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LOW REYNOLDS NUMBER SWIMMING AND CONTROLLABILITY

F. Alouges[1]

Abstract. The modeling of swimming at the micro scale, either for biological understanding of microorganisms' motility, e.g. bacteria, or for the conceptual building of small mechanisms capable of underwater displacement, involves low Reynolds number flows. This paper is a short review of recent links unveiled by the author and his collaborators between this problem and control theory.

1. Introduction

The understanding of swimming capabilities of small objects or organisms is intimately linked to the study of self propulsion in a fluid at a low Reynolds number regime. Since the pioneering works by Taylor [40] and Lighthill [26], this problem has received a lot of attention both from the community of biophysicists and mathematicians (see the review papers [24] and the encyclopedia article [5] for a comprehensive list of references. Biological systems are also studied as in [19]).

Indeed, the Reynolds number $Re = LV/\nu$ used to estimate the ratio of the inertial effects to the viscous ones for a swimmer of size L moving at speed V in a fluid of kinematic viscosity ν can vary over many orders of magnitude. In water for instance, where $\nu \sim 1 \cdot 10^{-6} m^2 s^{-1}$ at room temperature, since the velocity rarely exceeds a few body lengths L per second, this number is small in the regime where $L \ll 1mm$ meaning that inertial effects become negligible, while viscous forces are dominant. The swimmer is forced to use only the viscous resistance of the surrounding fluid in order to move. The situation is comparable, at human scale, if water is replaced by a much more viscous fluid like honey or silicon.

Bacteria and unicellular organisms *are* of micron size, while artificial robots to be used non-invasively inside human bodies e.g. for medical purposes *must* be small. Understanding the secret of low Reynolds number swimming is therefore of prime interest for applications.

Being at low Reynolds number not only changes a lot the physics but also the mathematical picture of the problem. In particular, since viscous flows are reversible, it was remarked by Purcell [34] that a *reciprocal deformation*[I] of the swimmers could not induce any net motion. This property, nicely called *The scallop theorem* in reference to a scallop which can only move by opening and closing its valve is a crucial obstruction that the swimmer needs to circumvent in order to move. Moreover, the motion of microswimmers is geometric: the trajectory of a low Re swimmer is entirely determined by the sequence of shapes that the swimmer assumes. Doubling the rate of shape changes simply doubles the speed at which the same trajectory is traversed. As observed in [38], this suggests that there must be a natural, attractive mathematical framework for this problem.

The problems we face in order to proceed are numerous. First, we have to understand the dynamics of the system. Namely, given a (periodic) time history of shapes of a swimmer (a stroke), we need to determine the

[1] CMAP UMR 7641, École Polytechnique CNRS, Route de Saclay, 91128 Palaiseau Cedex - France (`francois.alouges@polytechnique.edu`)

[I] By this, we mean that during its stroke the swimmer follows the same sequence of shapes once forward and then backward to its original shape.

corresponding evolution of the position and orientation of the swimmer in time. Then, a typical question that one has to study is: starting from a given position and orientation, can the swimmer achieve any prescribed position and orientation by performing a suitable sequence of strokes? This is a question of *controllability*.

Once controllability is established, i.e. it is possible to go from A to B with a suitable sequence of strokes, one can ask the question of how to go from A to B *at minimal energetic cost*. This is typically a question of *optimal control*.

Although, like for all locomotion problems, there is a clear connection between swimming, and in particular low Reynolds number swimming and control theory, this subject has only emerged quite recently, see for instance [23], and the more recent contributions [8], [12], [22] or [36]. Controllability and optimal control problems of low Reynolds number swimming are adressed e.g. in [2], [5] and [6] where the authors study in detail both controllability and optimal control for axisymmetric swimmers whose varying shapes are described by few (in particular, two) scalar parameters.

The purpose of this paper is to make a short review of recent results obtained by the author and collaborators on the subject. In the sequel, we show the basic structure of the equations for the dynamics of self-propelled swimmers at low Reynolds number, show the basic tools of geometric control theory that are needed to solve the controllability problem and discuss a little how optimal control questions can be naturally derived from the model and solved through numerical procedures.

2. Modelization of the fluid

At low Reynolds number, inertial terms can be neglected in Navier-Stokes equations. Calling Ω the domain filled by the fluid, namely the exterior of the swimmer, the fluid obeys Stokes equations in Ω

$$\left\{\begin{array}{rcll} -\eta\Delta\mathbf{u}+\nabla p &=& 0 & \text{in } \Omega\,, \\ \nabla\cdot\mathbf{u} &=& 0 & \text{in } \Omega\,, \\ \boldsymbol{\sigma}\mathbf{n} &=& \mathbf{f} & \text{on } \Gamma\,, \\ \mathbf{u} &\to& 0 & \text{at } \infty\,, \end{array}\right. \tag{1}$$

where $(\mathbf{u},p)$ are respectively the velocity and the pressure of the fluid, η its viscosity, $\boldsymbol{\sigma}:=\eta(\nabla\mathbf{u}+\nabla\mathbf{u}^t)-p\mathbf{Id}$ is the Cauchy stress tensor and $\mathbf{n}$ is the outer unit normal to $\Gamma=\partial\Omega$ (hence, $\mathbf{n}$ points from the fluid to the interior of the swimmer). Existence and uniqueness of a solution to (1) is classical in the Hilbert space

$$\mathcal{V} \;:=\; \left\{\mathbf{u}\in\mathcal{D}'(\Omega,\mathbf{R}^3)\mid\nabla\mathbf{u}\in L^2(\Omega),\;\frac{\mathbf{u}}{\sqrt{1+|\mathbf{r}|^2}}\in L^2(\Omega)\right\},$$

endowed with the norm

$$\|\mathbf{u}\|^2_{\mathcal{V}} \;:=\; \int_\Omega|\nabla\mathbf{u}|^2\,.$$

If we assume that the force field $\mathbf{f}$ that the swimmer imposes to the fluid belongs to $\mathcal{H}^{-1/2}(\partial\Omega)$, it is well-known that the solution of (1) can be expressed in terms of the stokeslet

$$\mathbf{G}(\mathbf{r}) \;:=\; \frac{1}{8\pi\eta}\left(\frac{1}{|\mathbf{r}|}+\frac{\mathbf{r}\otimes\mathbf{r}}{|\mathbf{r}|^3}\right) \tag{2}$$

as

$$\mathbf{u}(\mathbf{r})=\int_\Gamma\mathbf{G}(\mathbf{r}-\mathbf{s})\mathbf{f}(\mathbf{s})\,d\mathbf{s}\,. \tag{3}$$

The Neumann-to-Dirichlet map

$$\begin{aligned} \mathcal{T}_\Gamma : \mathcal{H}^{-1/2}(\Gamma) &\longrightarrow \mathcal{H}^{1/2}(\Gamma) \\ \mathbf{f} &\longmapsto \mathbf{u}|_\Gamma = \left(\int_\Gamma \mathbf{G}(\mathbf{r}-\mathbf{s})\mathbf{f}(\mathbf{s})\, d\mathbf{s} \right) |_\Gamma \end{aligned} \tag{4}$$

links linearly the velocity distribution $\mathbf{u}$ to the force distribution $\mathbf{f}$. Notice however that $\mathcal{T}_\Gamma$ highly depends (and in a non trivial and certainly non linear way) on Γ.

3. Dynamics of low Reynolds self-propelled swimmers

3.1. A universal structure

The swimmer is a deformable body and we focus on situations where its state is completely determined by:

- the *shape variables*, denoted by $\xi \in \mathcal{S}$, where $\mathcal{S}$ is an open connected subset of $\mathbf{R}^M$;
- the *position variables*, denoted by $p \in \mathcal{P}$, which describe the global position and orientation in space of the swimmer. For instance, one can have in mind the dimension 6 manifold

  $$\mathcal{P} = \mathbf{R}^3 \times \mathrm{SO}(3)$$

  which represents translations and rotations in the three-dimensional space.

Physically speaking, the swimmer is able to change its shape. Doing so, it pushes the fluid, which reacts, obeying Stokes equations, and pushes and rotates the swimmer, changing its position variables p. We show that there is a universal structure which is highlighted by the following lemma.

Lemma 3.1. *The system that describes the dynamics of low Re swimmers can be written in the form*

$$\frac{d}{dt}\begin{pmatrix} \xi \\ p \end{pmatrix} = \sum_{i=1}^{M} \mathbf{F}_i(\xi, p)\dot{\xi}_i \,. \tag{1}$$

It is therefore a control problem linear in the controls and without drift where the controls are the rate of shape changes.

Notice, that this is a very general statement which is not specific to the actual variables that have been chosen to describe the shape or the position of the swimmer.

Proof
Neglecting also the inertia of the swimmer itself, and keeping in mind that self propulsion forbids the swimmer to use external forces, the equations that govern the dynamics of the swimmer are simply given by saying that the total force and torque applied to the swimmer by the surrounding fluid[II] should vanish

$$\begin{cases} \mathbf{F}_{\text{tot}} = \displaystyle\int_\Gamma \boldsymbol{\sigma}(\mathbf{s})\mathbf{n}(\mathbf{s})\, d\mathbf{s} = 0\,, \\ \mathbf{T}_{\text{tot}} = \displaystyle\int_\Gamma \mathbf{s} \times \boldsymbol{\sigma}(\mathbf{s})\mathbf{n}(\mathbf{s})\, d\mathbf{s} = 0\,. \end{cases} \tag{2}$$

[II] Notice that by action-reaction principle, these are also the (opposite of the) force and torque applied to the fluid by the swimmer.

Remark that we have here computed the torque with respect to the origin of the fixed lab frame. However, due to the vanishing of the total force, this torque does not depend on the chosen point with reference to which it is computed. Using the inverse of the Neumann-to-Dirichlet map $\mathcal{T}_\Gamma$, we rewrite these equations in terms of the velocity of the fluid $\mathbf{v}$, and thus, due to non-slip boundary condition, of the boundary of the swimmer itself

$$\begin{cases} \displaystyle\int_\Gamma \left(\mathcal{T}_\Gamma^{-1}\mathbf{v}\right)(\mathbf{s})\, d\mathbf{s} = 0\,, \\[2ex] \displaystyle\int_\Gamma \mathbf{s} \times \left(\mathcal{T}_\Gamma^{-1}\mathbf{v}\right)(\mathbf{s})\, d\mathbf{s} = 0\,. \end{cases} \tag{3}$$

Calling Γ_0 a reference configuration of the swimmer which is used to parameterize the current configuration Γ, the system being uniquely determined by the variables (ξ, p), there exists a function

$$\begin{aligned} \mathbf{X} : \mathcal{S} \times \mathcal{P} \times \Gamma_0 \quad &\longrightarrow \quad \Gamma \subset \mathbf{R}^3 \\ (\xi, p, \mathbf{s}_0) \quad &\mapsto \quad \mathbf{X}(\xi, p, \mathbf{s}_0) \end{aligned}$$

which gives the current position of a point $\mathbf{s}_0$ of the reference shape Γ_0 in Γ when the swimmer is in the state $(\xi, p) \in \mathcal{S} \times \mathcal{P}$. The non-slip boundary condition on Γ imposes that the velocity $\mathbf{v}(\mathbf{s})$ of the fluid at the point $\mathbf{s} = \mathbf{X}(\xi, p, \mathbf{s}_0)$ is given by

$$\mathbf{v}(\mathbf{s}) = \frac{d}{dt}\mathbf{X}(\xi, p, \mathbf{s}_0) = (\dot{\xi} \cdot \nabla_\xi)\mathbf{X}(\xi, p, \mathbf{s}_0) + (\dot{p} \cdot \nabla_p)\mathbf{X}(\xi, p, \mathbf{s}_0) \tag{4}$$

which is a linear combination of $(\dot{\xi}, \dot{p})$. Using (4) into (3), leads to a linear system which links $\dot{\xi}$ and $\dot{p}$

$$\begin{cases} \displaystyle\int_\Gamma \left(\mathcal{T}_\Gamma^{-1}\left[(\dot{\xi} \cdot \nabla_\xi)\mathbf{X}(\xi, p, \cdot) + (\dot{p} \cdot \nabla_p)\mathbf{X}(\xi, p, \cdot)\right]\right)(\mathbf{s})\ d\mathbf{s} = 0\,, \\[2ex] \displaystyle\int_\Gamma \mathbf{s} \times \left(\mathcal{T}_\Gamma^{-1}\left[(\dot{\xi} \cdot \nabla_\xi)\mathbf{X}(\xi, p, \cdot) + (\dot{p} \cdot \nabla_p)\mathbf{X}(\xi, p, \cdot)\right]\right)(\mathbf{s})\ d\mathbf{s} = 0\,. \end{cases} \tag{5}$$

This system of equations has to be understood in the following way: changing its shape, the swimmer is able to produce a non zero $\dot{p}$. This change of shape produces a change of position $\dot{p}$ which can be computed in such a way that (5) is satisfied. It turns out that this is always possible (there is a unique solution $\dot{p}$ to (5)) since it can be written formally as a linear system

$$A\dot{p} = b$$

where the matrix A happens to be the *grand resistance matrix* of the system (the matrix that produces the force and torque due to translations and rotations of the object). This matrix is known to be symmetric and negative definite (see [20] for instance). Solving (5) in terms of $\dot{p}$ leads to (1). □

3.2. **The scallop theorem**

It was remarked in [34] that a swimmer which possesses only one degree of freedom, essentially always returns to its starting position after a stroke. This feature, known as the *scallop theorem* in the literature, is physically speaking a consequence of the reversibility of viscous flows. To be more precise the scallop theorem is more general than that and expresses the fact that any swimmer – with an arbitrary number of degree of freedom – which executes a *reciprocal motion* (a deformation back and forth along the same path of shapes) does not undergo any net displacement at the end of the stroke.

Mathematically, the scallop theorem can be easily proved from the structure of the equation of motion. Indeed if $(\xi(s), p(s))$ is a solution of (1) for all $s \in [0, 1]$ then it is obvious that $(\xi(\varphi(s)), p(\varphi(s)))$ is also a solution for every change of variable $\varphi : [0,1] \longrightarrow [0,1]$ of class $\mathcal{C}^1$. Therefore, since in a reciprocal shape change, the shape history in the recovery phase is obtained from the one of the forward phase by a simple change of variable, the position history will follow the same rule and the swimmer at the end of the stroke returns to its original position.

An even simpler case is obtained by considering a one degree of freedom swimmer, and rewriting (1) as

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ p \end{pmatrix} = \mathbf{F}_1(\xi, p)\dot{\xi}_1 = \begin{pmatrix} 1 \\ \mathbf{F}_{1,p}(\xi_1, p) \end{pmatrix} \dot{\xi}_1 \,. \tag{6}$$

Imagine furthermore that the vectorfield $\mathbf{F}_1$ do not depend on p, then the dynamics the position equation becomes

$$\frac{d}{dt}p = \mathbf{F}_{1,p}(\xi_1)\dot{\xi}_1 \,. \tag{7}$$

where now $\xi_1 \in \mathbf{R}$. Integrating the former expression leads to

$$p(t) = \mathcal{F}_{1,p}(\xi_1(t)) \,. \tag{8}$$

where $\mathcal{F}_{1,p}$ is a primitive of $\mathbf{F}_{1,p}$. Therefore a periodic in time curve of ξ_1 leads to a periodic in time position p, or, in other words, the swimmer returns to its original position at the end of the stroke.

Equation (8) is a so-called *holonomic constraint* linking ξ_1 and p. If the system has such a constraint then it is clear that no net displacement can be obtained by any stroke. The swimming problem, i.e., is the swimmer capable to change its position with a suitable shape change, can thus be rephrased as the fact that the equation of motion (1) is nonholonomic.

3.3. Geometric control theory

When using $(\dot{\xi}_i)_{1\le i\le M}$ as control variables, the preceding system belongs to the class of linear control systems of the form

$$\frac{dX}{dt} = \sum_{i=1}^{M} \alpha_i(t) F_i(X) \tag{9}$$

where X is a point of a $d-$dimensional manifold $\mathcal{M}$, and $(F_i)_{1\le i\le M}$ are p vector fields defined on $T\mathcal{M}$, the tangent bundle of $\mathcal{M}$. For such systems, the tools of geometric control theory apply. Moreover, if, as it will be our case in the examples below, $\mathcal{M}$ and the vector fields F_i are furthermore analytic, very strong results exist. An exposition of Geometric Control Theory can be found for instance in [1, 13, 21].

In (9), the system is governed by the controls $(\alpha_1, \cdots, \alpha_p)$, and the simplest controllability question that one can state is as follows:

Question 1. *For any pair of states $(X_0, X_1) \in \mathcal{M}^2$, are there M functions $\alpha_i : [0,T] \to \mathbf{R}$ such that solving (9) starting from $X(0) = X_0$ leads to $X(T) = X_1$?*

A classical result by Rashevsky and Chow [1, 21] states that the preceding question has a positive answer for any X_1 in a suitable neighborhood of X_0 with piecewise constant controls α_i (we say that the system (9) is locally controllable near $X_0 \in \mathcal{M}$) if the Lie algebra generated by $(F_1, \cdots, F_M)$ is of full rank at X_0

$$\dim \mathit{Lie}(F_1, \cdots, F_M)(X_0) = \dim T_{X_0}\mathcal{M} \,. \tag{10}$$

Here, $T_{X_0}\mathcal{M}$ is the tangent space to $\mathcal{M}$ at X_0 and the Lie algebra $\mathit{Lie}(F_1, \cdots, F_M)(X_0)$ is the algebra obtained from the vector fields $(F_1, \cdots, F_M)$ by successive applications of the Lie bracket operation defined as

$$[F, G](X) = (F \cdot \nabla)G(X) - (G \cdot \nabla)F(X) \,.$$

Let us explain why the Lie algebra plays a role in this setting.

Given a vector field $F : \mathcal{M} \longrightarrow T\mathcal{M}$, we call $\exp(tF)(X_0)$ the solution at time t of

$$\left[\begin{array}{l} \dfrac{dX}{dt} = F(X)\,, \\ X(0) = X_0\,. \end{array}\right. \tag{11}$$

The main idea is that one can "move" from X_0 in the direction $g = \sum_i \beta_i F_i(X_0)$ by using (11) with $F = \sum_i \beta_i F_i$ since we have, to first order, for small $|t| \ll 1$,

$$\exp(tF)(X_0) = X_0 + tF(X_0) + O(t^2) = X_0 + tg + O(t^2)\,.$$

This formula shows that the global displacement of the solution of the dynamical system is proportional to time for these directions.

A more subtle move allows one to reach points in the direction $[F_i, F_j]$. Namely, we now need to consider

$$\exp(-tF_i) \circ \exp(-tF_j) \circ \exp(tF_i) \circ \exp(tF_j)(X_0) = X_0 + t^2[F_i, F_j](X_0) + O(t^3)\,,$$

which also shows that in time t, one can reach a displacement in the desired direction which is proportional to t^2. Lie bracket directions of higher order are also attainable at the price of higher order expressions in t, leading to smaller displacements (for small values of t). If the Lie algebra has a full rank, every direction in $\mathcal{M}$ is attainable and the system is locally controllable.

A more precise result is also known in the case where the vectorfields F_i are furthermore analytic. Indeed the Hermann-Nagano theorem (see [21]).

Theorem 3.2 (Hermann-Nagano)**.** *Let $\mathcal{M}$ an analytic manifold, and $\mathcal{F}$ a family of analytic vectorfields on $\mathcal{M}$. Then*

1. *each orbit of $\mathcal{F}$ is an analytic submanifold of $\mathcal{M}$, and*
2. *if $\mathcal{N}$ is an orbit of $\mathcal{F}$, then the tangent space of $\mathcal{N}$ at X is given by $Lie_X(\mathcal{F})$. In particular, the dimension of $Lie_X(\mathcal{F})$ is constant as X varies over $\mathcal{N}$.*

4. The 3 sphere swimmer of Najafi and Golestanian

4.1. **The swimmer**

This swimmer, has been initially proposed in [31] (see also [18]) in order to produce an example simpler than the 3 link swimmer of Purcell. It has been studied thoroughly in [2] and [3]. It is composed of 3 aligned spheres of same radius $a > 0$, as depicted in Fig. 1.

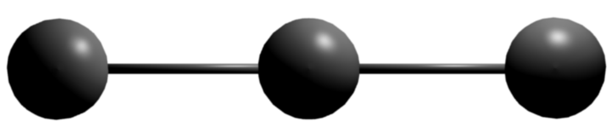

Figure 1. The Three Sphere swimmer of Najafi and Golestanian.

This swimmer is capable of changing the lengths of its arms by shortening and elongating thin jacks that link the spheres, as shown in Fig. 2. Calling $\xi = (\xi_1, \xi_2)$ the lengths of the arms, and p the position of the central sphere, we are led to consider the shape set $\mathcal{S} = (2a, +\infty)^2$ (the lower bound is chosen in order to avoid any overlap of the spheres), and the position set $\mathcal{P} = \mathbf{R}$. Indeed, due to axial symmetry, it is obviously seen that this swimmer may only move straight along its direction.

Theorem 4.1. *Assume the three sphere swimmer is self-propelled in a three dimensional infinite viscous flow modeled by Stokes equations. Then for any initial configuration $(\xi^i, p^i) \in \mathcal{S}\times\mathcal{P}$ any final configuration $(\xi^f, p^f) \in \mathcal{S}\times\mathcal{P}$ and any final time $T > 0$, there exists a stroke $\xi \in \mathcal{C}^0([0,T],\mathcal{S})$, piecewise $\mathcal{C}^1([0,T],\mathcal{S})$ satisfying $\xi(0) = \xi^i$ and $\xi(T) = \xi^f$ such that if the self-propelled swimmer starts in position p^i with the shape ξ^i at time $t = 0$, it ends at position p^f and shape ξ^f at time $t = T$ by changing its shape along $\xi(t)$.*

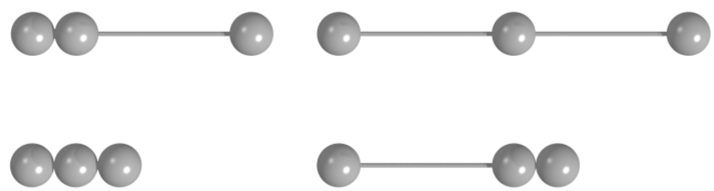

FIGURE 2. The Three Sphere swimmer of Najafi and Golestanian is capable of elongating or shortening its arms.

We refer the reader to [2] for the proof of this theorem. Let us however give a hint of the procedure used. At first sight, this seems very tempting to apply Chow's theorem since the system we want to study writes as a control problem, linear in the controls and without drift. However, the vectorfields $(\mathbf{F}_i)_{1\leq i\leq 2}$ are not explicit and depend on the outer Stokes problem, which itself depends on the shape of the swimmer parameterized by ξ. We proceed in the following steps.

- We first remark that because of the translational invariance of Stokes problem, the vectorfields $(\mathbf{F}_i)_{1\leq i\leq 2}$ do not depend on p. The system thus reads

$$\frac{d}{dt}\begin{pmatrix}\xi_1\\ \xi_2\\ p\end{pmatrix} = \begin{pmatrix}1\\ 0\\ V_1(\xi)\end{pmatrix}\dot{\xi}_1 + \begin{pmatrix}0\\ 1\\ V_2(\xi)\end{pmatrix}\dot{\xi}_2\,, \tag{1}$$

in which we have clarified the vectorfields $(\mathbf{F}_i)_{1\leq i\leq 2}$ as

$$\mathbf{F}_1(\xi) = \begin{pmatrix}1\\ 0\\ V_1(\xi)\end{pmatrix} \text{ and } \mathbf{F}_2(\xi) = \begin{pmatrix}0\\ 1\\ V_2(\xi)\end{pmatrix}.$$

Computing the Lie bracket between $\mathbf{F}_1$ and $\mathbf{F}_2$ gives

$$\begin{aligned}[\mathbf{F}_1,\mathbf{F}_2] &= F_1^1\frac{\partial\mathbf{F}_2}{\partial\xi_1} - F_2^1\frac{\partial\mathbf{F}_1}{\partial\xi_1} + F_1^2\frac{\partial\mathbf{F}_2}{\partial\xi_2} - F_2^2\frac{\partial\mathbf{F}_1}{\partial\xi_2} + F_1^3\frac{\partial\mathbf{F}_2}{\partial p} - F_2^3\frac{\partial\mathbf{F}_1}{\partial p}\\ &= \begin{pmatrix}0\\ 0\\ \frac{\partial}{\partial\xi_1}V_2(\xi) - \frac{\partial}{\partial\xi_2}V_1(\xi)\end{pmatrix}\end{aligned}$$

where we have denoted by F_i^j the $j-$th component of F_i.

We can thus write the *sufficient* condition for controllability which ensures that at (ξ, p) (for any p) the Lie algebra is of dimension 3 (or full rank)

$$\det(\mathbf{F}_1(\xi), \mathbf{F}_2(\xi), [\mathbf{F}_1(\xi), \mathbf{F}_2(\xi)]) \neq 0\,. \tag{2}$$

Here, an obvious computation shows that

$$\det(\mathbf{F}_1(\xi), \mathbf{F}_2(\xi), [\mathbf{F}_1(\xi), \mathbf{F}_2(\xi)]) = \frac{\partial}{\partial \xi_1} V_2(\xi) - \frac{\partial}{\partial \xi_2} V_1(\xi)\,. \tag{3}$$

- The next step has been to show that the vectofield $(\mathbf{F}_1, \mathbf{F}_2)$ are analytic functions in the shape variable ξ. This is not fully classical, since, it amounts to show that the solution to Stokes equation is analytic as a function of the *domain*, namely here, of the two variables (ξ_1, ξ_2) which parameterize it. This fact was originally proven in [2] and further much simplified in [4]. Same strategies of proofs for controllability was also used afterwards in the context of perfect fluids in [12].

  Knowing this, the determinant in (3) is an analytic function of ξ and vanish either everywhere or on a submanifold of $\mathbb{R}^3$. We eventually show that it does not vanish identically by computing an expansion of the determinant as $\xi_1, \xi_2 \to +\infty$.
- The proof is finished by considering the Martinet surface

$$\mathcal{V} = \{(\xi, p) \in \mathcal{S} \times \mathcal{P} \text{ such that } \det(\mathbf{F}_1, \mathbf{F}_2, [\mathbf{F}_1, \mathbf{F}_2])(\xi) = 0\}\,,$$

  and show that at any $\bar{\xi} \in \mathcal{V}$, at least one of the vectorfields $\mathbf{F}_1$ or $\mathbf{F}_2$ is transverse to the Martinet surface.

4.2. **generalizations**

This result has been generalized to other swimmers and situations. Namely, in [4] it has been proposed new swimmers composed of balls linked by arms which are able to swim in a plane or the whole space.

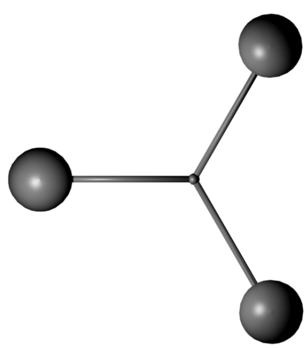

FIGURE 3. The three-sphere planar swimmer.

For the plane swimmer, shown in Fig. 3, we consider a reference equilateral triangle (S_1, S_2, S_3) with center $O \in \mathbf{R}^3$ in the horizontal plane (O, x, y) such that $\mathrm{dist}(O, S_i) = 1$ and define $\mathbf{t}_i = \vec{OS_i}$. Position and orientation in the horizontal plane are described by the coordinates of the center $\mathbf{c} \in \mathbf{R}^3$ (but $\mathbf{c}$ stays confined to the horizontal plane) and the horizontal angle θ that one arm, say arm number 1, makes with a fixed direction, say (O, x), in such a way that $d = 3$. Therefore, the swimmer is defined by the three balls (B_1, B_2, B_3) where we place the center of the ball B_i at $\mathbf{x}_i = \mathbf{c} + \xi_i \mathcal{R}_\theta \mathbf{t}_i$ with $\xi_i > 0$ for $i = 1, 2, 3$, where $\mathcal{R}_\theta$ stands for the horizontal rotation of angle θ given for instance by the matrix:

$$\mathcal{R}_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The swimmer is then fully described by the parameters $X = (\xi, \mathbf{c}, \theta) \in \mathcal{S} \times \mathcal{P}$, where $\mathcal{S} := (\frac{2a}{\sqrt{3}}, +\infty)^3$, the lower bound being chosen in order to avoid overlaps of the balls, $\mathcal{P} = \mathbf{R}^2 \times \mathbf{R}$.

For the 3D case, the situation is somehow more involved. In this case, we fix $N = 4$ and we consider a regular reference tetrahedron (S_1, S_2, S_3, S_4) with center $O \in \mathbf{R}^3$ such that $\mathrm{dist}(O, S_i) = 1$ and as before, we call $\mathbf{t}_i = \vec{OS_i}$ for $i = 1, 2, 3, 4$.

The position and orientation in the three dimensional space of the tetrahedron are described by the coordinates of the center $\mathbf{c} \in \mathbf{R}^3$ and a rotation $\mathcal{R} \in \mathrm{SO}(3)$, in such a way that $d = 6$.

The swimmer is now defined by the four balls (B_1, B_2, B_3, B_4) where the center of the ball B_i is placed at $\mathbf{x}_i = \mathbf{c} + \xi_i \mathcal{R} \mathbf{t}_i$ with $\xi_i > 0$ for $i = 1, 2, 3, 4$ as depicted in Fig. 4. We also forbid possible rotation of the spheres around the axes. A global rotation $(\mathcal{R} \neq \mathbf{Id})$ of the swimmer is however allowed.

The four ball cluster is now completely described by the list of parameters $X = (\xi, \mathbf{c}, \mathcal{R}) \in \mathcal{S} \times \mathcal{P}$, where $\mathcal{S} := (\sqrt{\frac{3}{2}}, +\infty)^4$ and $\mathcal{P} = \mathbf{R}^3 \times \mathrm{SO}(3)$. Again, the lower bound for ξ_i is chosen in order to avoid overlaps of the balls.

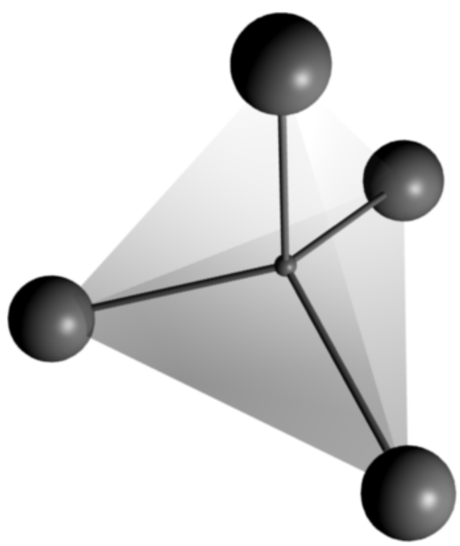

FIGURE 4. The four sphere swimmer (4S).

The same theorem of controllability than before has been proved in [4]. Although the strategy is the same than for the three sphere swimmer, the proof is much more involved in terms of calculations at least.

Theorem 4.2. *Consider any of the swimmers described before and assume it is self-propelled in a three dimensional infinite viscous flow modeled by Stokes equations. Then for any initial configuration $(\xi^i, p^i) \in \mathcal{S} \times \mathcal{P}$ any final configuration $(\xi^f, p^f) \in \mathcal{S} \times \mathcal{P}$ and any final time $T > 0$, there exists a stroke $\xi \in \mathcal{C}^0([0, T], \mathcal{S})$, piecewise*

$\mathcal{C}^1([0,T],\mathcal{S})$ *satisfying* $\xi(0)=\xi^i$ *and* $\xi(T)=\xi^f$ *such that if the self-propelled swimmer starts in position* p^i *with the shape* ξ^i *at time* $t=0$*, it ends at position* p^f *and shape* ξ^f *at time* $t=T$ *by changing its shape along* $\xi(t)$*.*

4.3. **Swimming with boundaries**

Other swimmers have been proposed and studied as the Purcell's three link swimmer [9, 34], the Purcell's rotator [15], the Purcell's "toroïdal" swimmer [25], the stick and donut (see Fig. 5) or other axisymmetric swimmers [3, 6, 27]. They all enter in the framework we have explained above, and understanding the geometry and more precisely the Lie algebra generated by the underlying vectorfields turns out to be a crucial step for the rigorous proof of the swimming capabilities of such systems.

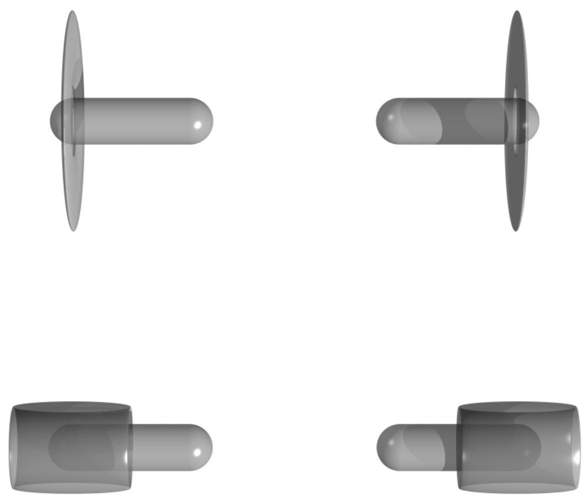

FIGURE 5. The stick and donut swimmer in which a central part in able to slide into an outer "parachute". This is also a "two control" swimmer and the picture shows the shapes obtained along a typical stroke.

Yet another example in which the same theory applies, though to the price of even more complex computations is the case where the swimmer is assumed to swim in a bounded environment, e.g. a half 3D-space. To our knowledge, the very first attempt to tackle mathematically this problem is given in [7] (compare also with [32, 33, 43]) where it is shown a quite surprising result. Indeed, using the techniques shown before, the authors have been able to prove that the hydrodynamic interactions with the boundary are responsible of a bigger reachable set for the swimmer. Considering the three sphere swimmer, swimming in a half space, at low Reynolds number, a symmetry argument shows immediately that it stays in the vertical plane it has started in. However, although in the whole 3D space the swimmer was constrained to stay on a straight line (still by symmetry), the presence of the boundary breaks this symmetry and increases the reachable set, which becomes for almost every starting point a set of dimension 5 (2 for the change of arm's lengths, 3 for position's change). This result was shown by a subtle combination of analytic and symbolic (using the software Maple) calculations. Again, the analyticity of the vectorfields is proven, using the decomposition of the Green function with the so-called *method of images* [11] and the authors make use of Hermann-Nagano Theorem.

This seems contradictory to results of he biological literature where it is shown that the boundary *attracts* the swimmer, and thus seems to reduce its movement capabilities [10, 17, 39, 42].

5. Optimal control problems

Once controllability is known, i.e. it is possible for a given swimmer to reach any final state from any initial one with a suitable shape change, a very natural question consists in considering optimal control problems in which is sought the shape change which produces the least possible cost among those which produce the given displacement.

Of course, this kind of questions is highly dependent on the *cost* which is taken in the preceding considerations. Several possible costs can be used. In the literature, the authors focus namely on the two following:

- the time to reach the objective;
- the mechanical energy spent by the swimmer in a given stroke.

In the former case, it is obvious that if a stroke is performed twice as fast, then the position also changes twice faster. In order for such a problem to have a solution, a constraint must be enforced on the control as for instance a bound. Those problems can therefore be written as

Problem 1. *For given (ξ_i, p_i) and (ξ_f, p_f), calling*

$$\begin{aligned}\mathcal{A}_T &= \big\{\xi : [0,T] \to \mathcal{S}, p \textit{ solution to } (1) \\ &\qquad \textit{s.t. } (\xi(0), p(0)) = (\xi_i, p_i) \textit{ and } (\xi(T), p(T)) = (\xi_f, p_f)\big\}\,,\end{aligned}$$

the problem we want to consider is

$$\min_{T>0, \xi \in \mathcal{A}_T, |\dot{\xi}| \leq 1} T\,. \tag{1}$$

Such problems have been considered in the context of low Reynolds swimmers in [28] for instance.

The latter class of optimization problems has a very interesting mathematical structure. Again, reparameterizing a stroke in time leading to different energies, one has to phrase the problem as time independent. A classical way to do so is to introduce the notion of *efficiency* due to Lighthill [26]. For a given stroke that steer the system from (ξ_i, p_i) to $(\xi_f = \xi_i, p_f)$ in time T, the efficiency is defined as

$$\mathrm{Eff}^{-1}(\xi) = \frac{1}{T}\int_0^T \left(\int_{\Gamma_t} \mathbf{f}(\mathbf{s},t)\cdot \mathbf{v}(\mathbf{s},t)\, d\mathbf{s}\right) dt\,. \tag{2}$$

In this expression, $\mathbf{f}$ and $\mathbf{v}$ are respectively the force and velocity densities on the surface of the swimmer. The problem that one considers is now to maximize the efficiency of a stroke.

Problem 2. *For given (ξ_i, p_i) and $(\xi_f = \xi_i, p_f)$, calling*

$$\begin{aligned}\mathcal{B}_T &= \big\{\xi : [0,T] \to \mathcal{S}, p \textit{ solution to } (1) \\ &\qquad \textit{s.t. } (\xi(0), p(0)) = (\xi_i, p_i) \textit{ and } (\xi(T), p(T)) = (\xi_f, p_f)\big\}\,,\end{aligned}$$

find a stroke $\bar{\xi}$ of maximal efficiency

$$\bar{\xi} = \arg\max_{T>0, \xi\in\mathcal{B}_T} \mathit{Eff}(\xi)\,. \tag{3}$$

It is easily seen that the efficiency does not depend on the time spent for making the stroke, in the sense that a linear reparameterization of the stroke does not change the efficiency. From the linearity of Stokes equations one can remark

- that the force density can be expressed as a linear function of $\mathbf{v}$ through the Dirichlet to Neumann map;
- that the velocity can be expressed as a linear function of $\dot{\xi}$ and $\dot{p}$ (equation (4)), and since $\dot{p}$ is itself a linear function of $\dot{\xi}$ due to (1), $\mathbf{v}$ can be written as a linear function of $\dot{\xi}$.

Thus, both forces and velocities are linear in $\dot{\xi}$ which, after integration, leads to an expression for (the inverse of) the efficiency as

$$\mathrm{Eff}^{-1} = \frac{1}{T}\int_0^T \mathbf{G}(\xi,p)(\dot{\xi(t)},\dot{\xi(t)})\,dt\,,$$

where $\mathbf{G}(\xi,p)$ is a metric defined on $T_\xi\mathcal{S}$. Pay attention that again, the metric $\mathbf{G}$ is not explicit and depends on the resolution of the outer Stokes problem.

In this case, the cost is quadratic in the control, and the problem is naturally recast in terms of minimizing a quadratic energy along the strokes which produce a given displacement. This is the classical framework of *subriemannian geometry* (see [29]), and optimal strokes are therefore *subriemannian geodesics* that link (ξ_i,p_i) and (ξ_f,p_f). The name refers to the class of problem in which the metric is singular and defined only on a subspace of the tangent bundle. This is clearly the case here, since admissible trajectories must satisfy (1) at all time, or, in other words be tangent at every (ξ,p) to the subspace defined by

$$dp = \sum_{i=1}^{M}\mathbf{F}_i(\xi,p)d\xi_i\,.$$

The numerical approximation of problem 2 for the swimmers described in section 4 was considered in [2–4,6]. Many methods can be used which roughly speaking enter into two categories. On the one hand, it is possible to write an ODE that subriemannian geodesics satisfy, and solve it with a Runge-Kutta like method. Pay attention that the problem is more complicated than what it seems. Indeed, we want to solve a boundary problem and not a Cauchy problem. To that aim, one usually implements a shooting method. Also, the coefficients of the ODE depend on the outer Stokes problem. What has been done in [2] for instance, was to tabulate those coefficients by solving beforehand a family of Stokes problems for all possible shapes in the shape space[III]. This is only possible in low dimension shape space. For higher dimension (typically larger than 3 or 4) this is no longer possible and one has to turn to use optimization packages [4].

As a sake of example we show in Fig. 6 three geodesic strokes that have been computed for the three sphere swimmer of Najafi and Golestanian.

6. Conclusion

This paper is a short review of recent results that have been obtained by the author and his collaborators in the study of low Reynolds number swimmers. This subject turns out to have a growing importance in the community of mathematicians, probably because it possesses many links between several fields (control theory, PDE and ODE analysis, numerical tools, subriemannian geometry, etc.), and clear applications in robotics and biology. Moreover, the methods used are generalized to slightly different context as, for instance, the case of perfect fluid swimmers [12,30]. Many extensions of these works are of considerable importance and are currently under thorough study as for instance the collective swimming of organisms [35], the practical building of such small swimmers (in that direction, a very nice submillimiter motor is proposed in [41]), the interaction of the swimmers with their environment, the biological motors [37], etc. We believe that the geometric structure enlightened in this paper would be of importance for the understanding of such phenomena.

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[III] Actually only a discretization of the possible shapes in $\mathcal{S}$ is considered from which the coefficients are interpolated.

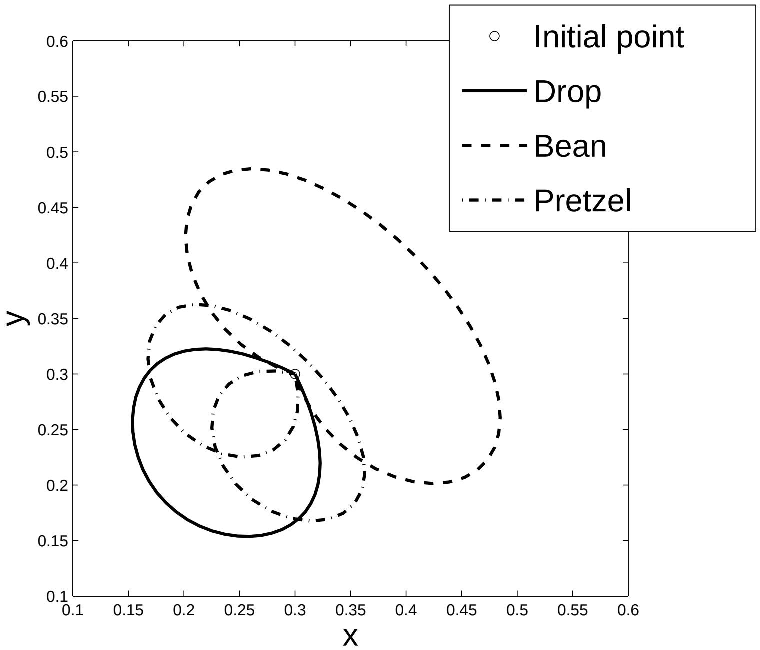

FIGURE 6. Geodesic strokes for the 3 sphere swimmer that all start at shape $(\xi_1, \xi_2) = (0.3, 0.3)$. Among the three strokes, only the one corresponding to the plain line is optimal. The radius of the balls is $a = 0.1$.

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