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INVARIANT GRAPHS OF FUNCTIONS FOR THE MEAN-TYPE MAPPINGS

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Abstract. Let I be a real interval, J a subinterval of I, $p \geq 2$ an integer number, and $M_1, \ldots, M_p : I^p \to I$ the continuous means. We consider the problem of invariance of the graphs of functions $\varphi : J^{p-1} \to I$ with respect to the mean-type mapping $\mathbf{M} = (M_1, \ldots, M_p)$.

Applying a result on the existence and uniqueness of an $\mathbf{M}$-invariant mean [7], we prove that if the graph of a continuous function $\varphi : J^{p-1} \to I$ is $\mathbf{M}$-invariant, then φ satisfies a simple functional equation. As a conclusion we obtain a theorem which, in particular, allows to determine all the continuous and decreasing in each variable functions φ of the $\mathbf{M}$-invariant graphs. This improves some recent results on invariant curves [8] where the case $p = 2$ is considered.

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Keywords: mean; mean-type mapping; invariant mean; function of invariant graph; iteration; functional equation.

Résumé. Soit I un intervalle réel, J un sous-intervalle de I, $p \geq 2$ un entier, et $M_1, \ldots, M_p : I^p \to I$ les moyennes continues. Nous considérons le problème de l'invariance des graphes des fonctions $\varphi : J^{p-1} \to I$ par rapport aux applications de type moyenne $\mathbf{M} = (M_1, \ldots, M_p)$.

En appliquant un résultat d'existence et unicité d'une moyenne $\mathbf{M}$-invariante [7], nous montrons que si le graphe d'une fonction continue $\varphi : J^{p-1} \to I$ est $\mathbf{M}$-invariant, alors φ vérifie une équation fonctionnelle simple. En conclusion, nous obtenons un théorème qui, en particulier, permet de déterminer toutes les fonctions φ des graphes $\mathbf{M}$-invariant continues et décroissantes en chaque variable. Ceci améliore les résultats récents sur les courbes invariantes [8] où le cas $p = 2$ était considéré.

Mots clefs: moyenne; application de type moyenne; moyenne invariante; function de graphe invariant; itération; équation fonctionnelle.

Introduction

Let X, Y be arbitrary sets and $T : X \times Y \to X \times Y$, $T = (f, g)$, be a given mapping. If a function $\varphi : U \to Y$, for some $U \subset X$, is such that

$$\varphi[f(x, \varphi(x))] = g(x, \varphi(x)), \qquad x \in U, \tag{1}$$

then the graph of φ is invariant under the mapping T. In the case when U is a real interval, Y is a metric space and φ is continuous, the graph of φ is a T- invariant curve. If φ is unknown, the composite equation (1), called a *functional equation on invariant curves,* under some stronger regularity assumptions, was treated by several authors (cf. Montel [11] (also Hadamard [2], Kuczma [4]). In some special cases, the continuous solutions of equation (1) were considered, for instance, by Jarczyk [3], Matkowski & Okrzesik [9], [10].

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Let us fix $p \in \mathbb{N}$, $p \geq 2$, and an interval $I \subset \mathbb{R}$. Take some arbitrary continuous means $M_1, \ldots, M_p : I^p \to I$ and put $\mathbf{M} = (M_1, \ldots, M_p)$. In the present paper we consider the problem of $\mathbf{M}$-invariant graphs of functions $\varphi : J^{p-1} \to I$, where $J \subset I$ is a subinterval of I. More precisely, we consider equation (1) where $X := I^{p-1}$, $Y := I$, $T := \mathbf{M}$ with $f = (M_1, \ldots, M_{p-1})$, $g = M_p$, and $U = J$.

According to a theorem on invariant means (section 1, Theorem 1, [7]; cf. also [5] and [6]), under general conditions, for the mean-type mapping $\mathbf{M} : I^p \to I^p$, there exists a unique continuous mean $K : I^p \to I$ such that

$$K \circ \mathbf{M} = K$$

and, moreover, the sequence $(\mathbf{M}^n)_{n \in \mathbb{N}}$ of iterates of $\mathbf{M}$ converges to the mean-type mapping $\mathbf{K} = (K_1, \ldots, K_p)$ with $K_1, \ldots, K_p = K$. Applying this result in section 2 we prove that if the graph of a continuous function $\varphi : J^{p-1} \to I$ is $\mathbf{M}$-invariant, then φ satisfies the following simple functional equation

$$\varphi[K(\bar{x}, \varphi(\bar{x})), \ldots, K(\bar{x}, \varphi(\bar{x}))] = K(\bar{x}, \varphi(\bar{x}))$$

for all $\bar{x} := (x_1, \ldots, x_{p-1}) \in J^{p-1}$ (Theorem 2). As a conclusion we obtain Theorem 3 allowing, in particular, to determine all the continuous and decreasing in each variable functions φ of the $\mathbf{M}$-invariant graphs and stating that any increasing with respect to each variable solution φ must be a mean. This extends and improves a recent result of [8] where the case $p = 2$ is considered. Some examples are also given.

1. Mean-type mappings, their iterates and invariant means

Let $I \subset \mathbb{R}$ be an interval and $p \in \mathbb{N}$, $p \geq 2$, be fixed. A function $M : I^p \to \mathbb{R}$ is said to be a *mean* on I (cf., for instance, Bullen [1]) if , for all $x_1, \ldots, x_p \in I$,

$$\min(x_1, \ldots, x_p) \leq M(x_1, .., x_p) \leq \max(x_1, \ldots, x_p).$$

A mean M in I is called *strict* if these inequalities are sharp whenever

$$\min(x_1, \ldots, x_p) < \max(x_1, \ldots, x_p).$$

We say that a mean M in I is *positively homogeneous,* if for all $x_1, \ldots, x_p \in I$ and $t > 0$,

$$(tx_1, \ldots, tx_p \in I) \Longrightarrow M(tx_1, \ldots, tx_p) = tM(x_1, .., x_p).$$

The remarks below are easy to verify.

Remark 1. *Let $M : I^p \to \mathbb{R}$ be an arbitrary function. Then the following conditions are equivalent*
1. *M is a mean;*
2. *$M(J^p) \subset J$ for every subinterval $J \subset I$,*
3. *$M(J^p) = J$ for every subinterval $J \subset I$.*

Remark 2. *If $M : I^p \to \mathbb{R}$ is a mean then M maps I^p onto I and, moreover, M is reflexive, that is, for all $x \in I$,*

$$M(x, \ldots, x) = x.$$

Remark 3. *If a function $M : I^p \to \mathbb{R}$ is reflexive and (strictly) increasing with respect to each variable, then M is a (strict) mean I.*

A mapping $\mathbf{M} : I^p \to I^p$ is referred to as *mean-type* if there are some means $M_i : I^p \to I$, $i = 1, \ldots, p$, such that $\mathbf{M} = (M_1, \ldots, M_p)$. We say that the mean-type mapping $\mathbf{M}$ is *strict (positively homogeneous)* if each of its coordinate means $M_1, \ldots, M_p$ is strict (positively homogeneous).

If $\mathbf{M} : I^p \to I^p$ is a mean-type mapping then, clearly, the sequence $(\mathbf{M}^n)_{n=1}^{\infty}$ of the iterates of $\mathbf{M}$,

$$\mathbf{M}^1 := \mathbf{M}; \qquad \mathbf{M}^{n+1} := \mathbf{M} \circ \mathbf{M}^n \quad \text{for} \quad n \in \mathbb{N},$$

is well defined.

Let us note the following obvious

Remark 4. *Suppose that* $\mathbf{M} : I^p \to I^p$, $\mathbf{M} = (M_1, \ldots, M_p)$, *is a mean-type mapping of* I^p. *Then, for each* $n \in \mathbb{N}$,

$$\mathbf{M}^n = \left(M_1^n, \ldots, M_p^n\right)$$

where, for all $i = 1, \ldots, p$,

$$M_i^1 = M_i,$$

and, for all $n \in \mathbb{N}_0$, $i = 1, \ldots, p$, $(x_1, \ldots, x_p) \in I^p$,

$$M_i^{n+1}(x_1, \ldots, x_p) = M_i\left(M_1^n(x_1, \ldots, x_p), \ldots, M_p^n(x_1, \ldots, x_p)\right).$$

Given a mean-type mapping $\mathbf{M} : I^p \to I^p$ and a mean $K : I^p \to I$ we say that K is *invariant with respect to the mean-type mapping* $\mathbf{M}$, briefly, $\mathbf{M}$*-invariant*, if

$$K \circ \mathbf{M} = K.$$

Remark 5. *Note that a mean* $K : I^p \to I$ *is* $\mathbf{M}$*-invariant iff the mean-type mapping* $\mathbf{K} : I^p \to I^p$ *defined by* $\mathbf{K} = (K, \ldots, K)$ *is* $\mathbf{M}$*-invariant, that is, iff* $\mathbf{K} = \mathbf{K} \circ \mathbf{M}$.

Put

$$\Delta(I^p) := \left\{(x_1, \ldots, x_p) \in I^p : x_1 = \ldots = x_p\right\}.$$

We shall need the following

Theorem 1. *([7]) Let* $I \subset \mathbb{R}$ *be an interval and* $p \in \mathbb{N}$, $p \geq 2$, *fixed. Suppose that* $\mathbf{M} : I^p \to I^p$, $\mathbf{M} = (M_1, \ldots, M_p)$, *is a continuous mean-type mapping of* I^p *such that, for all* $(x_1, \ldots, x_p) \in I^p \backslash \Delta(I^p)$,

$$\begin{aligned}&\max(M_1(x_1, \ldots, x_p), \ldots, M_p(x_1, \ldots, x_p)) - \min(M_1(x_1, \ldots, x_p), \ldots, M_p(x_1, \ldots, x_p)) \\ &< \max(x_1, \ldots, x_p) - \min(x_1, \ldots, x_p).\end{aligned}$$

Then

1. for every $n \in \mathbb{N}$, *the* n*-th iterate* $\mathbf{M}^n = (M_{n,1}, \ldots, M_{n,p})$ *is a mean-type mapping of* I^p;

2. there is a continuous mean $K : I^p \to I$ *such that the sequence of iterates* $(\mathbf{M}^n)_{n=1}^{\infty}$ *converges, uniformly on compact subsets of* I^p, *to the mean-type mapping* $\mathbf{K} : I^p \to I^p$, $\mathbf{K} = (K_1, \ldots, K_p)$ *such that*

$$K_1 = \ldots = K_p = K;$$

3. $\mathbf{K} : I^p \to I^p$ *is* $\mathbf{M}$*-invariant, that is,*

$$\mathbf{K} = \mathbf{K} \circ \mathbf{M}$$

or, equivalently, the mean K *is* $\mathbf{M}$*-invariant;*

4. a continuous $\mathbf{M}$*-invariant mean (mean-type mapping) is unique;*

5. if $\mathbf{M}$ *is strict then so is* K *(and* $\mathbf{K}$*);*

6. if $M_1, \ldots, M_p$ *are (strictly) increasing with respect to each variable then so is* K*;*

7. if $\mathbf{M}$ *is positively homogeneous, then every iterate of* $\mathbf{M}$ *and* K *are positively homogeneous.*

Note that the inequality assumed in this theorem is satisfied if at most one of the coordinate means $M_1, \ldots, M_p$ is not strict.

2. Invariant graphs of functions

The main result reads as follows

Theorem 2. *Let $p \in \mathbb{N}$, $p \geq 2$, and let $I, J \subset \mathbb{R}$, $J \subset I$, be intervals. Suppose that $M_1, \ldots, M_p : I^p \to I$ are continuous means such that*

$$\begin{aligned} &\max(M_1(x_1, \ldots, x_p), \ldots, M_p(x_1, \ldots, x_p)) - \min(M_1(x_1, \ldots, x_p), \ldots, M_p(x_1, \ldots, x_p)) \\ &< \max(x_1, \ldots, x_p) - \min(x_1, \ldots, x_p), \end{aligned}$$

for all $(x_1, \ldots, x_p) \in I^p \backslash \Delta(I^p)$.

If the graph of a continuous mapping $\varphi : J^{p-1} \to I$ is an invariant under the mean-type mapping $(M_1, \ldots, M_p)$, that is, if

$$\begin{aligned} \varphi[M_1(x_1, \ldots, x_{p-1}, \varphi(x_1, \ldots, x_{p-1})), \ldots, M_{p-1}(x_1, \ldots, x_{p-1}, \varphi(x_1, \ldots, x_{p-1})] \\ = M_p(x_1, \ldots, x_{p-1}, \varphi(x_1, \ldots, x_{p-1})) \end{aligned} \tag{2}$$

for all $x_1, \ldots, x_{p-1} \in J$, then φ satisfies the functional equation

$$\begin{aligned} \varphi[K(x_1, \ldots, x_{p-1}, \varphi(x_1, \ldots, x_{p-1})), \ldots, K(x_1, \ldots, x_{p-1}, \varphi(x_1, \ldots, x_{p-1})] \\ = K(x_1, \ldots, x_{p-1}, \varphi(x_1, \ldots, x_{p-1})) \end{aligned} \tag{3}$$

for all $x_1, \ldots, x_{p-1} \in J$, where $K : I^p \to I$ is a unique continuous $(M_1, \ldots, M_p)$-invariant mean.

Proof. Put $\mathbf{M} := (M_1, \ldots, M_p)$. In view of Theorem 1, there is a unique continuous $\mathbf{M}$-invariant mean K in I. According to Remark 4 we write

$$\mathbf{M}^n = (M_1^n, \ldots, M_p^n)$$

for the nth iterate of $\mathbf{M}$ ($n \in \mathbb{N}$). Assume that the graph of a continuous function $\varphi : J^{p-1} \to I$ is invariant under the transformation $\mathbf{M}$, that is (2) holds true. Denote

$$\bar{x} := (x_1, \ldots, x_{p-1}).$$

We shall show that

$$\varphi[M_1^n(\bar{x}, \varphi(\bar{x})), \ldots, M_{p-1}^n(\bar{x}, \varphi(\bar{x}))] = M_p^n(\bar{x}, \varphi(\bar{x})) \tag{4}$$

for all $\bar{x} = (x_1, \ldots, x_{p-1}) \in J^{p-1}$ and $n \in \mathbb{N}$. Indeed, from (2) we have

$$\begin{aligned} \varphi[M_1^1(\bar{x}, \varphi(\bar{x})), \ldots, M_{p-1}^1(\bar{x}, \varphi(\bar{x}))] &= \varphi[M_1(\bar{x}, \varphi(\bar{x})), \ldots, M_{p-1}(\bar{x}, \varphi(\bar{x}))] \\ &= M_p(\bar{x}, \varphi(\bar{x})) = M_p^1(\bar{x}, \varphi(\bar{x})) \end{aligned}$$

for all $\bar{x} = (x_1, \ldots, x_{p-1}) \in J^{p-1}$, so the equality (4) holds true for $n = 1$. Assume that (4) holds true for some $n \in \mathbb{N}$. Since

$$(M_1^{n+1}, \ldots, M_p^{n+1}) = (M_1 \circ (M_1^n, \ldots, M_p^n), \ldots, M_p \circ (M_1^n, \ldots, M_p^n)), \quad n \in \mathbb{N},$$

applying in turn: Remark 4, (4), (2), again (2) and Remark 4, we get

$$\begin{aligned} &\varphi[M_1^{n+1}(\bar{x}, \varphi(\bar{x})), \ldots, M_{p-1}^{n+1}(\bar{x}, \varphi(\bar{x}))] \\ &= \varphi\left\{M_1\left[\left(M_1^n(\bar{x}, \varphi(\bar{x})), \ldots, M_{p-1}^n(\bar{x}, \varphi(\bar{x})), M_p^n(\bar{x}, \varphi(\bar{x}))\right)\right], \ldots,\right. \\ &\qquad \left. M_{p-1}\left[\left(M_1^n(\bar{x}, \varphi(\bar{x})), \ldots, M_{p-1}^n(\bar{x}, \varphi(\bar{x})), M_p^n(\bar{x}, \varphi(\bar{x}))\right)\right]\right\} \\ &= \varphi\left\{M_1\left[\left(M_1^n(\bar{x}, \varphi(\bar{x})), \ldots, M_{p-1}^n(\bar{x}, \varphi(\bar{x})), \varphi[M_1^n(\bar{x}, \varphi(\bar{x})), \ldots, M_{p-1}^n(\bar{x}, \varphi(\bar{x}))]\right)\right], \ldots,\right. \end{aligned}$$

$$M_{p-1}\left[M_1^n(\bar{x},\varphi(\bar{x})),\dots,M_{p-1}^n(\bar{x},\varphi(\bar{x})),\varphi[M_1^n(\bar{x},\varphi(\bar{x})),\dots,M_{p-1}^n(\bar{x},\varphi(\bar{x}))]\right]\}$$
$$= M_p\left(M_1^n(\bar{x},\varphi(\bar{x})),\dots,M_{p-1}^n(\bar{x},\varphi(\bar{x})),\varphi[M_1^n(\bar{x},\varphi(\bar{x})),\dots,M_{p-1}^n(\bar{x},\varphi(\bar{x}))]\right)$$
$$= M_p\left(M_1^n(\bar{x},\varphi(\bar{x})),\dots,M_{p-1}^n(\bar{x},\varphi(\bar{x})),M_p^n(\bar{x},\varphi(\bar{x}))\right)$$
$$= M_p\left(M_1^n(\bar{x},\varphi(\bar{x})),\dots,M_p^n(\bar{x},\varphi(\bar{x}))\right) = M_p\left(\mathbf{M}^n(\bar{x},\varphi(\bar{x}))\right)$$
$$= M_p^{n+1}((\bar{x},\varphi(\bar{x})))$$

for all $\bar{x} = (x_1,\dots,x_{p-1}) \in J^{p-1}$, and the induction completes the proof of (4). Now, by the continuity of φ, letting $n \to \infty$ and applying Theorem 1, we obtain (3). □

Theorem 3. *Let $p \in \mathbb{N}$, $p \geq 2$, the interval $I \subset \mathbb{R}$, and the means $M_1,\dots,M_p : I^p \to I$ satisfy the conditions of Theorem 2. Let $K : I^p \to I$ be a unique $(M_1,\dots,M_p)$-invariant mean. Suppose that K is strictly increasing with respect to each variable. Then:*

(i) *For every $c \in I$, the set*

$$I_K(c) := \left\{(x_1,\dots,x_{p-1}) \in I^{p-1} : K(x_1,\dots,x_{p-1},x_p) = c \text{ for some } x_p \in I\right\}$$

is a connected subset of I^{p-1},

$$(c,\dots,c) \in \operatorname{Int} I_K(c),$$

and there exists exactly one function $\varphi_{K,c} : I_K(c) \to I$ such that

$$K(x_1,\dots,x_{p-1},\varphi_{K,c}(x_1,\dots,x_{p-1})) = c, \qquad (x_1,\dots,x_{p-1}) \in I_K(c); \tag{5}$$

moreover $\varphi_{K,c}$ is continuous, strictly decreasing with respect to each variable, and

$$\varphi_{K,c}(c,\dots,c) = c.$$

For every $c \in I$, the graph of the function $\varphi_{K,c}$ is invariant with respect to the mean-type mapping $(M_1,\dots,M_p)$.

(ii) *Suppose that for some interval $J \subset I$, the function $\varphi : J^{p-1} \to I$ is continuous, decreasing with respect to each variable, and graph of φ is invariant with respect to the mapping $(M_1,\dots,M_p)$.*

If the function

$$J^{p-1} \ni (x_1,\dots,x_{p-1}) \to K(x_1,\dots,x_{p-1},\varphi(x_1,\dots,x_{p-1})) \tag{6}$$

is monotonic with respect to each variable, then there is $c \in I$ such that

$$K(x_1,\dots,x_{p-1},\varphi(x_1,\dots,x_{p-1})) = c, \qquad x_1,\dots,x_{p-1} \in J.$$

(iii) *If $\varphi : I^{p-1} \to I$ is continuous, increasing with respect to each variable, and its graph is invariant with respect to the mapping $(M_1,\dots,M_p)$, then φ is a mean in the set J_φ^{p-1} where J_φ is the interval defined by*

$$J_\varphi := \{K(x_1,\dots,x_{p-1},\varphi(x_1,\dots,x_{p-1})) : x_1,\dots,x_{p-1} \in I\}.$$

Proof. Ad (i). Since the first statement of (i) is easy to show, we prove the second. Note that, for any fixed $c \in I$, by the $(M_1,\dots,M_p)$-invariance of K and (5) we have

$$K\left(M_1(\bar{x},\varphi_{K,c}(\bar{x})),\dots,M_p(\bar{x},\varphi_{K,c}(\bar{x}))\right) = K(\bar{x},\varphi_{K,c}(\bar{x})) = c,$$

for all $\bar{x} = (x_1,\dots,x_{p-1}) \in I_K(c)$, whence, by the uniqueness of $\varphi_{K,c}$,

$$\varphi_{K,c}\left(M_1(\bar{x},\varphi_{K,c}(\bar{x})),\dots,M_{p-1}(\bar{x},\varphi_{K,c}(\bar{x}))\right) = M_p(\bar{x},\varphi_{K,c}(\bar{x})), \qquad \bar{x} \in I_K(c).$$

Ad (ii). Assume that the function (6) is decreasing. Then, by the assumed increasing monotonicity of K, the left-hand side of equality (3) is an increasing function and the right-hand side is a decreasing function. It follows that (6) must be a constant function. In the case when the function (6) is nondecreasing we argue similarly.

Ad (iii). Take $x \in J_\varphi$. Then $x := K(x_1, \ldots, x_{p-1}, \varphi(x_1, \ldots, x_{p-1}))$ for some $x_1, \ldots, x_{p-1} \in I$. From Theorem 2, setting x in (3), we get

$$\varphi(x, \ldots, x) = x, \qquad x \in J_\varphi,$$

which proves that φ is reflexive. Now the result follows from Remark 2. □

Example 1. *For $p = 3$ consider the mean-type mapping $\mathbf{M} = (M_1, M_2, M_3) : \mathbb{R}^3 \to \mathbb{R}^3$ such that*

$$M_1(x, y, z) = ax + by + (1 - a - b)z, \qquad M_2(x, y, z) = cx + dy + (1 - a - d)z,$$

$$M_3(x, y, z) = kx + my + (1 - k - m)z$$

where a, b, c, d, k, m are positive and $a + b, c + d, k + m$ are less than 1. It easy to verify that the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$,

$$\varphi(x, y) := rx + (1 - r)y,$$

satisfies the following special case of equation (2):

$$\varphi[M_1(x, y, \varphi(x, y)), M_2(x, y, \varphi(x, y))] = M_3(x, y, \varphi(x, y)), \qquad x, y \in \mathbb{R},$$

if, and only if,

$$(a + b - c - d)r^2 - (a - 2c - d + k + m)r - c + m = 0.$$

Taking $d = a$, $c = b$ and $m = k$ we get $r = \frac{1}{2}$. Thus, in this case the function φ, the arithmetic mean

$$\varphi(x, y) = \frac{x + y}{2},$$

satisfies this equation and its graph is invariant with respect to the mapping (M_1, M_2, M_3).

Taking here $a = \frac{1}{2}$, $b = \frac{1}{4}$, $k = m = \frac{1}{4}$, we obtain that

$$K(x, y, z) = \frac{x + y + z}{3}$$

is (a unique continuous) (M_1, M_2, M_3)-invariant mean. In view of the first part of Theorem 3, for any $C \in \mathbb{R}$, the graph of the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi = \varphi_{K,c}$,

$$\varphi(x, y) = 3C - x - y$$

is invariant with respect to the mean-type mapping (M_1, M_2, M_3).

Example 2. *It is easy to verify that $\mathbf{M} : (0, \infty)^3 \to (0, \infty)^3$, $\mathbf{M} = (A, N, H)$, where*

$$A(x, y, z) \ : \ = \frac{x + y + z}{3}, \quad N(x, y, z) := \frac{yx + zy + xz}{x + y + z},$$
$$H(x, y, z) \ : \ = \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} = \frac{3xyz}{yz + zx + xy},$$

is a continuous, homogeneous and strict mean-type mapping. Let $J \subset (0, \infty)$ be a fixed interval. The problem to determine the continuous functions $\varphi : J^2 \to (0, \infty)$ such that the graph of φ is invariant with respect to $\mathbf{M}$ leads to the functional equation

$$\varphi\left(\frac{x + y + \varphi(x, y)}{3}, \frac{yx + \varphi(x, y)y + x\varphi(x, y)}{x + y + z}\right) = \frac{3xy\varphi(x, y)}{y\varphi(x, y) + \varphi(x, y)x + xy}$$

for all $x, y \in I$.

Let $G : (0,\infty)^3 \to (0,\infty)$ *denote the geometric mean, that is*

$$G(x,y,z) = \sqrt[3]{x,y,z}, \qquad x,y,z > 0.$$

Of course, G *is continuous. Since*

$$\begin{aligned} & G(A(x,y,z), N(x,y,z), H(x,y,z)) \\ = & \sqrt[3]{\frac{x+y+\varphi(x,y)}{3} \cdot \frac{yx+zy+xz}{x+y+z} \cdot \frac{3xyz}{yz+zx+xy}} = \sqrt[3]{xyz} \\ = & G(x,y,z) \end{aligned}$$

for all $x, y, z > 0$, *the geometric mean* G *is (a unique) invariant with respect to the mean-type mapping* $\mathbf{M}$. *Let* $J \subset (0,\infty)$ *be an interval. Applying Theorem 3 with* $K = G$, *we conclude that, if* $\varphi : J^2 \to (0,\infty)$ *is continuous, decreasing with respect to each variable, and its graph is invariant with respect to the map* (A,N,H), *then there is* $c > 0$ *such that* $\sqrt[3]{xy\varphi(x,y)} = c$ *for all* $x, y \in J$, *whence*

$$\varphi(x,y) = \frac{c^3}{xy}, \qquad x,y \in J.$$

Taking $n = 2$ in Theorem 2 we obtain

Corollary 1. *([8]) Let* $I, J \subset \mathbb{R}$, $J \subset I$, *be intervals. Suppose that* $M, N : I^2 \to I$ *are continuous means such that, for all* $x, y \in I$, $x \neq y$,

$$\max(M(x,y), N(x,y)) - \min(M(x,y), N(x,y)) < \max(x,y)) - \min(x,y).$$

If the graph of a continuous function $\varphi : J \to I$ *is an invariant curve under the mean-type mapping* (M,N), *that is, if*

$$\varphi[M(x,\varphi(x))] = N(x,\varphi(x)), \qquad x \in J,$$

then φ *satisfies the functional equation*

$$\varphi[K(x,\varphi(x))] = K(x,\varphi(x)), \qquad x \in J,$$

where K *is a unique continuous* (M,N)*-invariant mean.*

Thus Theorem 2 extends the main result of [8].

Taking $n = 2$ in Theorem 3 we obtain the following

Corollary 2. *Let* $I \subset \mathbb{R}$ *be an interval and let the means* $M, N : I^2 \to I$ *satisfy the conditions of Corollary 2. Let* $K : I^2 \to I$ *be a unique* (M,N)*-invariant mean. Suppose that* K *is strictly increasing with respect to each variable. Then:*

(i) *For every* $c \in I$, *the set*

$$I_K(c) := \{x \in I : K(x,y) = c \text{ for some } y \in I\}$$

is a subinterval of I,

$$c \in \operatorname{Int} I_K(c),$$

and there exists exactly one function $\varphi_{K,c} : I_K(c) \to I$ *such that*

$$K(x,\varphi(x)) = c, \qquad x \in I_K(c);$$

moreover $\varphi_{K,c}$ *is continuous, strictly decreasing with respect to each variable, and*

$$\varphi_{K,c}(c) = c.$$

For every $c \in I$, *the graph of the function* $\varphi_{K,c}$ *is invariant with respect to the mean-type mapping* (M, N).

(ii) *Suppose that for some interval* $J \subset I$, *the function* $\varphi : J \to I$ *is continuous, decreasing with respect to each variable, and graph of* φ *is invariant with respect to the mapping* (M, N).

If the function

$$J \ni x \to K(x, \varphi(x))$$

is monotonic with respect to each variable, then there is $c \in I$ *such that*

$$K(x, \varphi(x)) = c, \qquad x \in J.$$

(iii) *If* $\varphi : I \to I$ *is continuous, increasing with respect to each variable, and its graph is invariant with respect to the mapping* (M, N), *then* φ *is a mean in the set* J_φ^2 *where* J_φ *is the interval defined by*

$$J_\varphi := \{K(x, \varphi(x)) : x \in I\}.$$

This result essentially improves Theorem 3 in [8].

Note that Corollaries 1 and 2 cannot be deduced from the Montel theorem [11] (cf. also Kuczma [4]), where the Lipschitz-continuity of the mapping is assumed.

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