ESAIM: PROCEEDINGS, April 2012, Vol. 36, p. 68-72
Danièle Fournier-Prunaret, Laura Gardini, & Ludwig Reich, Editors

ON EVOLUTION OF SMALL SPHERES IN THE PHASE SPACE OF A DYNAMICAL SYSTEM *

Boris Gurevich[1] and Sergei Komech[2]

Abstract. We study the connection between the entropy of a dynamical system and the boundary distortion rate of regions in the phase space of the system.

AMS (2000) subject classification. 54C70, 37D20, 34D08.
Keywords. Kolmogorov entropy, Lyapunov exponents, toral automorphisms, synchronized system.

Résumé. Nous étudions la connexion entre l'entropie d'un système dynamique et le taux de distortion au bord dans l'espace des phases du système.

Mots clefs. Entropie de Kolmogorov, exposants de Lyapunov, automorphismes du tore, système synchronisé.

Introduction

Let $X = (X, \rho)$ be a metric space. For every $A \subset X$ and $\varepsilon > 0$, we denote the ε-neighborhood of A by $\mathcal{O}_\varepsilon(A)$. If $A = \{x\}$ is a single point, then $\mathcal{O}_\varepsilon(A)$ is the ball of radius ε centered at x, and we denote it by $B(x, \varepsilon)$. Let $\tau : X \to X$ be a continuous map and let $h_\mu(\tau)$ be the Kolmogorov entropy of τ with respect to an invariant probability measure μ. The following conjecture is motivated by some remarks in [7]:

If τ, μ and a function $\varepsilon \mapsto k(\varepsilon) \in \mathbb{N}$ satisfy some general conditions, then

$$\lim_{\varepsilon\to 0} \frac{1}{k(\varepsilon)} \ln \frac{\mu(\mathcal{O}_\varepsilon(\tau^{k(\varepsilon)} B(x, \varepsilon)))}{\mu(B(x, \varepsilon))} = h_\mu(\tau), \tag{1}$$

where the convergence holds at least in measure.

In particular, we assume that

$$\lim_{\varepsilon\to 0} k(\varepsilon) = \infty, \quad \lim_{\varepsilon\to 0} k(\varepsilon)/\ln \varepsilon = 0. \tag{2}$$

(One cannot look forward to reasonable results when k and ε vary independently.)

It turns out that the conjecture is true at least within two classes of dynamical systems in some sense opposite of each other: symbolic systems (the first rigorous results for them were obtained in [2]) and smooth maps. More precisely, (1) can be proved for so-called synchronized subshifts (and hence for all sofic systems) under natural restrictions on μ, and for Anosov diffeomorphisms with SRB (see, e.g., [4]) measures μ .

* *The work is partially supported by RFBR grant 11-01-00485-a.*

[1] Moscow State University & Institute for Information Transmission Problems of the Russian Academy of Sciences; email: gurevich@mech.math.msu.su.

[2] Institute for Information Transmission Problems of the Russian Academy of Sciences; email: komech@iitp.ru.

We begin with formulating a precise result for synchronized subshifts. Then we will prove the above conjecture for hyperbolic automorphisms of tori to show some basic ideas also employed in a nonlinear case.

1. Statement of results

Let $\mathcal{A}$ be a finite alphabet, S a shift transformation defined on $\mathcal{A}^{\mathbb{Z}}$, and let X be an S-invariant compact subset of $\mathcal{A}^{\mathbb{Z}}$. We define a metric ρ on X by

$$\rho(x,y) = \theta^{n(x,y)}, \quad x, y \in X,$$

where $\theta \in (0,1)$ and

$$n(x,y) = \min\{k \in \mathbb{Z}_+ : x(-k) \neq y(-k) \text{ or } x(k) \neq y(k)\}$$

if $x \neq y$, and $n(x,x) = \infty$.

We recall the definition of a synchronized system (see, e.g., [1]).

Definition 1.1. Let (X,S) be a transitive subshift. A word in the alphabet $\mathcal{A}$ is an X-block if it is a subblock in some $x \in X$. The pair (X,S) is a synchronized system if there exists an X-block w (a 'magic' word) such that if uw and wv (and hence u, v) are X-blocks, then uwv is also an X-block.

It is known that the family of synchronized systems contains all subshifts of finite type and, more generally, the sofic systems introduced by Weiss [6].

Theorem 1.2. *Let (X,S) be a synchronized system and let μ be an S-invariant ergodic probability measure on X. Assume that the function $k : \mathbb{R}^+ \to \mathbb{Z}^+$ satisfies (2) and that there exists a 'magic' word w such that*

$$\mu(\{y \in X : y_1 \dots y_{|w|} = w\}) > 0 \ ,$$

where $|w|$ is the length of the word w. Then for all $\theta \in (0,1)$ equation (1) holds with convergence in the sense of L^1_μ.

This theorem is proved in [8].

We now remind the reader of several well-known facts concerning algebraic automorphisms of tori (see, e.g., [3,5]).

Let τ be a hyperbolic automorphism of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and let μ be the Lebesgue (Haar) measure on $\mathbb{T}^n$, which is clearly τ-invariant.

We denote the matrix that induces τ by A_τ. If $\lambda_1, \dots, \lambda_{max}$ are its eigenvalues, then

$$h_\mu(\tau) = \log \prod_{i:|\lambda_i|>1} |\lambda_i|. \tag{3}$$

Hyperbolicity of τ implies that $|\lambda_i| \neq 1$ for all i.

Let H_s (H_u) be the direct sum of the subspaces in $\mathbb{R}^n$ corresponding to λ_i with $|\lambda_i| < 1$ ($|\lambda_i| > 1$, respectively); H_s and H_u are invariant under A_τ, $H_s \cap H_u = \{0\}$, and $\mathbb{R}^n = H_s \oplus H_u$.

We will use the following simple statement: there exist constants $a_s > 0$, $a_u > 0$, $\lambda_s < 1$ and $\lambda_u > 1$ such that for all $y_1 \in H_s$, $y_2 \in H_u$ and $m \in \mathbb{N}$,

$$||A_\tau^m y_1|| \leq a_s \lambda_s^m ||y_1||, \quad ||A_\tau^{-m} y_2|| \leq a_u \lambda_u^{-m} ||y_2||, \tag{4}$$

where $||\cdot||$ denotes the Euclidean norm in $\mathbb{R}^n$.

We denote the Lebesgue measures on H_s and H_u by ν_s and ν_u, respectively. Let $\lambda = \prod_{i:|\lambda_i|>1} |\lambda_i|$. For any measurable sets $B_s \subset H_s$ and $B_u \subset H_u$,

$$\nu_s(A_\tau B_s) = \lambda^{-1} \nu_s(B_s), \quad \nu_u(A_\tau B_u) = \lambda \nu_u(B_u). \tag{5}$$

Theorem 1.3. *If $k(\varepsilon)$ satisfies condition (2) and if τ and μ are as above, then for each $x \in \mathbb{T}^n$ equation (1) holds.*

2. Proof of Theorem 1.3

Let
$$H_s(y) := y + H_s, \quad H_u(y) = y + H_u, \quad y \in \mathbb{R}^n.$$
Observe that the sets $H_s(y)$ and $H_u(y)$, $y \in \mathbb{R}^n$, induce two partitions of $\mathbb{R}^n$ that are invariant under the action of A_τ. We will keep the same notation ν_s and ν_u for the Lebesgue measures on $H_s(y)$ and $H_u(y)$, respectively.

For a set $D \subset \mathbb{R}^n$ we denote its ε-neighbourhood in $\mathbb{R}^n$ by $O^\varepsilon(D)$. If for some $y \in \mathbb{R}^n$, $D \subset H_s(y)$ or $D \subset H_u(y)$, we denote the ε-neighbourhood of D in the intrinsic metrics of $H_s(y)$ or $H_u(y)$ by $O_s^\varepsilon(D)$ or $O_u^\varepsilon(D)$, respectively.

Let $G_{s,y}$ and $G_{u,y}$ be measurable subsets of $H_s(y)$ and $H_u(y)$, respectively. We say the set

$$G_{s,y} \times G_{u,y} := \{z \in \mathbb{R}^n : \ z = y_s + y_u - y, \ \ y_s \in G_{s,y}, \ \ y_u \in G_{u,y}\}$$

is a parallelogram in $\mathbb{R}^n$.

For all $y \in \mathbb{R}^n$, $\delta > 0$, we put
$$P^\delta(y) = O_s^\delta(y) \times O_u^\delta(y).$$
Clearly, for a given $y \in \mathbb{R}^n$, there exist constants C_1, C_2 depending only on the angle between H_s and H_u such that
$$P^{C_1\varepsilon}(y) \subset O^\varepsilon(y) \subset P^{C_2\varepsilon}(y). \tag{6}$$
Upper bound. From (6) we obtain

$$A_\tau^m O^\varepsilon(y) \subset A_\tau^m \Big(P^{C_2\varepsilon}(y)\Big) = A_\tau^m \Big(O_s^{C_2\varepsilon}(y) \times O_u^{C_2\varepsilon}(y)\Big), \quad m \in \mathbb{Z}_+. \tag{7}$$

It is easy to see that there exists $\gamma_1 > 0$ depending only on the angle between H_s and H_u such that

$$O^\varepsilon(A_\tau^m P^{C_2\varepsilon}(y)) \subset O_s^{\gamma_1\varepsilon}(A_\tau^m O_s^{C_2\varepsilon}(y)) \times O_u^{\gamma_1\varepsilon}(A_\tau^m O_u^{C_2\varepsilon}(y)), \quad m \in \mathbb{Z}_+. \tag{8}$$

The first inequality in (4) implies that

$$\operatorname{diam}(A_\tau^m O_s^{C_2\varepsilon}(y)) \le 2a_s \lambda_s^m C_2 \varepsilon.$$

Since $\lambda_s < 1$ and $\lim_{\varepsilon\to 0} k(\varepsilon) = \infty$, for sufficiently small ε (sufficiently large $k(\varepsilon)$) we have

$$A_\tau^{k(\varepsilon)} O_s^{C_2\varepsilon}(y) \subset O_s^\varepsilon(A_\tau^{k(\varepsilon)} y)$$

and hence
$$O_s^{\gamma_1\varepsilon}(A_\tau^{k(\varepsilon)} O_s^{C_2\varepsilon}(y)) \subset O_s^{\varepsilon(1+\gamma_1)}(A_\tau^{k(\varepsilon)} y).$$
Therefore
$$\nu_s(O_s^{\gamma_1\varepsilon}(A_\tau^{k(\varepsilon)} O_s^{C_2\varepsilon}(y))) \le \nu_s(O_s^{\varepsilon(1+\gamma_1)}(A_\tau^{k(\varepsilon)} y)). \tag{9}$$
Now we can estimate $\nu_u\Big(O_u^{\gamma_1\varepsilon}(A_\tau^m O_u^{C_2\varepsilon}(y))\Big)$ from above. By (4)

$$d(A_\tau^m y, \partial(A_\tau^m O_u^{C_2\varepsilon}(y))) \ge a_u^{-1} \lambda_u^m C_2 \varepsilon,$$

where d is the Euclidean metric in $\mathbb{R}^n$. Then the set $\Theta(A_\tau^{k(\varepsilon)} y)$ obtained from $A_\tau^{k(\varepsilon)} O_u^{C_2\varepsilon}(y)$ by the homothety with coefficient 2 centered at $A_\tau^{k(\varepsilon)} y$ contains, for a sufficiently small ε, $O_u^{\gamma_1\varepsilon}(A_\tau^{k(\varepsilon)} O_u^{C_2\varepsilon}(y))$. It is evident that

$$\nu_u(\Theta(A_\tau^{k(\varepsilon)} y)) = 2^{n-l} \nu_u(A_\tau^{k(\varepsilon)} O_u^{C_2\varepsilon}(y)),$$

where l is the dimension of H_s. Now (5) implies that

$$\nu_u(O_u^{\gamma_1\varepsilon}(A_\tau^{k(\varepsilon)}O_u^{C_2\varepsilon}(y))) \leq \nu_u(\Theta(A_\tau^{k(\varepsilon)}y)) = 2^{n-l}\lambda^{k(\varepsilon)}\nu_u(O_u^{C_2\varepsilon}(y)). \tag{10}$$

Let ν be the Lebesgue measure on $\mathbb{R}^n$. From (8) – (10) we conclude that

$$\begin{aligned} &\nu(O^\varepsilon(A_\tau^{k(\varepsilon)}P^{C_2\varepsilon}(y))) \leq \\ &\leq \gamma_0\nu_s(O_s^{\gamma_1\varepsilon}(A_\tau^{k(\varepsilon)}O_s^{C_2\varepsilon}(y)))\nu_u(O_u^{\gamma_1\varepsilon}(A_\tau^{k(\varepsilon)}O_u^{C_2\varepsilon}(y))) \\ &\leq \gamma_0\alpha_1(\varepsilon(1+\gamma_1))^l\ 2^{n-l}\ \lambda^{k(\varepsilon)}\alpha_2\ (C_2\varepsilon)^{n-l}, \end{aligned} \tag{11}$$

where γ_0 depends only on the angle between H_u and H_s, and α_1, α_2 depend only on l.

Lower bound. By (6) and the definition of the parallelogram $P^{C_1\varepsilon}(y)$, for each $m \in \mathbb{Z}_+$,

$$O_s^\varepsilon(A_\tau^m m) \times A_\tau^m O_u^{C_1\varepsilon}(y) \subset O^\varepsilon(A_\tau^m P^{C_1\varepsilon}(y)) \subset O^\varepsilon(A_\tau^m O^\varepsilon(y)). \tag{12}$$

Let π denote the natural projection of $\mathbb{R}^n$ onto $\mathbb{T}^n$. We will use the following lemma to estimate the measure of the set on the left-hand side of (12).

Lemma 2.1. *If ε is small enough, then the restriction of π to the set $O^\varepsilon(A_\tau^{k(\varepsilon)}P^{C_2\varepsilon}(y))$ is a bijection.*

Proof. From the definition of the parallelogram $P^{C_2\varepsilon}(y)$ it follows that

$$\operatorname{diam}(P^{C_2\varepsilon}(y)) \leq 2C_2\varepsilon.$$

Therefore

$$\operatorname{diam}(O^\varepsilon(A_\tau^{k(\varepsilon)}P^{C_2\varepsilon}(y))) \leq 2C_2\varepsilon\|A_\tau\|^{k(\varepsilon)} + 2\varepsilon. \tag{13}$$

By (2) $k(\varepsilon) = \mathrm{o}(\ln(\varepsilon))$, hence the right-hand side of (13) tends to 0 as $\varepsilon \to 0$. But on every set of diameter smaller than 1 the projection π is bijective.

□

The lemma just proved implies that if ε is small enough, then

$$\nu(O^\varepsilon(A_\tau^{k(\varepsilon)}P^{C_1\varepsilon}(y))) = \mu\Big(\pi\Big(O^\varepsilon(A_\tau^{k(\varepsilon)}P^{C_1\varepsilon}(y))\Big)\Big). \tag{14}$$

By (5) and (12)

$$\begin{aligned} \nu(O^\varepsilon(A_\tau^{k(\varepsilon)}P^{C_1\varepsilon}(y))) &\geq \gamma_0\nu_s(O_s^\varepsilon A_\tau^{k(\varepsilon)}(y))\nu_u(O_u^{C_1\varepsilon}(y))\lambda^{k(\varepsilon)} \\ &\geq \gamma_0\alpha_1\varepsilon^l\ \alpha_2\ (C_1\varepsilon)^{n-l}\lambda^{k(\varepsilon)}, \end{aligned} \tag{15}$$

where γ_0, as before, depends only on the angle between H_s and H_u.

Given $x \in \mathbb{T}^n$ and $y \in \pi^{-1}x$, from (6) and (14) it follows that

$$\begin{aligned} \nu(O^\varepsilon(A_\tau^{k(\varepsilon)}P^{C_1\varepsilon}(y))) &\leq \mu\left(O_\varepsilon(\tau^{k(\varepsilon)}B(x,\varepsilon))\right) \\ &\leq \nu(O^\varepsilon(A_\tau^{k(\varepsilon)}P^{C_2\varepsilon}(y))). \end{aligned} \tag{16}$$

We substitute (11) and (15) into (16) to obtain

$$\begin{aligned} \tilde{\gamma}\varepsilon^n\ \lambda^{k(\varepsilon)}\ {C_1}^{n-l} &\leq \mu\left(O_\varepsilon(\tau^{k(\varepsilon)}B(x,\varepsilon))\right) \\ &\leq \tilde{\gamma}(1+\gamma_1)^l\ 2^{n-l}\ \lambda^{k(\varepsilon)}\ {C_2}^{n-l}\varepsilon^n, \end{aligned} \tag{17}$$

where $\tilde{\gamma} = \gamma_0 \alpha_1 \alpha_2$.

The measure of the ball $B(x, \varepsilon)$ with a small ε equals

$$\mu(B(x, \varepsilon)) = \nu(O^{\varepsilon}(y)) = \alpha_3 \varepsilon^n, \tag{18}$$

where $\alpha_3 = const$.

By putting (11), (17), (18) together and taking (3) into account we arrive at (1).

Remark 2.2. For a nonlinear hyperbolic map of a Riemann manifold, the proof inherits these basic constructions. Of course, it is more delicate to estimate the measure of the projections on the stable and unstable manifolds (in general, with nonzero curvature), and we have to use some additional technics to complete the proof. Moreover, we cannot prove (1) for all x and have to content ourself with a weaker kind of convergence.

References

[1] D. Fiebig, U.-R. Fiebig, *The authomorphism group of a coded system.* Trans. Amer. Math. Soc. **348**, 3173–3191, 1996.

[2] B. M. Gurevich, *Geometric interpretation of entropy for random processes.* AMS Transl. **171**, 81–87, 1996.

[3] B. M. Gurevich, Ya. G. Sinai *Algebraic automorphisms of the torus and Markov chains*, Appendix to the Russian translation of: P. Billingsley, *Ergodic theory and information*, Wiley, New York, 1965; (Translation: Ergodicheskaya teoriya i informatsiya, Mir, Moscow, 1969).

[4] B. Hasselblat, A. Katok, *A first course in dynamics: with a panorama of recent developments*, Camb. Univ. Press, Cambridge, 2003.

[5] P. Walters, *An introduction to ergodic theory*, Springer-Verlag, New York-Berlin, 1982.

[6] B. Weiss, *Subshifts of finite type and sofic systems*, Monats. Math. 77, 1973, pp. 462–474.

[7] G. M. Zaslavsky, *Chaos in dynamic systems.* Harwood, Chur, 1985.

[8] S. Komech, *Boundary distortion rate in synchronized systems: Geometrical meaning of entropy*, Problems of Information Transmission, Volume 48, Number 1 (March 2012), pp. 11–20.