures (Definition 1.2) and recall a consequence of their existence (Theorem 1.5).
2) Following [3] and [5], we give an example of an application of Young's results (Theorem 2.7).
3) We give a brief summary of the work [2], in which it is shown that, in great generality, the existence of an expanding invariant measure is sufficient to imply the existence of GMY structures (Theorem 3.2).

1. GMY Structures

Our setting will be a compact Riemannian manifold M of finite dimension endowed with the normalized Riemannian volume Leb (Lebesgue measure) and a measurable map $f : M \to M$ which is differentiable almost everywhere.

* *I would like to thank Marco Zambon for encouragement, for reading this work and making several useful comments.*

** *I would like to thank CMUP for supporting the trip to ECIT 2010.*

*** *Many thanks also to the referees for their very valuable comments.*

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Definition 1.1. Given a ball $\Delta \subseteq M$, we say that $F : \Delta \to \Delta$ is an *induced map* if $F(x) = f^{R(x)}(x)$ and $R : \Delta \to \mathbb{N}$ is an *return time* function with the property that $f^{R(x)}(x) \in \Delta$ whenever $x \in \Delta$.

The structures defined by Young, also known as Young's Tower, are defined below.

Definition 1.2. We say that an induced map $F : \Delta \to \Delta$ is *Gibbs-Markov-Young (GMY)* if there exists a (Leb mod 0) partition $\mathcal{P}$ of Δ into open subsets such that the return time function R is constant on each element $U \in \mathcal{P}$ and $F|_U$ is a uniformly expanding diffeomorphism onto Δ with uniformly bounded volume distortion: more precisely, there are $0 < \kappa < 1$ and $K > 0$ such that for all $U \in \mathcal{P}$ and all $x, y \in U$

i) $\|DF(x)^{-1}\| < \kappa$;

ii) $\log \left| \dfrac{\det DF(x)}{\det DF(y)} \right| \leq K \operatorname{dist}(F(x), F(y))$.

Moreover, if the return time function R is integrable with respect to Leb, then we say that the induced map has integrable return times.
We say that f *admits a GMY structure if it admits a GMY induced map with integrable return times.*

Definition 1.3. We say that an f-invariant probability measure μ is *mixing* if, for all measurable sets A and B

$$\mu(f^{-n}A \cap B) \to \mu(A)\mu(B) \text{ as } n \to \infty.$$

A generalization of this concept is given in the following definition.

Definition 1.4. Let μ be an f-invariant Borel probability measure and $\varphi, \psi : M \to \mathbb{R}$ real valued measurable functions. We define the *correlation function*

$$\mathcal{C}_n(\varphi, \psi) = \left| \int \psi(\varphi \circ f^n) d\mu - \int \varphi d\mu \int \psi d\mu \right|. \tag{1}$$

Notice that in the special case in which φ, ψ are characteristic functions then we have the condition of mixing.

By the *speed of mixing*, or the *rate of correlation decay*, we refer to the speed with which the quantity (1) decreases to zero when n goes to infinity. GMY structures were used to prove that a bound on the decay of correlations for Holder continuous observables can be obtained from bounds on the rate of decay of the tail of the return times.

Theorem 1.5. *[9, 10] Suppose that $f : M \to M$ admits a GMY structure. Then it admits an ergodic absolutely continuous invariant probability measure. Moreover the correlation function for Holder continuous observables satisfies the following bounds:*

- *Exponential tail: If $\alpha > 0$ such that $\mathrm{Leb}(\{R > n\}) = \mathcal{O}(e^{-\alpha n})$, then there exists $\tilde{\alpha}$ such that $\mathcal{C}_n = \mathcal{O}(e^{-\tilde{\alpha} n})$.*
- *Polynomial tail: If $\alpha > 0$ such that $\mathrm{Leb}(\{R > n\}) = \mathcal{O}(n^{-\alpha})$, then $\mathcal{C}_n = \mathcal{O}(n^{-\alpha})$.*

2. Applications

In this section we discuss two results in [3] and [5], where the GMY structures were useful to study the decay of correlations in the setting of non-uniformly expanding maps. First we give some non-degeneracy conditions.

Definition 2.1. We say that x is a *critical point* if $Df(x)$ is not invertible. We denote the set of critical points by $\mathcal{S}$ and let $\operatorname{dist}(x, \mathcal{S})$ denote the distance between the point x and the set $\mathcal{S}$. We say that a critical set $\mathcal{S}$ is *non-degenerate* if there are constants $B > 1$ and $\beta > 0$ such that for every $x \in M \setminus \mathcal{S}$

(C1) $\quad B^{-1} \operatorname{dist}(x, \mathcal{S})^{\beta} \leq \frac{\|Df(x)v\|}{\|v\|} \leq B \operatorname{dist}(x, \mathcal{S})^{-\beta}$.

Moreover, the functions $\log|\det Df|$ and $\log\|(Df)^{-1}\|$ are *locally Lipschitz* at points $x \in M \setminus \mathcal{S}$: for every $x, y \in M \setminus \mathcal{S}$ with $\operatorname{dist}(x, y) < \operatorname{dist}(x, \mathcal{S})/2$ we have

(C2) $\big|\log \|Df(x)^{-1}\| - \log \|Df(y)^{-1}\|\,\big| \le B\mathrm{dist}(x,y)/\mathrm{dist}(x,\mathcal{S})^{\beta}$;
(C3) $|\log|\det Df(x)| - \log|\det Df(y)|\,| \le B\mathrm{dist}(x,y)/\mathrm{dist}(x,\mathcal{S})^{\beta}$.

Now, we define a special type of map which generalizes the uniformly expanding maps.

Definition 2.2. Let $f : M \to M$ be a C^2 local diffeomorphism outside a non-degenerate critical set $\mathcal{S}$. We say that f is *non-uniformly expanding* (*NUE*) on a set $A \subset M$ if there is $\lambda > 0$ such that for every $x \in A$ one has

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{n} \log \|Df(f^j(x))^{-1}\| < -\lambda. \tag{2}$$

We say that f has *slow recurrence* (*SR*) if given any $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in A$ we have

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{n} -\log \mathrm{dist}_\delta(f^j(x),\mathcal{S}) \le \epsilon, \tag{3}$$

where $\mathrm{dist}_\delta(x,\mathcal{S}) = 1$ if $\mathrm{dist}(x,\mathcal{S}) \ge \delta$ and $\mathrm{dist}(x,\mathcal{S})$ otherwise.

An example nontrivial of such maps are Viana maps, see [8].

Definition 2.3. For $x \in M$, we define the *expansion time function*

$$\xi(x) = \min\left\{ N : \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df^{-1}_{f^j(x)}\|^{-1} \ge \frac{\lambda}{2}\,\forall n \ge N \right\} \tag{4}$$

and the *recurrence time function*

$$\mathcal{R}(x) = \min\left\{ N \ge 1 : \frac{1}{n}\sum_{i=0}^{n-1} -\log \mathrm{dist}_\delta(f^j(x),\mathcal{S}) \le 2\epsilon\,\forall n \ge N \right\} \tag{5}$$

The above functions are well defined and finite Lebesgue almost everywhere by conditions (2) and (3). Consider also the tail set (at time n)

$$\Gamma_n = \{x : \xi(x) > n \text{ or } \mathcal{R}(x) > n\}. \tag{6}$$

This set Γ_n can be seen as a quantitative way to measure the non-uniformity of a map because is the set of points that in time n have not yet reached the exponential growth or slow recurrence ensured by conditions (4) and (5).

An important property of non-uniformly expanding maps is the existence of times at which the map looks like an uniformly expanding map on some neighborhood. But, first let $B > 1$ and $\beta > 0$ be as in the Definition 2.1 and fix b such that $0 < b < \min\{1/2, 1/(2\beta)\}$.

Definition 2.4. Given $0 < \sigma < 1$ and $\delta > 0$, we say that n is a σ-*hyperbolic time* for $x \in M$ if for all $1 \le k \le n$,

$$\prod_{j=n-k+1}^{n} \|Df(f^j(x))^{-1}\| \le \sigma^k \quad \text{and} \quad \mathrm{dist}_\delta(f^{n-k}(x),\mathcal{S}) \ge \sigma^{bk}. \tag{7}$$

In the case $\mathcal{S} = \emptyset$ the definition of hyperbolic time reduces to the first condition in (7).

We denote

$$H_j(\sigma) = \{x \in M : j \text{ is a } \sigma\text{-hyperbolic time for } x\}.$$

The next result state an important feature of NUE maps: the existence of infinitely many hyperbolic times for Lebesgue almost every point. For the proof see [1].

Proposition 2.5. *Suppose that f is NUE and has SR on a set $A \subset M$. Then there are $\delta > 0$, $0 < \sigma < 1$ and $\theta > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \#\{1 \leq j \leq n : x \in H_j(\sigma)\} \geq \theta,$$

for Leb almost all $x \in A$.

Often it is convenient to use the notation H_n rather than $H_n(\sigma)$. The next Lemma summarizes important properties of hyperbolic times, such as growth to a fixed size and bounded distortion. For the proof see [1].

Lemma 2.6. *There exists $\delta_1, C_1 > 0$ such that if n is a σ-hyperbolic time for x, then there is neighborhood $V_n(x)$ of x such that:*

(1) *f^n maps $V_n(x)$ diffeomorphically onto a ball of radius δ_1 around $f^n(x)$;*
(2) *for every $1 \leq k \leq n$ and $y, z \in V_n(x)$,*

$$\operatorname{dist}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2} \operatorname{dist}(f^n(y), f^n(z));$$

(3) *for any $y, z \in V_n(x)$*

$$\log \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \leq C_1 \operatorname{dist}(f^n(y), f^n(z)).$$

(4) *there is $\tau_n > 0$ such that for any $x \in H_n$ one has $B(x, \tau_n) \subset V_n(x)$.*

We call the sets V_n *hyperbolic pre-balls* and their images $f^n(V_n)$ *hyperbolic balls.* The latter are actually balls of radius $\delta_1 > 0$. The last item in particular says that the set H_n is covered by a finite number of sets V_n.

The hyperbolic times played a fundamental role in the construction of GMY structures of the next two results. In both, using complicated algorithms and combinatorial arguments, the authors constructed a partition and obtained information about the tail of the return time function R in terms of the tail set Γ_n.

Theorem 2.7. *Let $f : M \to M$ be a transitive non-uniformly expanding C^2 map outside a non-degenerate critical set $\mathcal{S}$. Then,*

- *[3] The map f admits a GMY structure and if there exists $\alpha > 0$ such that* $\mathrm{Leb}(\Gamma_n) = \mathcal{O}(n^{-\alpha})$, *then the correlation decay satisfies $\mathcal{C}_n = \mathcal{O}(n^{-\alpha})$.*
- *[5] The map f admits a GMY structure and if there exists $\alpha > 0$ such that* $\mathrm{Leb}(\Gamma_n) = \mathcal{O}(e^{-\alpha n})$, *then there exists $\alpha' > 0$ such that the correlation decay satisfies $\mathcal{C}_n = \mathcal{O}(e^{-\alpha' n})$.*

3. Existence of GMY structures

In the previous section we discussed the application of GMY structures in the setting of non-uniform expansion. Now, we talk over a necessary and sufficient condition for the existence of a GMY structure. This is a brief summary of the work [2] with J.F.Alves and S. Luzzatto.

Definition 3.1. We say that an invariant probability measure μ is *expanding* if all its Lyapunov exponents are positive, i.e. for μ-almost every x and every $v \in T_xM \setminus \{0\}$,

$$\lambda(x, v) := \limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| > 0. \tag{8}$$

We say that μ is *regularly expanding* if it is expanding and in addition we have

$$\log \|Df^{-1}\| \in L^1(\mu). \tag{9}$$

Notice that condition (9) implies in particular that the limit in (8) converges.

In the next result we relate the existence of GMY structures and expanding measures. Henceforth we shall use the standard abbreviation of the term "absolutely continuous (with respect to Lebesgue) invariant probability" to *acip*.

Theorem 3.2. *[2] Let $f : M \to M$ be a C^2 local diffeomorphism. Then f admits a GMY structure if and only if it admits an ergodic regularly expanding acip.*

In the proof of this theorem we highlight the use of efficient algorithm to implement the partition. We also note that there are versions of this result if the map f has a critical or singular set (see [2]), but here we present only the simplest result .

3.1. The proof of Theorem 3.2

Now we present the main ingredients in the proof of Theorem 3.2. Detailed proofs will appear in [2].

The fact that the existence of a GMY structure implies the existence of expanding measure is almost immediate. Hereafter we focus on the proof of the other implication.

Two independent propositions. The result is proved in two independent steps. First, we prove that the existence of an expanding acip implies that some power of f satisfies the NUE condition on a forward invariant set A having a positive Lebesgue measure subset of points whose orbit is dense in A. For this we apply classical results of Ergodic Theory. Second, assuming the setting of non-uniformly expanding maps, we construct a GMY map with integrable return times as we shall see in the next items.

The existence of a ball. The existence of infinite hyperbolic times for NUE maps (see Proposition 2.5) gives us the important topological fact that the forward invariant set A contains a ball Δ.

Return in finite iterations. Choosing the radius of the ball Δ properly, it is possible to show that every ball of radius at least δ_1 (as in the Lemma 2.6) has a subset which maps diffeomorphically with bounded distortion onto Δ within a uniformly bounded N_0 number of iterations.

The elements of the partition. Given a point $x \in \Delta$ with hyperbolic time n, then there exists a hyperbolic pre-ball $V_n(x)$ such that $f^n(V_n(x)) = B(f^n(x), \delta_1)$. From the previous item we have that there are a set $V \subset B(f^n(x), \delta_1)$ and an integer $0 \le m \le N_0$ such that $f^m(V) = \Delta$. Define

$$U^x_{n,m} = (f|^{n+m}_{V_n(x)})^{-1}(\Delta). \tag{10}$$

These sets $U^x_{n,m}$ are the candidates for elements of the partition of Δ corresponding to the induced map F since they are mapped onto Δ with uniform expansion and bounded distortion. Notice that the sets $U^x_{n,m}$ and $U^{x'}_{n',m'}$ for distinct points x, x' are not necessarily disjoint and also that $U^x_{n,m}$ does not necessarily contain the point x.

The algorithm partitioning. Now we describe inductively the partitioning algorithm for the construction of the $\mathcal{P}$ partition (Leb mod 0) of Δ.

First step of induction

We fix some large $n_0 \in \mathbb{N}$ and ignore any dynamics occurring up to time n_0. Define $\Delta^c = M \setminus \Delta$. By the fourth item of Lemma 2.6 , H_{n_0} can be covered by a finite number of hyperbolic pre-balls, and thus there are $z_1, \dots, z_{N_{n_0}} \in H_{n_0}$ such that

$$H_{n_0} \subset V_{n_0}(z_1) \cup \cdots \cup V_{n_0}(z_{N_{n_0}}).$$

Consider a maximal family

$$\mathcal{U}_{n_0} = \{U^{x_1}_{n_0,m_1}, \dots, U^{x_{k_{n_0}}}_{n_0,m_{k_{n_0}}}\}$$

of pairwise disjoint sets of type (10) contained in Δ with $x_1, \dots, x_{k_{n_0}} \in \{z_1, \dots, z_{N_{n_0}}\}$. These are the elements of the partition $\mathcal{P}$ constructed in the n_0-step of the algorithm. Set $R(x) = n_0 + m_i$ for each $x \in U^{x_i}_{n_0,m_i}$ with $0 \le i \le k_{n_0}$. Now let

$$\widetilde{H}_{n_0} = \{z_1, \dots, z_{N_{n_0}}\} \setminus \{x_1, \dots, x_{k_{n_0}}\}.$$

be the set of points in $\{z_1, \dots, z_{N_{n_0}}\}$ which were not "used" in the construction of $\mathcal{U}_{n_0}$. Notice that the reason they were not used is that the associated sets of the form $U_{n_0,m}$ overlap one of the sets in $\mathcal{U}_{n_0}$ which were selected. We want to keep track of which sets overlap which and so, for each given $U \in \mathcal{U}_{n_0}$, and each $0 \le m \le N_0$, we define

$$H^m_{n_0}(U) = \left\{x \in \widetilde{H}_{n_0} : U^x_{n_0,m} \cap U \ne \emptyset\right\} \tag{11}$$

and the n_0-*satellite*

$$S_{n_0}(U) = \bigcup_{m=0}^{N_0} \bigcup_{x \in H_{n_0}^m(U)} V_{n_0}(x) \cap (\Delta \setminus U).$$

Thus, the n_0-satellite of U is the union of all hyperbolic pre-balls which "could have" had a subset returning to Δ but were unlucky in that such a subset overlaps the set U which was chosen instead. It will be convenient to consider also the n_0-satellite associated to Δ^c

$$S_{n_0}(\Delta^c) = \bigcup_{m=0}^{N_0} \bigcup_{x \in H_{n_0}^m(\Delta^c)} V_{n_0}(x) \cap \Delta.$$

We also define *global* n_0-*satellite*

$$S_{n_0} = \bigcup_{U \in \mathcal{U}_{n_0}} S_{n_0}(U) \cup S_{n_0}(\Delta^c). \tag{12}$$

We point out that this is a technical definition and will be useful to demonstrate that the algorithm actually results in a partition.

Finally we define the part of Δ that has not been partitioned up to time n_0

$$\Delta_{n_0} = \Delta \setminus \bigcup_{U \in \mathcal{U}_{n_0}} U. \tag{13}$$

General step of induction

Assume that the sets $\mathcal{U}_s$, Δ_s and S_s are defined for each $n_0 \leq s \leq n-1$. By the fourth item of Lemma 2.6 there are $z_1, \ldots, z_{N_n} \in H_n$ such that

$$H_n \subset V_n(z_1) \cup \cdots \cup V_n(z_{N_n}).$$

Consider a maximal family

$$\mathcal{U}_n = \{U_{n,m_1}^{x_1}, \ldots, U_{n,m_{k_n}}^{x_{k_n}}\}$$

of pairwise disjoint sets of type (10) contained in Δ_{n-1} with $x_1, \ldots, x_{k_n} \in \{z_1, \ldots, z_{N_n}\}$. These are the elements of the partition $\mathcal{P}$ constructed in the n-step of algorithm. Set $R(x) = n + m_i$ for each $x \in U_{n,m_i}^{x_i}$ with $0 \leq i \leq k_n$. Let

$$\widetilde{H}_n = \{z_1, \ldots, z_{N_n}\} \setminus \{x_1, \ldots, x_{k_n}\}.$$

Given $U \in \mathcal{U}_{n_0} \cup \cdots \cup \mathcal{U}_n$, we define for $0 \leq m \leq N_0$

$$H_n^m(U) = \left\{x \in \widetilde{H}_n : U_{n,m}^x \cap U \neq \emptyset\right\} \tag{14}$$

and its n-*satellite*

$$S_n(U) = \bigcup_{m=0}^{N_0} \bigcup_{x \in H_n^m(U)} V_n(x) \cap (\Delta \setminus U).$$

This set corresponds to the portion of a reference hyperbolic pre-ball that was not used for constructing an element of the partition. It will be convenient to consider also the n-*satellite* associated to Δ^c

$$S_n(\Delta^c) = \bigcup_{m=0}^{N_0} \bigcup_{x \in H_n^m(\Delta^c)} V_n(x) \cap (\Delta \setminus \Delta^c).$$

Finally we define the *global n-satellite*

$$S_n = \bigcup_{U \in \mathcal{U}_{n_0} \cup \cdots \cup \mathcal{U}_n} S_n(U) \cup S_n(\Delta^c) \tag{15}$$

and the part of Δ that has not been partitioned up to time n:

$$\Delta_n = \Delta \setminus \bigcup_{U \in \mathcal{U}_{n_0} \cup \cdots \cup \mathcal{U}_n} U. \tag{16}$$

It is important to note that each step of the algorithm is associated to a unique hyperbolic time and possibly several distinct return times.

Expansion and bounded distortion. The expansion and uniformly bounded distortion is an immediate consequence of properties of hyperbolic times, the choice of the radius of Δ and the constant n_0.

The measure of satellites. Using the bounded distortion property we can prove that the sum of Lebesgue measure of the global satellites sets $\sum_{n=n_0}^{\infty} \mathrm{Leb}(S_n)$ is finite and it follows from this fact and the existence of infinitely many hyperbolic times that the algorithm described above indeed yields a (Leb mod 0) partition.

Integrability of return times. The map $F : \Delta \to \Delta$ defined for $F(x) = f^{R(x)}(x)$ is a C^2 piecewise uniformly expanding map with uniform bounded distortion. By [9, Lemma 2] such a map has a unique absolutely continuous (with respect to Lebesgue measure) ergodic invariant probability measure ν whose density is bounded away from zero and infinity by constants. Thus in particular the integrability with respect to Lebesgue of the return time function R is equivalent to the integrability with respect to ν. It is therefore sufficient to show that the time function R is ν-integrable and this follows from the construction of partition.

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