EXPONENTIAL WEALTH DISTRIBUTION:
A NEW APPROACH FROM FUNCTIONAL ITERATION THEORY [*]

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R. L.-R. dedicates this paper, in memoriam,
to José Félix Sáenz Lorenzo, the former Director of the BIFI Institute.

Abstract. Different approaches are possible in order to derive the exponential regime in statistical systems. Here, a new functional equation is proposed in an economic context to explain the wealth exponential distribution. Concretely, the new iteration [1] given by

$$f_{n+1}(x) = \iint_{u+v>x} \frac{f_n(u)f_n(v)}{u+v}\,\mathrm{d}u\mathrm{d}v\,.$$

It is found that the exponential distribution is a stable fixed point of this functional iteration equation. From this point of view, it is easily understood why the exponential wealth distribution (or by extension, other kind of distributions) is asymptotically obtained in different multi-agent economic models.

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Résumé. Différentes approches pour dériver le régime asymptotique exponentiel dans les systèmes statistiques sont possibles. Ici une nouvelle équation fonctionnelle est proposée, dans le cadre des systèmes économiques, pour expliquer la distribution exponentielle. Nous montrons que cette distribution est le seul point fixe vers lequel la dynamique de cette équation fonctionnelle évolue quand l'itération va vers l'infini. De ce point de vue, il est facile de comprendre l'ubiquité de cette distribution (ou d'autres) en différents problèmes statistiques réels.

Mots clefs: Econophysique, Distribution des richesses, Modèle aleatoire, Equilibre statistique.

INTRODUCTION

Exponential distribution is ubiquitous in the framework of multi-agent systems. Usually, it appears as an equilibrium state in the asymptotic time evolution of statistical systems. It has been explained from very different perspectives. In statistical physics, it is obtained from the principle of maximum entropy. In the same

context, it can also be derived without any consideration about information theory, only from geometrical arguments under the hypothesis of equiprobability in phase space. Also, several multi-agent economic models based on mappings, with random, deterministic or chaotic interactions, can give rise to the asymptotic appearance of the exponential wealth distribution. For instance, in a random context, if someone were to make the similarity of a gas of particles with human society we certainly would think that it is a ridiculous option. If this comparison is restricted only to economic aspects, maybe the simile is not so disproportionate. Let us think, for instance, that human or economic agents exchange money or wealth in commercial transactions in the same way that particles in a gas exchange energy in collisions. It is clear that we could always argue that economic agents are endowed with intelligence and a certain foresight and determination in their decisions to purchase and sale, characteristics not present in the random interactions of particles in a gas. But even ignoring these details about the intelligence of the agents, the similarity between markets and gases has been successfully introduced in recent years. It has managed to reproduce surprisingly some of the features observed in the real economy, such as for example the distribution of wealth in society. This approach that applies methods of statistical physics to describe the economy is what has been called Econophysics [2]. Likewise, models that compare the behavior of the economy in a system of trading agents with which occurs with energy or other quantities in a gas of particles are called gas-type models [3].

Nowadays it is known that the society of the Western (capitalist) economies can be divided into two distinct groups if we follow the distribution of wealth (wages, property, etc.) [4]. On the one hand, the 95% of the population, the middle and lower economic classes of society, allocate their wealth according to an exponential distribution. Furthermore, there is a group of 5% of individuals, those with privileged incomes, whose wealth is distributed according to a power law distribution. (See Figure 1).
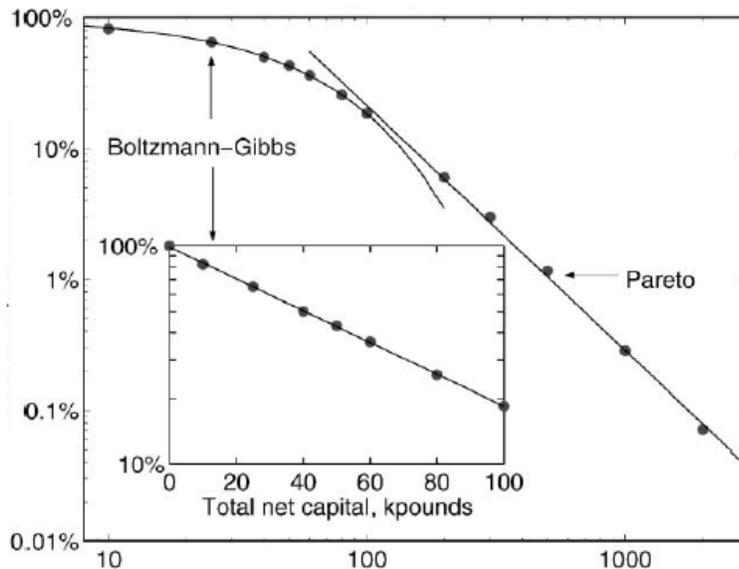

FIGURE 1. Cumulative probability distribution of the total net capital (wealth) in kpounds in UK (data for 1996) shown in log-log, and log-linear the inset. The exponential (Boltzmann-Gibbs) and power (Pareto) laws fit for the 95% and 5% of the population, respectively. (Figure extracted from Ref. [4]).

In this work, we are interested in an alternative explanation of the exponential behavior of the wealth distribution in western societies. First, let us see what an exponential distribution of income means. If we assume for example that in Spain the average salary per month is 1500 euros, we can consider the low incomes those salaries below the half of the average, i.e. 750 euros per month, and the high incomes those revenues

above the double of the average, i.e. 3000 euros per month. Then, the middle economic classes are formed by those individuals with incomes in the range between 750 and 3000 euros per month. If now we know that wealth is distributed in an exponential way , it means, roughly speaking, that the 40% of the population are low economic class and receive the 10% of the total payroll, the 47% of the population are middle economic class and pocket the 50% of the total salary mass , and finally the remaining 13% are high economic class and allocate the 40% of the total wealth. (See Figure 2).
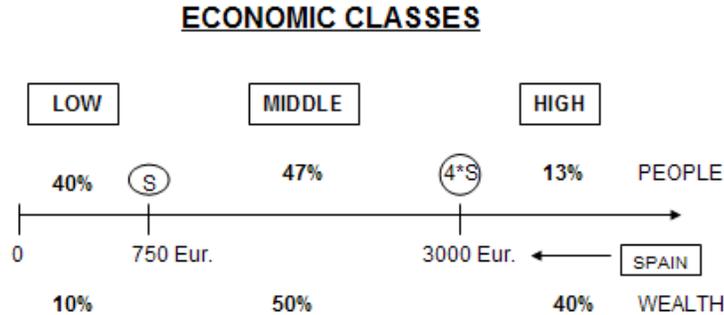

FIGURE 2. Economic partition generated in the society by an exponential wealth distribution. The minimum salary S is considered as the reference level for the low economic class, and $4S$ is taken as the reference level for the high economic class. The average salary in this case will be $2S$.

1. STRATEGIES EXPLAINING THE EXPONENTIAL BEHAVIOR IN STATISTICAL SYSTEMS

Different strategies can generate the exponential wealth distribution with a gas-type model. One of them is as follows [5]: (1) a set of agents is placed each with the same initial amount of money, (2) transactions between agents are allowed in a random manner, thus every time two agents are randomly chosen, (3) they put their money all together and they share it in a random way too, (5) the gas evolves with this type of interactions and the asymptotic wealth distribution appears to be the exponential distribution. Note that in this model all the components are random, the pairs of interacting agents at each moment and the way they divide their money. As local interactions conserve the money, the global dynamics is also conservative and the total amount of money is constant in time.
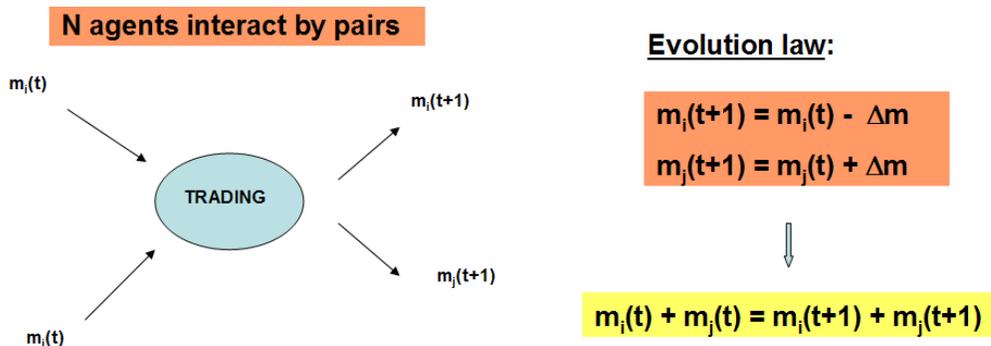

FIGURE 3. Visual representation of the trading mechanism between pairs (m_i, m_j) of economic agents in the gas-like models. The time t is discrete.

Then the time evolution of this system in phase space is made on the hyperplane defined by the constraint that imposes the conservation of the total amount of money. If we suppose that none of the points of the hyperplane is privileged over any other one, i.e. if the equiprobability hypothesis is assumed, then it can be newly calculated that the distribution of wealth in such a system is exponential. Let us remark that this method (presented in [6]) derives the exponential distribution by simple geometric arguments. This strategy is in some way related with the variational method proposed by Jaynes [7], which obtains the exponential distribution in a statistical system as the result of maximizing the entropy of the system when the average value of the energy (or other variables) in the ensemble is fixed. Although equiprobability seems to be strongly related to the exponential behavior of the measurable variables of a system, it is not mandatory that equiprobability be reached by random mechanisms. In fact, the exponential distribution has also been found in multi-agent economic systems with just strictly deterministic elements [8] or even in gas-like models with chaotic ingredients [9] .

2. ANOTHER EXPLANATION FOR THE EXPONENTIAL DISTRIBUTION: A FUNCTIONAL ITERATION MODEL

If we assume an ensemble of N economic agents trading with each other as explained in the former section, the evolution equations of this gas-like model can be written as

$$
\begin{aligned}
m_i' &= \epsilon\,(m_i + m_j), \\
m_j' &= (1 - \epsilon)(m_i + m_j), \\
i, j &= 1 \ldots N,
\end{aligned}
\tag{1}
$$

where ϵ is a random number in the interval $(0, 1)$. The agents (i, j) are randomly chosen and they trade with money (or goods, commodities, etc.) in such a way that their initial money, (m_i, m_j), at time t is transformed in (m_i', m_j') at time $(t + 1)$. It is found that the asymptotic distribution p_f, obtained by numerical simulations, is the exponential (Boltzmann-Gibbs) distribution [5],

$$
p_f(m) = \beta \exp(-\beta\,m), \quad \text{with} \quad \beta = 1/\langle m \rangle,
\tag{2}
$$

where $p_f(m)\mathrm{d}m$ denotes the PDF (probability density function), i.e. the probability of finding an agent with money (or energy in a gas system) between m and $m + \mathrm{d}m$. Evidently, this PDF is normalized, $\| p_f \| = \int_0^\infty p_f(m)\mathrm{d}m = 1$. The mean value of the wealth, $\langle m \rangle$, can be easily calculated by $\langle m \rangle = \sum_i m_i/N$.

We can regard this model as an evolution problem in the space of PDFs, such as

$$
\lim_{n \to \infty} \mathcal{T}^n\,(p_0(m)) \to p_f(m),
\tag{3}
$$

where an initial wealth distribution $p_0(m)$ with a mean wealth value $\langle p_0 \rangle$ evolves in time under the action of an operator $\mathcal{T}$ to asymptotically reach the equilibrium distribution $p_f(m)$, that in this particular case is the exponential one which presents the same average value $\langle p_f \rangle = \langle p_0 \rangle$. This is a macroscopic interpretation of Eqs. (1) in the sense that, under this point of view, each iteration of the operator $\mathcal{T}$ means that many interactions, of order $N/2$, have taken place between different pair of agents. If the subindex n indicates the time evolution of $\mathcal{T}$, we can roughly assume that $t \approx N * n/2$, where t follows the microscopic evolution of the individual tradings (or collisions) between the agents (this alternative microscopic interpretation can be seen in [10]).

Thus the question to answer is if it is possible to find $\mathcal{T}$ and, if this is the case, how $\mathcal{T}$ looks like. Now, we proceed to derive $\mathcal{T}$ in an easy way (see Fig. 4, presented in [1]). Suppose that p_n is the wealth distribution in the ensemble at time n. The probability that two agents with money (u, v) interact will be $p_n(u) * p_n(v)$. As the trading is totally random, their exchange can give rise with equal probability to any value x comprised in the interval $(0, u + v)$. Then, the probability to obtain a particular x (with $x < u + v$) for the interacting pair (u, v) will be $p_n(u) * p_n(v)/(u + v)$. Finally we can obtain the probability to have money x at time $(n + 1)$. It
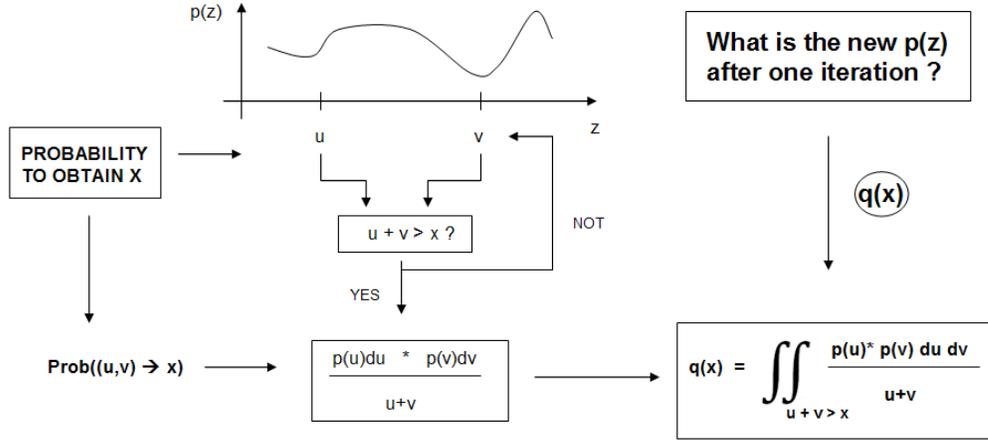
FIGURE 4. Derivation of the nonlinear evolution operator $\mathcal{T}$ for the agent-based economic model given by Eqs. (1). See explanation in the text.

will be the sum of the probabilities for all the pairs of agents (u, v) able to generate the quantity x, that is, all the pairs verifying $u + v > x$. $\mathcal{T}$ has then the form of a nonlinear integral operator,

$$p_{n+1}(x) = \mathcal{T}p_n(x) = \int\int_{u+v>x} \frac{p_n(u)p_n(v)}{u+v}\, dudv. \tag{4}$$

If we suppose $\mathcal{T}$ acting in the PDFs space, it can be proved that $\mathcal{T}$ conserves the norm ($\|\ \|$), i.e. $\mathcal{T}$ maintains the total richness of the system, $\| \mathcal{T}p \| = \| p \| = 1$. By extension, it also conserves the mean wealth of the system, $\langle \mathcal{T}p \rangle = \langle p \rangle$. Then, it can be seen that the exponential distribution $p_f(x)$ with the right average value is the steady state of $\mathcal{T}$, i.e. $\mathcal{T}p_f = p_f$. Also it can be proved that this exponential distribution p_f is the only fixed point of $\mathcal{T}$. In consequence, the relation (3) is true. Finally, it is also found that the entropy is always an increasing quantity with time. Some of these properties [11, 12] are explicitly shown in the next section.

3. SOME PROPERTIES OF THE OPERATOR $\mathcal{T}$

In order to prove some of the properties of the operator $\mathcal{T}$, we provide an adequate mathematical framework with the following definitions.

First, we introduce the space L_1^+ of positive functions (wealth distributions) in the interval $[0, \infty)$,

$$L_1^+[0, \infty) = \{y : [0, \infty) \to R^+ \cup \{0\}, \ \ ||y|| < \infty\},$$

with norm

$$||y|| = \int_0^\infty y(x)dx.$$

Also, we define the mean richness $< x >_y$ associated to a wealth distribution $y \in L_1^+[0, \infty)$ as the mean value of x for the distribution y,

$$< x >_y = ||xy(x)|| = \int_0^\infty xy(x)dx.$$

Then, for $x \geq 0$ and $y \in L_1^+[0, \infty)$, the action of operator $\mathcal{T}$ on y is defined by

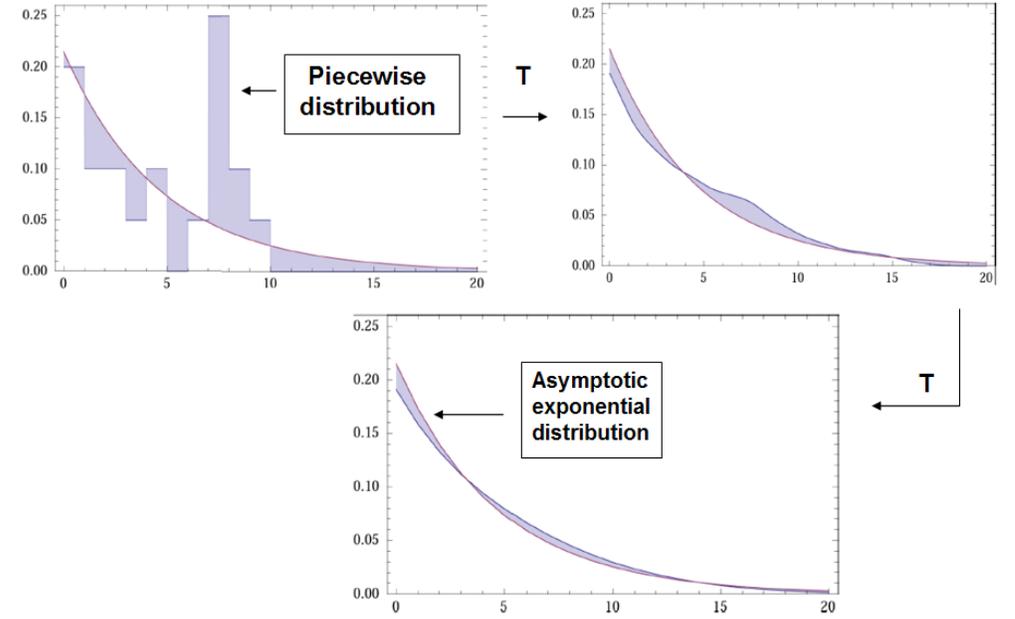$$\mathcal{T}(y(x)) = \int\int_{S(x)} dudv \frac{y(u)y(v)}{u+v},$$

FIGURE 5. An example of the action of $\mathcal{T}$ in the space of PDFs. In this case, $p_0(x)$ is chosen as a normalized piecewise distribution. Observe that after two iterations of $\mathcal{T}$ the system has practically reached the asymptotic state, i.e. the exponential distribution $p_f(x)$. (Figure avanced from [12]).

with $S(x)$ the region of the plane representing the pairs of agents (u, v) which can generate a richness x after their trading, i.e.

$$S(x) = \{(u, v), \quad u, v > 0, \quad u + v > x\}.$$

Now, we establish the following exact results.

The conservation of the number of agents in the economic system is established in the next theorem.

Theorem 1. *For any $y \in L_1^+[0, \infty)$ we have that $||\mathcal{T}y|| = ||y||^2$. In particular, consider the subset of PDFs in $L_1^+[0, \infty)$, i.e. the unit sphere $B = \{y \in L_1^+[0, \infty), ||y|| = 1\}$. Observe that if $y \in B$ then $\mathcal{T}y \in B$.*

Proof. Take $y \in L_1^+[0, \infty)$. Then

$$||\mathcal{T}y|| = \int_0^\infty dx \int\int_{S(x)} dudv \frac{y(u)y(v)}{u + v} =$$

$$= \int_0^\infty du \int_0^\infty dv \int_0^{u+v} dx \frac{y(u)y(v)}{u + v} =$$

$$= \int_0^\infty y(u)du \int_0^\infty y(v)dv = ||y||^2.$$

$\square$

The mean wealth, and by extension the total richness, of the economic system are preserved in time. This is shown in the next theorem.

Theorem 2. *The mean value $<x>_y$ of a PDF y is conserved, that is $<x>_{\mathcal{T}y}=<x>_y$ for any $y \in B$.*

Proof.

$$< x >_{\mathcal{T}y}= ||x(\mathcal{T}y)(x)|| = \int_0^\infty xdx \int \int_{S(x)} dtds\frac{y(t)y(s)}{t+s} =$$

$$= \int_0^\infty dt \int_0^\infty ds \int_0^{t+s} xdx\frac{y(t)y(s)}{t+s} =$$

$$= \frac{1}{2}\int_0^\infty dt \int_0^\infty ds(t+s)y(t)y(s) =$$

$$= \frac{1}{2}\int_0^\infty ty(t)dt + \frac{1}{2}\int_0^\infty sy(s)ds =$$

$$= \int_0^\infty xy(x)dx = ||xy(x)|| =< x >_y .$$

$\square$

In the next theorem, it is proven that the exponential distribution is a fixed point of the operator $\mathcal{T}$.

Theorem 3. *Apart from $y = 0$, the one-parameter family of functions $y_\alpha(x) = \alpha e^{-\alpha x}$, $\alpha > 0$, are fixed points of $\mathcal{T}$ in the space $L_1^+[0, \infty)$.*

Proof. After a change of variables to radial coordinates, it is possible to see that

$$\mathcal{T}\left(\alpha e^{-\alpha x}\right) = \frac{\alpha^2}{2}\int_x^\infty dr \int_{-r}^r dt \frac{e^{-\alpha r}}{r} =$$

$$= \alpha^2 \int_x^\infty e^{-\alpha r}dr = \alpha e^{-\alpha x}.$$

$\square$

Then, as a consequence of these exact results and by computational inspection of many examples, we can conclude that for any $y \in B$ it is verified the following conjecture:

$$\lim_{n\to\infty} \mathcal{T}^n y(x) = \begin{cases} \delta e^{-\delta x} & with \quad \delta = 1/ < x >_y , \\ & or \\ 0^+ & when \quad < x >_y= +\infty . \end{cases}$$

Let us observe that the above pointwise limit of $\mathcal{T}^n y$ when $n \to \infty$ can be outside of B in the case that $< x >_y= +\infty$. A more complete and detailed study of the properties of $\mathcal{T}$ is offered in [11].

4. CONCLUSIONS

From an economic point of view, different approaches to obtain the exponential (Boltzmann-Gibbs) distribution have been recalled. The gas-like models that interpret the market as an ensemble of agents trading with money in the same way as particles exchange energy in a gas can help to derive the asymptotic Gibbs regime. Following this insight, a new operator in the framework of functional iteration theory has been proposed. The uniparametric family of exponential distributions is the only family of attractive fixed points of this operator. Then, this model explains in a straightforward way the ubiquity of the exponential distribution in the asymptotic regime of the more diverse problems, those with some economic inspiration included.

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