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DISSECTING THE CIRCLE, AT RANDOM *

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Abstract. Random laminations of the disk are the continuous limits of random non-crossing configurations of regular polygons. We provide an expository account on this subject. Initiated by the work of Aldous on the Brownian triangulation, this field now possesses many characters such as the random recursive triangulation, the stable laminations and the Markovian hyperbolic triangulation of the disk. We will review the properties and constructions of these objects as well as the close relationships they enjoy with the theory of continuous random trees. Some open questions are scattered along the text.

Introduction

Let us begin our journey with the Brownian triangulation of Aldous. In the remaining of these pages, P_n denotes the regular polygon inscribed in the unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ whose vertices are the nth roots of unit. A *triangulation* of P_n is a subset of non-crossing (except at their endpoints) diagonals that triangulates P_n, see Fig. 1. These triangulations are counted by Catalan numbers and are connected to various combinatorial structures, see [22] for a beautiful application to the rotation distance problem.

We are interested here in *random* triangulations. For $n \geq 3$ we denote by T_n a uniform triangulation of P_n. Combinatorial properties of T_n have been investigated in length [4,12,15]. From a geometrical point of view, the random variable T_n can also be seen as a random closed subset of $\overline{\mathbb{D}}$ and one can investigate its limit geometry as $n \to \infty$. This has been proposed by David Aldous who proved the following:

Theorem (Aldous [2,3]) *We have the following convergence in distribution in the sense of Hausdorff distance on the closed subsets of* $\overline{\mathbb{D}}$

$$\mathsf{T}_n \xrightarrow[n\to\infty]{(\mathrm{d})} \mathcal{B}.$$

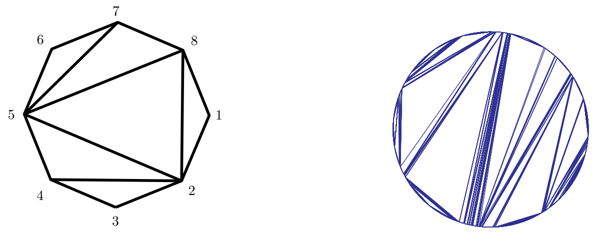

Figure 1. A triangulation of the octogon and a sample of $\mathcal{B}$.

* *Variation on Aldous' original title : "Triangulating the circle, at random".*

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The random closed subset $\mathcal{B}$ is called *the Brownian triangulation of the disk*. It is indeed a continuous *triangulation* since the complement of $\mathcal{B}$ inside $\overline{\mathbb{D}}$ is made of countably many disjoint Euclidean triangles a.s., see Fig. 1. This fractal object (it almost surely has Hausdorff dimension 3/2) has a fascinating structure and is connected to the Brownian continuum random tree [1] which can be thought of as its dual. The work of Aldous opened the doors for understanding the geometric structure of a large variety of random non-crossing structures yielding to a number of new objects such as the stable laminations ([17] and Section 1.2), the recursive triangulation ([10] and Section 2) or the Markovian hyperbolic triangulation ([11] and Section 4).

Our goal is to present these nice objects and to convince the reader that the framework adopted here could provide a way to investigate continuous limits of discrete random trees.

Disclaimer : This is an expository work which is not meant to be fully rigorous nor exhaustive. The complete proofs as well as precise definitions of the objects considered can be found in the references. May the reader forgive our wordy and sketchy style.

Acknowledgments: We thank Igor Kortchemski for comments on a first version of this note. Thanks also go to the anonymous referee for a careful reading.

1. The Brownian triangulation

In this section we give an overview of the construction of the Brownian triangulation $\mathcal{B}$. The tools introduced for this purpose are of great use throughout the paper. Let us begin with a precise definition of non-crossing configurations of regular polygons.

A *non-crossing configuration* (n.c.c. for short) of P_n is a subset of diagonals (and edges) of P_n that are not crossing except at their endpoints. They are many classes of non-crossing configurations (see e.g. [14]), let us list a few:

- A triangulation is a n.c.c. that triangulates P_n, and more-generally for $p \geq 3$ we speak of *p-angulation* (quadrangulation, pentagulation, ...) when all the faces of the configuration (except the external face) have p adjacent edges,
- A *dissection* is a n.c.c. formed by the edges of P_n and some non-crossing diagonals,
- A *non-crossing tree* is a n.c.c. that is also a spanning tree of the vertices of P_n.

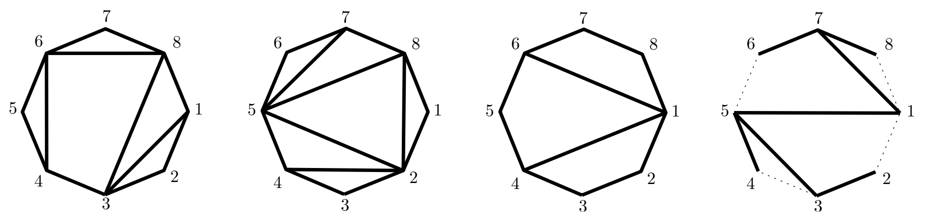

Figure 2. Examples of non-crossing configurations : from left to right, a dissection, a triangulation, a quadrangulation and a non-crossing tree of the octogon.

The general goal in these pages is to understand the geometric structure of random non-crossing configurations as $n \to \infty$. To do so, it is convenient to see a n.c.c. as a closed subset of the unit disk (we shall always do so) and to ask for a limit theorem in the sense of the Hausdorff metric on closed subsets of $\overline{\mathbb{D}}$. We remind the reader that the Hausdorff distance $\mathrm{d_H}$ between two closed subsets $A, B \subset \overline{\mathbb{D}}$ is the least $\varepsilon > 0$ such that A is contained in the ε-enlargement of B and vice-versa. Recall also that

$$\text{the set of closed subsets of } \overline{\mathbb{D}} \text{ is } \textit{compact} \text{ for } \mathrm{d_H}.$$

See the second part of the discussion in Section 3. Using this topology, the Brownian triangulation appears as a *universal* limit of uniform non-crossing configurations:

Theorem 1 ([17] and [9]). *If C_n is a uniform n.c.c. chosen in the class of p-angulations*[1] *or dissections or non-crossing trees of* P_n *then we have the convergence for* $\mathrm{d_H}$

$$\mathsf{C}_n \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \mathcal{B}.$$

The combinatorial details of the class of n.c.c. considered thus vanish as $n \to \infty$ and give rise to the Brownian triangulation. This limit result can be used to compute asymptotic quantities on n.c.c. For example the law of the arc (normalized by $1/2\pi$) intercepted by the longest diagonal in a random uniform $\{p$-angulation or dissection or non-crossing tree$\}$ of P_n converges as $n \to \infty$ towards that of the Brownian triangulation which is given [3, 12] by

$$\frac{1}{\pi}\frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}}\mathbf{1}_{\frac{1}{3}\le x\le\frac{1}{2}}\mathrm{d}x.$$

We let the reader think about many other applications of Theorem 1.

Laminations and continuous triangulations. Using terminology of geometers, we call a geodesic *lamination* of the disk (lamination for short) any closed subset of the unit disk $\overline{\mathbb{D}}$ that can be written as a disjoint union of diagonals $[e^{\mathrm{i}x}, e^{\mathrm{i}y}]$ for $x, y \in \mathbb{R}$ that are not intersecting inside $\mathbb{D}$, see [6]. In particular, any n.c.c. of P_n can be seen as a finite lamination. It is fairly easy to see that the set of laminations is closed for $\mathrm{d_H}$. A *continuous triangulation* is a lamination whose complement in $\overline{\mathbb{D}}$ is made of disjoint open Euclidean triangles. Equivalently, they consist of the laminations that are maximal for the inclusion relation, see [6, 20].

Using this vocabulary, the Brownian triangulation is indeed a continuous triangulation (and a lamination) almost surely. This fact could seem obvious when $\mathcal{B}$ is considered as the limit of uniform discrete triangulations of P_n but less clear when considered as the limit of uniform quadrangulations! As all characters of the Brownian family, the Brownian triangulation is a fractal object. Indeed, almost surely, no distinct triangles of $\overline{\mathbb{D}}\backslash\mathcal{B}$ share an edge and it is shown in [20] (and sketched in [3]) that

$$\dim(\mathcal{B}) \quad = \quad \frac{3}{2}, \quad \text{a.s.} \tag{1}$$

Without giving a proof of Theorem 1, let us introduce the main techniques and ideas it contains. For sake of simplicity we stick to the case of discrete triangulations.

1.1. Duality and contour function

There is an obvious (once remarked) duality between triangulations of P_n and rooted oriented binary trees with $n-1$ leaves: take the dual of the triangulation, see Fig. 3. Binary trees with $n-1$ leaves are themselves in bijection with their contour functions of length $4n-6$ (the definition of the contour function should be clear from Fig.3).

Hence chosing a uniform triangulation of P_n boils down to chosing a uniform binary tree with $n-1$ leaves or equivalently picking its contour function. Let $D^{(n)}$ denote the contour function of a uniform binary tree with $n-1$ leaves and write T_n for the dual triangulation of P_n associated with it (in particular T_n is indeed a uniform triangulation of P_n). Let us see how to relate these objects. Firstly, it is clear to see that local maxima of $D^{(n)}$ are associated with the leaves of the binary tree or equivalently with the sides of the polygon P_n. Secondly, with any chord $[e^{-2\mathrm{i}\pi x_n/n}, e^{-2\mathrm{i}\pi y_n/n}]$ with $x_n < y_n \in \{0, 1, ..., n-1\}$ of T_n we can associate two unique instants $a_n < b_n$ in $\{0, 1, ..., 4n-7\}$ such that

$$D^{(n)}_{a_n+\frac{1}{2}} = D^{(n)}_{b_n+\frac{1}{2}} = \min_{t\in[a_n+\frac{1}{2}, b_n+\frac{1}{2}]} D^{(n)}_t. \tag{2}$$

They form an up-step and a down-step of the path that can “see” each other below the curve and correspond to the two visits of the edge dual of the chord $[e^{-2\mathrm{i}\pi x_n/n}, e^{-2\mathrm{i}\pi y_n/n}]$ by the contour process of

[1] in this case $n-p$ has to be divisible by $p-2$

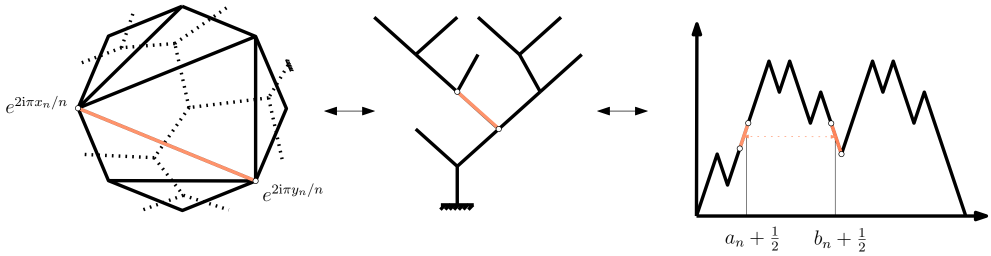

Figure 3. Duality with trees and excursion.

the tree, see Fig. 3. Also x_n is equal to the number of local maxima in $D^{(n)}$ before time a_n and similarly y_n is the number of local maxima of $D^{(n)}$ before time b_n.

Once this is digested, let us go to the continuous world. It is well known (see [19]) that the contour functions of uniform binary trees admit the Brownian excursion as scaling limit. More precisely we have[2]

$$\left(\frac{1}{2\sqrt{2n}} D^{(n)}_{4nt}\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(\mathrm{d})} (\mathbf{e}_t)_{0\leq t\leq 1}, \tag{3}$$

where $\mathbf{e}$ is the normalized Brownian excursion of duration 1. In fact, the last convergence, together with the forthcoming (4), implies the convergence in distribution of the T_n's towards a random continuous triangulation that is constructed from $\mathbf{e}$. This thus furnishes a definition of the Brownian triangulation from the Brownian excursion. Our goal here is only to make the reader guess this construction.

As we said, we need another ingredient: if $\mathrm{M}^{(n)}_t$ denotes the number of local maxima of $D^{(n)}$ before time t then we have (see e.g. [18])

$$\sup_{t\in[0,1]} \left(n^{-1}\mathrm{M}^{(n)}_{4nt} - t\right) \xrightarrow[n\to\infty]{(P)} 0. \tag{4}$$

By the Skorokhod representation theorem one can assume that (3) and (4) hold almost surely. Now pick a chord of T_n as above and assume that $n^{-1}x_n$ and $n^{-1}y_n$ converge towards $x < y \in [0,1]$. By (4) we have $(4n)^{-1}a_n \to x$ as well as $(4n)^{-1}b_n \to y$. Passing to the limit in (2) using (3) leads to $x \sim_{\mathbf{e}} y$ where $\sim_{\mathbf{e}}$ is the equivalence relation defined by

$$x \sim_{\mathbf{e}} y \quad \text{if and only if} \quad \mathbf{e}_x = \mathbf{e}_y = \min_{s\in[x\wedge y, y\vee y]} \mathbf{e}_s. \tag{5}$$

The suspected limit of the T_n's is then

$$\mathcal{L}_{\mathbf{e}} \quad := \quad \bigcup_{x\,\sim_{\mathbf{e}}\,y} [e^{-2i\pi x}, e^{-2i\pi y}]. \tag{6}$$

Though $\mathcal{L}_{\mathbf{e}}$ is clearly made of a union of diagonals of the unit disk, it is less clear that this union is disjoint inside $\mathbb{D}$. It is not the case in general, however a fairly easy exercise shows that if $\mathbf{e}$ is continuous and under the assumption

$$(\mathsf{H}_{\mathbf{e}}) \qquad \text{the local minima of } \mathbf{e} \text{ are distinct,}$$

then $\mathcal{L}_{\mathbf{e}}$ indeed is a lamination and furthermore a continuous triangulation. Since these hypotheses are a.s. fulfilled by the Brownian excursion, $\mathcal{L}_{\mathbf{e}}$ is a.s. a random continuous triangulation and it is possible to show that

$$\mathsf{T}_n \xrightarrow[n\to\infty]{a.s.} \mathcal{L}_{\mathbf{e}}$$

[2]putting $D^{(n)}_t = 0$ for $t \geq 4n - 6$

consequently $\mathcal{B} = \mathcal{L}_{\mathbf{e}}$ in distribution. See [2,9,17] for complete arguments.

This is in essence the idea of the proof of Theorem 1: for each class of non-crossing configurations, we find a bijection (usually the classical duality operation) with a class of trees. These random trees usually are simple enough (conditioned Galton-Watson trees or closely related) so that their contour functions converge in the scaling limit towards (a multiple) of the Brownian excursion. This convergence then finally implies the convergence of the random uniform non-crossing model towards the Brownian triangulation. This strategy has been implanted for a variety of n.c.c. in [9].

Open question 1 (Universality)**.** *Extend Theorem 1 to other classes of n.c.c., see [14].*

1.2. Stable laminations

The results of this section come from [17] to which the reader is referred for details.

As we saw in the last section, the universal limit of various classes of uniform non-crossing configurations is a random continuous *triangulation*. This is reminiscent of the fact that critical Galton-Watson trees with finite variance offspring reproduction law and condition to be large all admit a continuous random *binary* tree (the Brownian CRT [1]) as scaling limit: the only branching points that subsist in the scaling limit are at most of order three. However, if the offspring reproduction law has a heavy tail then branching points of infinite multiplicity remain in the scaling limit (the stable trees [13]), let us see how this phenomenon occurs in the context of random laminations.

Let $\mathbf{q} = (q_i)_{i\geq 1}$ be a sequence of non-negative weights with $q_1 = 0$. If ω is a dissection of P_n we associate a "Boltzmann" weight to ω by the formula

$$P^n_{\mathbf{q}}(\omega) = \frac{1}{Z_n} \prod_{f \text{ face of } \omega} q_{\deg(f)-1},$$

where $\deg(f)$ is the degree of the face f, that is the number of edges adjacent to f, and Z_n is a normalizing constant that makes $P^n_{\mathbf{q}}$ a probability measure. Under mild assumption this definition makes sense and one can consider a random dissection $\mathsf{D}^n_{\mathbf{q}}$ distributed according to $P^n_{\mathbf{q}}$. For example if $q_i = c^{i-1}$ for $c \geq 0$ the resulting probability $P^n_{\mathbf{q}}$ is uniform over all dissections and if $q_i = \mathbf{1}_{i=p-1}$ for $p \geq 3$, it is uniform over all p-angulations of P_n. In both cases we are back to the setting of Theorem 1. However if the measure $P^n_{\mathbf{q}}$ favors large faces then a different behavior occurs:

Theorem 2 ([17])**.** *If* $\mathbf{q}$ *is a probability measure on* $\{0, 2, 3, ...\}$ *of mean* 1 *in the domain of attraction of a stable law of parameter* $\theta \in (1, 2]$ [3] *then we have the following convergence in distribution for the Hausdorff metric*

$$\mathsf{D}^n_{\mathbf{q}} \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \mathcal{S}_\theta,$$

where $\mathcal{S}_\theta$ *is a random lamination of the disk called the stable lamination of parameter* θ*.*

In the case $\theta = 2$ the stable lamination of parameter 2 coincides with the Brownian triangulation. However when $\theta < 2$, the random lamination $\mathcal{S}_\theta$ is not a triangulation anymore and its complement in $\overline{\mathbb{D}}$ contains open polygons with infinitely many faces, see Fig. 4. Also, the dimension of $\mathcal{S}_\theta$ equals

$$\dim(\mathcal{S}_\theta) \quad = \quad 2 - \frac{1}{\theta}.$$

The strategy of the proof of Theorem 2 follows roughly that of Theorem 1. The dual tree associated to $D^n_{\mathbf{q}}$ is now a Galton-Watson tree with offspring distribution $\mathbf{q}$ conditioned on having $n-1$ leaves. This particular conditioning of Galton-Watson trees has recently been studied in [18] (see also [21]) and in particular it has been shown [18] that the rescaled contour functions of the last trees converge towards the height process of a stable Lévy process (see [13] for the definition). The main difficulty that arises when $\theta < 2$ is that these random excursion functions do not have distinct local minima (hypothesis $\mathsf{H}_{\mathbf{e}}$)

[3] e.g. $q_k \sim ck^{-1-\theta}$ as $k \to \infty$, for some $c > 0$

and thus the construction of last section breaks down. Still, it is possible to build the stable lamination from the stable height process in a way similar as $\mathcal{L}_{\mathbf{e}}$ is constructed from $\mathbf{e}$, see [17] for more details.

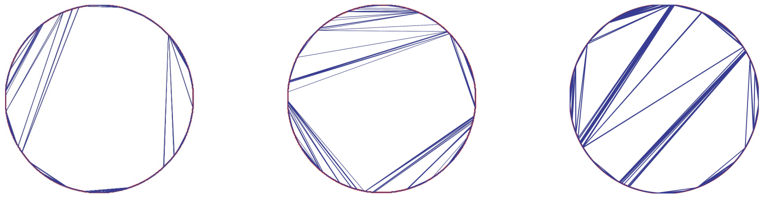

Figure 4. Stable laminations with parameters 1.1 (left), 1.5 (middle) and 1.9 (right). Simulations realized by Igor Kortchemski.

Many distributional properties of the stable laminations are still to be calculated. E.g.:

Open question 2 (I. Kortchemski)**.** *What is the distribution of the length of the longest diagonal of $\mathcal{S}_\theta$ for $\theta \in (1,2)$? The area of the largest face? What happens if the weight sequence $\mathbf{q} = \{q_i : i \in \{0, 2, 3, 4, ...\}\}$ is not a probability sequence or has infinite variance?*

2. Recursive triangulations

The results of this section come from [10] to which the reader is referred for details.
The last two sections studied the limit of n.c.c. under the uniform or "Boltzmann" type distributions. Another very natural probability measure, the recursive measure, arises when we actually try to draw a triangulation on a sheet of paper. A new object appears in the limit.

The recursive triangulation of P_n is the random discrete triangulation R_n obtained by the following process. We start with the empty n-gon P_n and draw a uniform diagonal of it. Iteratively, we draw a diagonal uniformly among those that do not intersect (inside $\mathbb{D}$) the previous drawn diagonals. The process stops after $n-3$ steps when no diagonal can be added to the configuration anymore. Although $\mathsf{R}_n = \mathsf{T}_n$ in law for $n = 3, 4$ and 5, the recursive and uniform triangulations differ much when n is large and a new object appears in the limit:

Theorem 3 ([10])**.** *We have the following convergence in distribution for* $\mathrm{d_H}$

$$\mathsf{R}_n \xrightarrow[n\to\infty]{(\mathrm{d})} \mathcal{R}.$$

The random lamination $\mathcal{R}$ is a continuous triangulation called the *random recursive triangulation* of the disk. Although $\mathcal{R}$ roughly looks like the Brownian triangulation, they are very different and $\mathcal{R}$ is "fatter"

$$\dim(\mathcal{R}) \quad = \quad 1 + \beta^*, \qquad \text{with} \quad \beta^* = \frac{\sqrt{17}-3}{2} \approx 0,56.$$

The recursive triangulation of the disk can be constructed directly as follows: we consider a sequence $U_1, V_1, U_2, V_2, \ldots$ of independent random variables, which are uniformly distributed over the unit circle $\mathbb{S}_1$. We then construct inductively a sequence $L_1, L_2, \ldots$ of random closed subsets of the (closed) unit disk $\overline{\mathbb{D}}$. To begin with, L_1 just consists of the chord $[U_1V_1]$ with endpoints U_1 and V_1. Then at step $n+1$, we consider two cases. Either the chord $[U_{n+1}V_{n+1}]$ intersects L_n, and we put $L_{n+1} = L_n$. Or the chord $[U_{n+1}V_{n+1}]$ does not intersect L_n, and we put $L_{n+1} = L_n \cup [U_{n+1}V_{n+1}]$. Thus, for every integer $n \geq 1$, L_n is a disjoint union of random chords. We then let $\mathcal{R}$ to be the closure of the increasing L_n's:

$$\mathcal{R} = \overline{\bigcup_{n\geq 1} L_n}.$$

The tools used to study $\mathcal{R}$ are very different from the ones used in the last sections. However the scenario is the same: we try to understand the contour function of the dual tree associated to L_n and prove a convergence of these contour processes (in a certain sense) towards a continuous non-negative process $(\mathbf{m}_x : 0 \leq x \leq 1)$ which finally encodes the lamination via

$$\mathcal{R} \overset{(\mathrm{d})}{=} \mathcal{L}_{\mathbf{m}}, \tag{7}$$

in the sense of (6). This coding of $\mathcal{R}$ by $\mathbf{m}$ then permits to deduce properties of $\mathcal{R}$ (e.g. Hausdorff dimension) from properties of $\mathbf{m}$ (e.g. Hölder continuity exponent). Let us clarify the construction of $\mathbf{m}$. For $x \in [0,1]$ we define (see Fig. 5)

$$\mathbf{m}_n(x) \quad := \quad \#\{\text{chords of } L_n \text{ that intersect } [1, e^{2i\pi x}]\}.$$

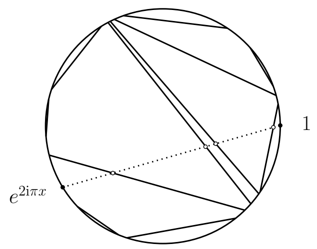

Figure 5. Definition of $\mathbf{m}_n(x)$ (four in this case).

This function plays the role of the contour function associated to the dual tree of L_n. It is then possible to prove that for all $x \in [0,1]$ we have the following convergence in probability

$$n^{-\beta^*/2}\mathbf{m}_n(x) \quad \xrightarrow[n\to\infty]{(P)} \quad \mathbf{m}_x, \tag{8}$$

where $(\mathbf{m}_x : 0 \leq x \leq 1)$ is a continuous random excursion which is Hölder continuous of exponent $\beta^* - \varepsilon$ for all $\varepsilon > 0$. Note that the last convergence is strictly weaker than a functional convergence in the type of (3):

$$n^{-\beta^*/2}\big(\mathbf{m}_n(x)\big)_{x\in[0,1]} \quad \xrightarrow[n\to\infty]{} \quad (\mathbf{m}_x)_{x\in[0,1]}$$

in probability for the L^∞-norm over $[0,1]$. This convergence, conjecture in [10] has been recently proved by Nicolas Broutin and Henning Sulzbach [7]. Even without this strong convergence, it is still possible to prove (7). Let us mention that the main tool used to prove (8) is fragmentation theory, see [5]. In particular the exponent β^* appears as the so-called Malthusian exponent of a fragmentation process intimately related to the construction of $\mathcal{R}$.

3. $\mathbb{R}$-trees, dendrites and laminations

Before introducing our last character in the next section, we show that laminations can, in a sense, be considered as weak versions of continuous trees thus giving an alternative approach to the classical Gromov-Hausdorff topology.

3.1. Laminations as limit of discrete trees

Over the last years a tremendous effort has been made to understand the continuous limits of discrete random trees. The general question is as follows:

Question: (Q) Assume that $(\tau_n)_{n\geq 0}$ is a sequence of random trees whose "size" grow to infinity with n. How can we make sense of a continuous limit of the τ_n?

Let us first remind the reader of the classical scaling limit approach to the last question based on the Gromov-Hausdorff topology: if (E,d) and (E',d') are two compact metric spaces, the Gromov-Hausdorff distance between them is

$$\mathrm{d_{GH}}\big((E,d),(E',d')\big) := \inf\{\mathrm{d_H}\big(\phi(E),\phi'(E')\big)\},$$

where the infimum is taken over all choices of a metric space (F,δ), isometric embeddings $\phi : E \to F$ and $\phi' : E' \to F$ where $\mathrm{d_H}$ denotes the Hausdorff distance in (F,δ). The Gromov-Hausdorff distance is indeed a distance on the set of equivalence classes of compact metric space (which is a Polish space). A discrete tree can obviously be seen as a metric space by endowing it with the graph metric $\mathrm{d_{gr}}$, hence an answer to (**Q**) is :

Answer 1: Find a scaling parameter α_n and show that the rescaled (finite) random compact metric space $(\tau_n, \alpha_n \cdot \mathrm{d_{gr}})$ converges in distribution for $\mathrm{d_{GH}}$.

The metric spaces arising as Gromov-Hausdorff limits of rescaled trees are known as $\mathbb{R}$-trees. They are compact metric spaces without cycles and such that the only geodesic between any two points is isometric to a real segment, see [19]. Several classes of random discrete trees have been investigated under that view point (Galton-Watson trees, Markov branching trees).

But this approach has a drawback: since the set of equivalence classes of compact metric spaces is not compact for $\mathrm{d_{GH}}$, one usually has to establish a thorny tightness property for the rescaled τ_n, even worse, some very natural sequences of random trees are not tight for $\mathrm{d_{GH}}$.

Let us give another approach to (**Q**). Assume for simplicity that the discrete trees we are dealing with are rooted ordered discrete trees with no vertices of degree 2. This class is particularly nice since it is in bijection with dissections of finite regular polygons as shown on Fig. 6. If τ is such a tree we denote by

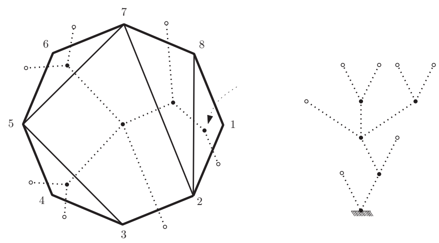

Figure 6. A tree and its dual dissection.

$\mathsf{Dis}(\tau)$ the dissection associated with τ. Viewing our random discrete trees τ_n through their associated dissections then gives a new point of view to (**Q**):

Answer 2: Show that $(\mathsf{Dis}(\tau_n))_{n\geq 1}$ converges for the Hausdorff topology on $\overline{\mathbb{D}}$.

The continuous limit of the discrete random trees (τ_n) is now a random lamination. Within this framework, the stable laminations of parameter $\theta \in [1,2]$ loosely speaking appear as the lamination limits of Galton-Watson trees whose offspring reproduction law is in the domain of attraction of a stable law of parameter θ. The great advantage of this approach lies in the compactness of the Hausdorff topology of $\overline{\mathbb{D}}$: we know *a priori* that $(\mathsf{Dis}(\tau_n))$ admits sub-sequential weak limits for $\mathrm{d_{GH}}$.

Let us give a trivial example. Consider τ_n the (deterministic) binary tree full up to level n, see Fig. 7. It is easy to see that the sequence τ_n cannot be rescaled to converge in the Gromov-Hausdorff sense towards a continuous $\mathbb{R}$-tree, however (τ_n) converges in the lamination sense towards the (deterministic) lamination of Fig. 7.

Open question 3 (Markov branching trees)**.** *Construct the limits in the sense of laminations of the discrete Markov branching trees [16] and study their properties.*

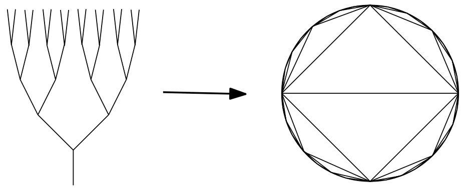

Figure 7. A binary tree full up to level 5 and its lamination limit.

3.2. **Laminations as measured dendrites**

As in the discrete setting, we will see that a lamination hides a "dual" topological tree. To simplify the exposition we present this construction in the case of the Brownian triangulation. The setting could be adapted to more general random laminations.

If $\mathcal{B}$ is the Brownian triangulation we define an equivalence relation $\approx$ on $\overline{\mathbb{D}}$ by putting $x \approx y$ if and only if x and y belong to a chord of $\mathcal{B}$ or if they both belong to the closure of a triangle of $\overline{\mathbb{D}}\backslash\mathcal{B}$. Since, a.s. there are no triangles of $\overline{\mathbb{D}}\backslash\mathcal{B}$ which are adjacent to eachother, the relation $\approx$ indeed is an equivalence relation. We then consider the random topological quotient space

$$\mathscr{T} \quad = \quad \overline{\mathbb{D}}/\approx$$

endowed with the quotient topology. We write π for the canonical projection. It is an exercise to check that $\mathscr{T}$ a.s. is a *dendrite*: a continuum (compact connected topological space) containing no simple closed curve. See [8] for a survey and for 32 equivalent characterizations of dendrites. This "topological tree" can be seen as the dual of $\mathcal{B}$. When $\mathcal{B} = \mathcal{L}_{\mathbf{e}}$ then $\mathscr{T}$ is homeomorphic to the Brownian tree $\mathscr{T}_{\mathbf{e}}$ coded by $\mathbf{e}$, see [20]. Unfortunately, we conjecture that in the Brownian case this topology is constant:

Open question 4 (Topology of Aldous' CRT)**.** *The topology of $\mathscr{T}$ is almost surely constant, i.e. two independent samples of the Brownian CRT are almost surely homeomorphic.*

But the lamination contains more information than this dendrite. Indeed, the push-forward by π of the uniform measure on $\mathbb{S}_1$ endows the dendrite $\mathscr{T}$ with a Borel probability measure μ. Also, the clockwise ordering of $\mathbb{S}_1\backslash\{1\}$ leads to a lexicographical order $\prec$ over $\mathscr{T}$ given by $a \prec b$ if and only if

$$\inf\big\{s \in [0,1) : \pi(e^{2i\pi s}) = a\big\} < \inf\big\{t \in [0,1) : \pi(e^{2i\pi t}) = b\big\}.$$

Actually, it is a simple but tedious exercise to see that in the Brownian case the information provided by the lamination $\mathcal{B}$ and $(\mathcal{T}, \mu, \prec, \pi(1))$ are equivalent, in other words one can reconstruct $\mathcal{B}$ from $(\mathcal{T}, \mu, \prec, \pi(1))$.

4. The Markovian hyperbolic triangulation

The results of this section come from [11] to which the reader is referred for details.

We finish this expository paper by introducing the Markovian hyperbolic triangulation $\mathcal{H}$. This continuous triangulation differs much from the previous ones and is related to hyperbolic geometry. In fact, contrary to $\mathcal{B}, \mathcal{S}_\theta$ or $\mathcal{R}$, the continuous triangulation $\mathcal{H}$ is not introduced as a limit of discrete n.c.c. neither it is constructed as a lamination associated with an excursion process in the sense of (6). Consequently it has no clearly defined continuous $\mathbb{R}$-tree associated with it.

In this section we have to work with hyperbolic geometry. The definition of lamination (and continuous triangulation) is slightly changed by considering hyperbolic chords instead of Euclidean ones. See Fig. 8. All the objects are denoted with an additional h- for "hyperbolic".

The advantage of working with h-laminations (except for obvious aesthetic reasons) is the fact that a maximal h-lamination (a h-triangulation) can be seen as a tilling of the hyperbolic plane by hyperbolic triangles. More precisely any open component of the complement of a h-triangulation in $\mathbb{D}$ is an open

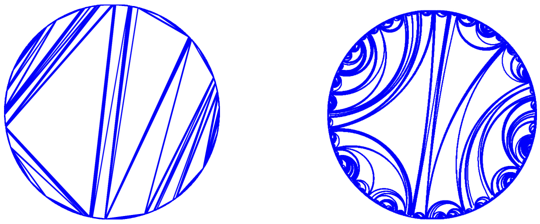

Figure 8. A continuous triangulation with Euclidean chords and its h-version.

h-triangle with its three apexes located on the boundary at infinity $\partial\mathbb{D}$, such a triangle is called *ideal*. We recall that the Möbius group Mob of all hyperbolic isometries acts transitively on the set of ideal triangles. In other words, all the triangles of a maximal h-triangulation are conformally equivalent and can be all seen as the same tiles!

In view of these remarks, it is legitimate to ask if there exists a random h-triangulation $\mathcal{T}$ whose law is invariant under the action of Mob that is, $\mathcal{T} = \phi(\mathcal{T})$ in distribution for every $\phi \in \mathsf{Mob}$. The answer is positive and it is fairly easy to construct many examples, see [11]. However there is an essentially unique random h-triangulation that is Mob-invariant and satisfies a spatial Markov property that can be roughly described as follows:

> **Spatial Markov Property :** Given a triangle $T = (abc)$ in $\mathcal{T}$, the triangulation restricted to the three connected components of the complement of T in $\mathbb{D}$ are conditionally independent, and moreover, the part that is beyond (bc) is independent of the position of a.

Theorem 4 ([11])**.** *There is a unique (law of a) random h-triangulation $\mathcal{H}$ such that*

- *the union of the triangles of $\mathcal{H}$ is of full Lebesgue measure in $\mathbb{D}$,*
- *$\phi(\mathcal{H}) = \mathcal{H}$ in distribution for every $\phi \in \mathsf{Mob}$,*
- *$\mathcal{H}$ satisfies the above spatial Markov property.*

The triangulation $\mathcal{H}$ is constructed in [11] using basic hyperbolic tools and subordinators. Although not locally finite (they are no triangles adjacent to each other), it is the thinest of all laminations considered in this paper in the sense that

$$\dim(\mathcal{H}) \quad = \quad 1.$$

Many open questions about this object remain open:

Open question 5 ([11])**.** *Is there a "natural" random discrete n.c.c. model that converges towards $\mathcal{H}$? Is there a random $\mathbb{R}$-tree dual to $\mathcal{H}$ as the Brownian tree or stable trees are dual to $\mathcal{B}$ and $\mathcal{S}_\theta$?*

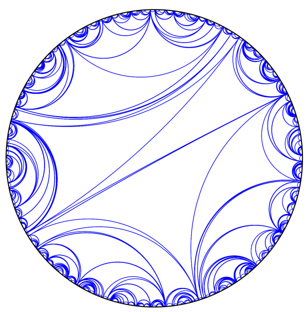

Figure 9. A sample of $\mathcal{H}$.

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