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NASH EQUILIBRIA FOR COUPLING OF CO_2 ALLOWANCES AND ELECTRICITY MARKETS *

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Abstract. In this note, we analyze Nash equilibria between electricity producers selling their production on an electricity market and buying CO_2 emission allowances on an auction carbon market. The producers' strategies integrate the coupling of the two markets via the cost functions of the electricity production. We set out clear Nash equilibria that can be used to compute equilibrium prices on both markets as well as the related electricity produced and CO_2 emissions released.

Résumé. Dans cette note, nous analysons les équilibres de Nash entre des producteurs d'électricité qui d'une part vendent leur production sur un marché d'achat-vente d'électricité, et d'autre part achètent des permis d'émission sur un marché d'enchère de permis de CO_2. Les stratégies des acteurs prennent en compte le couplage des deux marchés au travers de l'impact du coût d'émission sur les coûs de production des joueurs. Nous mettons en évidence des équilibres de Nash qui permettent d'expliciter et de calculer des prix d'équilibre sur les deux marchés, ainsi que les quantités d'électricité produites, et la répartition des couvertures en CO_2 achetées.

1. Introduction

The aim of this paper is to develop analytic tools in order to design a relevant mechanism for carbon markets, where relevant refers to emissions reduction. For this purpose, we focus on electricity producers in a power market linked to a carbon market. The link between markets is established through a market microstructure approach. In this context, where the number of agents is limited, a standard game theory applies. The producers are considered as players behaving on the two financial markets represented here by carbon and electricity. We establish a Nash equilibrium for this non-cooperative J-player game through a coupling mechanism between the two markets.

The original idea comes from the French electricity sector, where the spot electricity market is often used to satisfy peak demand. Producers' behavior is demand driven and linked to the maximum level of electricity production. Each producer then strives to maximize its market share. In the meantime, it has to manage the environmental burden associated with its electricity production through a mechanism inspired by the EU ETS[1] framework: each producer emission level must be counterbalanced by a permit or through the payment of a penalty. Emission permit allocations are simulated through a carbon market that allows the producers to buy allowances at an auction. Our focus on the electricity sector is motivated by its introduction in phase III of the

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EU ETS, and its prevalence in the emission share. In the present paper, the design assumptions made on the carbon market aim to foster emissions reduction in the entire electricity sector.

Based on a static elastic demand curve (referring to the time stages in an organized electricity market, mainly day-ahead and intra-day), we solve the local problem of establishing a non-cooperative Nash equilibrium for the two coupled markets.

While literature mainly addresses profit maximization, our share maximization approach deals with profit by making specific assumptions, i.e. no-loss sales, and a balance struck between the purchase of allowances and the carbon footprint of the electricity generated. Here the market is driven through demand dynamics rather than the electricity spot price dynamics used in recent works (see [4] [5] [6]).

In Section 2, we formalize the market (carbon and electricity) rules and the associated admissible set of players' coupled strategies.

We start by studying (in section 3.2) the set of Nash equilibria on the electricity market alone (see Proposition 3.2). This set constitutes an equivalence class (same prices and market shares) from which we exhibit a dominant strategy.

Section 3 is devoted to the analysis of coupled markets equilibria: given a specific carbon market design (in terms of penalty level and allowances), we compute the bounds of the interval where carbon prices (derived from the previous dominant strategy) evolve. We specify the properties of the associated equilibria.

2. Coupling markets mechanism

2.1. Electricity market

In the electricity market, demand is aggregated and summarized by a function $p \mapsto D(p)$, where $D(p)$ is the quantity of electricity that buyers are ready to obtain at maximal unit price p. We assume the following:

Assumption 2.1. *The demand function $D(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is decreasing, left continuous, and such that $D(0) > 0$.*

Each producer $j \in \{1, \dots, J\}$ is characterized by a finite production capacity κ_j and a bounded and increasing function $c_j : [0, \kappa_j] \longrightarrow \mathbb{R}^+$ that associates a marginal production cost to any quantity q of electricity. These marginal production costs depend on several exogenous parameters reflecting the technical costs associated with electricity production e.g. energy prices, O&M costs, taxes, carbon penalties *etc*. This parameter dependency makes possible to build different market coupling mechanisms. In the following we use it to link the carbon and electricity markets.

The merit order ranking features marginal cost functions sorted according to their production costs. These are therefore increasing staircase functions whereby each stair refers to the marginal production cost of a specific unit owned by the producer.

The producers trade their electricity on a dedicated market. For a given producer j, the strategy consists in a function that makes it possible to establish an asking price on the electricity market, defined as

$$s_j : \mathscr{C}_j \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+; \quad (c_j(\cdot), q) \longrightarrow s_j(c_j(\cdot), q),$$

where $\mathscr{C}_j$ is the set of marginal production cost functions, and will be explicitly given for the market coupling in Section 2.3. $s_j(c_j(\cdot), q)$ is the unit price at which the producer is ready to sell quantity q of electricity. An admissible strategy carries out the following sell at no loss constraint

$$s_j(c_j(\cdot), q) \geq c_j(q), \quad \forall q \in \mathrm{Dom}(c_j). \tag{1}$$

For example we can take $s_j(c_j(\cdot), q) = c_j(q)$ or $s_j(c_j(\cdot), q) = c_j(q) + \lambda(q)$, where $\lambda(q)$ is for any additional profit. As mentioned in the introduction, the constraint (1) guarantees profitable trade as much as the equilibrium established through this class of strategy will benefit each producer. This establishes a link between market share maximization and profit maximization paradigms.

Let us denote $\mathcal{S}$ the class of admissible strategy profiles on the electricity market. We have

$$\mathcal{S} = \left\{ \begin{array}{ccc} \mathbf{s} = (s_1, \dots, s_J);\ s_j : \mathscr{C}_j \times \mathbb{R}^+ & \longrightarrow & \mathbb{R}^+ \\ (c_j(\cdot), q) & \longrightarrow & s_j(c_j(\cdot), q) \end{array} \quad \text{s.t. } s_j(c_j(\cdot), q) \geq c_j(q), \quad \forall q \in \mathrm{Dom}(c_j) \right\} \tag{2}$$

As a function of q, $s_j(c_j(\cdot), q)$ is bounded on $\mathrm{Dom}(c_j)$. For the sake of clarity, we define for each $q \notin \mathrm{Dom}(c_j)$, $s_j(c_j(\cdot), q) = p_{\mathrm{lolc}}$, where p_{lolc} is the loss of load cost, chosen as any overestimation of the maximal production costs.

For producer j's strategy s_j, we define the associated asking size at price p as

$$\mathcal{O}(c_j(\cdot), s_j; p) := \sup\{q,\ s_j(c_j(\cdot), q) < p\}. \tag{3}$$

Hence $\mathcal{O}(c_j(\cdot), s_j; p)$ is the maximum quantity of electricity at unit price p supplied by producer j on the market.

Remark 2.2.
(i) The asking size function $p \mapsto \mathcal{O}(c_j(\cdot), s_j; p)$ is, with respect to p, an increasing surjection from $[0, +\infty)$ to $[0, \kappa_j]$, right continuous and such that $\mathcal{O}(c_j(\cdot), s_j; 0) = 0$. For an increasing strategy s_j, $\mathcal{O}(c_j(\cdot), s_j; .)$ is its generalized inverse function with respect to q.
(ii) Given two strategies $q \mapsto s_j(c_j(\cdot), q)$ and $q \mapsto s'_j(c_j(\cdot), q)$ such that $s_j(c_j(\cdot), q) \leq s'_j(c_j(\cdot), q)$, for all $q \in \mathrm{Dom}(c_j)$ we have for any positive p

$$\mathcal{O}(c_j(\cdot), s_j; p) \geq \mathcal{O}(c_j(\cdot), s'_j; p).$$

Indeed, if $p_1 \geq p_2$ then $\{q,\ s_j(c_j(\cdot), q) \leq p_2\} \subset \{q,\ s_j(c_j(\cdot), q) \leq p_1\}$ from which we deduce that $\mathcal{O}(c_j(\cdot), s_j; \cdot)$ is increasing. Next, if $s_j(c_j(\cdot), \cdot) \leq s'_j(c_j(\cdot), \cdot)$, for any fixed p, we have $\{q,\ s'_j(c_j(\cdot), q) \leq p\} \subset \{q,\ s_j(c_j(\cdot), q) \leq p\}$ from which the reverse order follows for the requests.

We shall now describe the electricity market clearing. Note that from a market view point, the dependency of the supply with respect to the marginal cost does not need to be explicit. For the sake of clarity, we write $s_j(q)$ and $\mathcal{O}(s_j; p)$ instead of $s_j(c_j(\cdot), q)$, $\mathcal{O}(c_j(\cdot), s_j; p)$. The dependency will be expressed explicitly whenever needed.

By aggregating the J asking size functions, we can define the overall asking function $p \mapsto \mathscr{O}(\mathbf{s}; p)$ a producer strategy profile $\mathbf{s} = (s_1, \dots, s_J)$ as:

$$\mathscr{O}(\mathbf{s}; p) = \sum_{j=1}^{J} \mathcal{O}(s_j; p). \tag{4}$$

Hence, for any producer strategy profile $\mathbf{s}$, $\mathscr{O}(\mathbf{s}; p)$ is the quantity of electricity that can be sold on the market at unit price p. The overall supply function $p \mapsto \mathscr{O}(\mathbf{s}; p)$ is an increasing surjection defined from $[0, +\infty)$ to $[0, \sum_{j=1}^{J} \kappa_j]$, such that $\mathscr{O}(\mathbf{s}; 0) = 0$.

2.1.1. *Electricity market clearing*

Taking producer strategy profile $\mathbf{s} = (s_1(\cdot), \dots, s_J(\cdot))$ the market sets the electricity market price $p^{\mathrm{elec}}(\mathbf{s})$ together with the quantities $(\varphi_1(\mathbf{s}), \dots, \varphi_J(\mathbf{s}))$ of electricity sold by each producer.

The market clearing price $p^{\mathrm{elec}}(\mathbf{s})$ is the unit price paid to each producer for the quantities $\varphi_j(\mathbf{s})$ of electricity. The price $p(\mathbf{s})$ may be defined as a price whereby supply satisfies demand. As we are working with a general non-increasing demand curve (possibly locally inelastic), the price that satisfies the demand is not necessarily unique. We thus define the clearing price generically with the following definition.

Definition 2.3 (The clearing electricity price)**.** Let us define

$$\underline{p}(\mathbf{s}) = \inf\{p > 0;\ \mathscr{O}(\mathbf{s}; p) > D(p)\} \ \text{ and } \quad \overline{p}(\mathbf{s}) = \sup\left\{p \in [\underline{p}(\mathbf{s}), p_{\mathrm{lolc}}]; D(p) = D(\underline{p}(\mathbf{s}))\right\} \tag{5}$$

with the convention that $\inf \emptyset = p_{\mathrm{lolc}}$. The clearing price may then be established as any $p^{\mathrm{elec}}(\mathbf{s}) \in [\underline{p}(\mathbf{s}), \overline{p}(\mathbf{s})]$ as an output of a specific market clearing rule. To keep the price consistency, the market rule must be such that for any two strategy profiles $\mathbf{s}$ and $\mathbf{s}'$,

$$\text{if } \underline{p}(\mathbf{s}) < \underline{p}(\mathbf{s}') \text{ then } p^{\text{elec}}(\mathbf{s}) < p^{\text{elec}}(\mathbf{s}'), \text{ and if } \underline{p}(\mathbf{s}) = \underline{p}(\mathbf{s}') \text{ then } p^{\text{elec}}(\mathbf{s}) = p^{\text{elec}}(\mathbf{s}'). \tag{6}$$

Note that $\underline{p}(\mathbf{s}) \neq \overline{p}(\mathbf{s})$ only if the demand curve $p \mapsto D(p)$ is constant on some intervals $[\underline{p}(\mathbf{s}), \underline{p}(\mathbf{s}) + \epsilon]$.

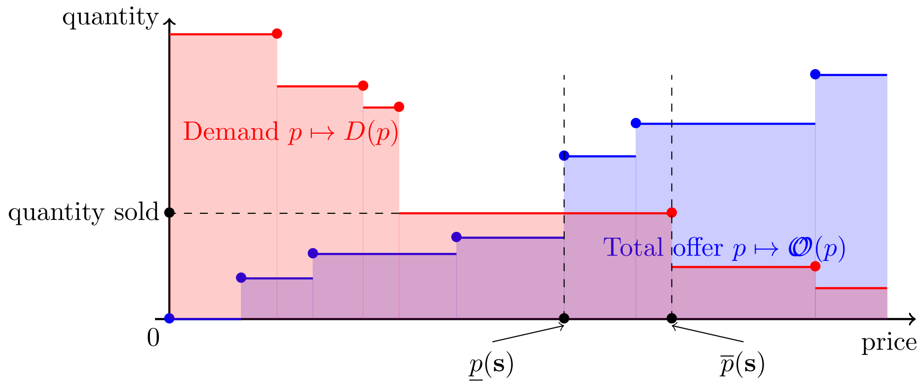

Figure 1. Electricity clearing price $\underline{p}(\mathbf{s})$ and $\overline{p}(\mathbf{s})$.

Note also that price $\underline{p}(\mathbf{s})$ is well defined in the case where demand does not strictly decrease. This includes the case where demand is constant. In such case, $\underline{p}(\mathbf{s}) = p_{\text{lolc}}$ only if the demand curve never crosses the supply.

Next, we define the quantity of electricity sold at price $p^{\text{elec}}(\mathbf{s})$. When $\mathcal{O}(\mathbf{s}; p^{\text{elec}}(\mathbf{s})) \leq D(p^{\text{elec}}(\mathbf{s}))$, each producer sells $\mathcal{O}(\mathbf{s}_j; p^{\text{elec}}(\mathbf{s}))$, but cases where $\mathcal{O}(\mathbf{s}; p^{\text{elec}}(\mathbf{s})) > D(p^{\text{elec}}(\mathbf{s}))$ may occur, requiring the introduction of an auxiliary rule to share $D(p^{\text{elec}}(\mathbf{s}))$ among the producers that propose $\mathcal{O}(\mathbf{s}; p^{\text{elec}}(\mathbf{s}))$. Note that in this last case, due to the clearing property (6) on $p^{\text{elec}}(\cdot)$, we have $\mathcal{O}(\mathbf{s}; \underline{p}(\mathbf{s})) \geq D(p^{\text{elec}}(\mathbf{s})) = D(\underline{p}(\mathbf{s}))$. Hence the $D(p^{\text{elec}}(\mathbf{s}))$ is totally provided by producers with non null offer at price $\underline{p}(\mathbf{s})$. The rule of the market is to share $D(p^{\text{elec}}(\mathbf{s}))$ among these producers only. This gives an explicit priority to the best offer prices $\underline{p}(\mathbf{s})$.

Let us break down supply as follows:

$$\mathcal{O}(\mathbf{s}; \underline{p}(\mathbf{s})) = \sum_{j=1}^{J} \mathcal{O}(s_j; \underline{p}(\mathbf{s})^-) + \sum_{j=1}^{J} \Delta^- \mathcal{O}(s_j; \underline{p}(\mathbf{s})),$$

where $\Delta^- \mathcal{O}(s_j; \underline{p}(\mathbf{s})) := \mathcal{O}(s_j; \underline{p}(\mathbf{s})) - \mathcal{O}(s_j; \underline{p}(\mathbf{s})^-)$.

The market's choice is to fully accept the asking size of producers with continuous asking size curve at point $\underline{p}(\mathbf{s})$. For producers with discontinuous asking size curve at $\underline{p}(\mathbf{s})$, a market rule based on proportionality that favors abundance is used to share the remaining part of the supply. We resume the market rule on quantities as follows.

Definition 2.4 (Clearing electricity quantities)**.** The quantity $\varphi_j(\mathbf{s})$ of electricity sold by Producer j on the electricity market is

$$\varphi_j(\mathbf{s}) = \begin{cases} \mathcal{O}(\mathbf{s}_j; p^{\text{elec}}(\mathbf{s})), & \text{if } D(p^{\text{elec}}(\mathbf{s})) \geq \mathcal{O}(\mathbf{s}; p^{\text{elec}}(\mathbf{s})), \\ \mathcal{O}(\mathbf{s}_j; \underline{p}(\mathbf{s})^-) + \Delta^- \mathcal{O}(\mathbf{s}_j; \underline{p}(\mathbf{s})) \dfrac{D(\underline{p}(\mathbf{s})) - \mathcal{O}(\mathbf{s}; \underline{p}(\mathbf{s})^-)}{\Delta^- \mathcal{O}(\mathbf{s}; \underline{p}(\mathbf{s}))}, & \text{if } D(p^{\text{elec}}(\mathbf{s})) < \mathcal{O}(\mathbf{s}; \underline{p}(\mathbf{s})), \end{cases} \tag{7}$$

where $\Delta^- \mathcal{O}(\mathbf{s}; \underline{p}(\mathbf{s})) := \sum_{j=1}^{J} \Delta^- \mathcal{O}(s_j; \underline{p}(\mathbf{s})) > 0$.

Note that, when $D(\underline{p}(\mathbf{s})) < \mathcal{O}(\mathbf{s};\underline{p}(\mathbf{s}))$, we have $\Delta^- \mathcal{O}(\mathbf{s};\underline{p}(\mathbf{s})) > 0$. Note also that we always have

$$\sum_{i=1}^{J} \varphi_j(\mathbf{s}) = D(p^{\mathrm{elec}}(\mathbf{s})) \wedge \mathcal{O}(\mathbf{s}; p^{\mathrm{elec}}(\mathbf{s})). \tag{8}$$

2.2. **Carbon market**

Let us recall the CO_2 regulation principle on which we base our analysis. Producers are penalized according to their emission level if they do not own allowances. Hence, in parallel to their position on the electricity market, producers buy CO_2 emission allowances on a separate CO_2 auction market.

In the following, we formalize producer strategy on the CO_2 market only.

Assumption 2.5 (Capped carbon market)**.**
(i) *The carbon market is capped and has a finite known quantity Ω of CO_2 emission allowances available.*
(ii) *Each producer j can buy a capped number of allowances $\mathcal{E}_j$, related to its own CO_2 emission capacity.*
(iii) *Emissions that are not covered by allowances are penalized at a unit rate $\mathfrak{p}$.*

On this market, producers adopt a strategy that consists in an offer function $\tau \mapsto A_j(\tau)$ defined from $[0, \mathfrak{p}]$ to $[0, \mathcal{E}_j]$. Quantity $A_j(\tau)$ is the quantity of allowances that producer j is ready to buy at price τ. This offer may not be a monotonic function. We denote $\mathscr{A}$ the strategy profile set on the CO_2 market,

$$\mathscr{A} := \{\mathbf{A} = (A_1, \dots, A_J);\ \text{s.t.}\ A_k : [0, \mathfrak{p}] \to [0, \mathcal{E}_j]\}.$$

The CO_2 market reacts by aggregating the J offers by $\boldsymbol{\mathcal{A}}(\tau) := \sum_{j=1}^{J} A_j(\tau)$, and the clearing market price is established following a *second item auction* as:

$$p^{\mathrm{CO}_2}(\mathbf{A}) := \sup\{\tau; \boldsymbol{\mathcal{A}}(\tau) > \Omega\}, \qquad \text{with the convention } \sup \emptyset = 0. \tag{9}$$

Note that $p^{\mathrm{CO}_2}(\mathbf{A}) = 0$ indicates that there are too many allowances to sell. It is worth a reminder here that the aim of allowances is to decrease emissions. In section 3.3, we discuss a design hypothesis (Assumption 3.5) that guarantees an equilibrium price $p^{\mathrm{CO}_2}(\mathbf{A}) > 0$. Therefore, in the following, we assume that the overall quantity Ω of allowances, is such that $p^{\mathrm{CO}_2}(\mathbf{A}) > 0$.

Next, we define the amount of allowances bought at price $p^{\mathrm{CO}_2}(\mathbf{A})$ by the producers. By Definition (9), we have $\boldsymbol{\mathcal{A}}(p^{\mathrm{CO}_2}(\mathbf{A})) \geq \Omega$ and $\boldsymbol{\mathcal{A}}(p^{\mathrm{CO}_2}(\mathbf{A})^+) \leq \Omega$. When $\boldsymbol{\mathcal{A}}(p^{\mathrm{CO}_2}(\mathbf{A})) > \Omega$, the CO_2 market must decide between the producers with an additional rule. We define

$$\Delta(A_i) := A_i(p^{\mathrm{CO}_2}(\mathbf{A})^+) - A_i(p^{\mathrm{CO}_2}(\mathbf{A})).$$

For a producer i, $\Delta(A_i) \geq 0$ means that its CO_2 demand does not decrease if the price increases. It is therefore ready to pay more to obtain the quantity of allowances it is asking for at price $p^{\mathrm{CO}_2}(\mathbf{A})$. The CO_2 market gives priority to this kind of producer, which will be fully served. The producers such that $\Delta(A_j) < 0$ share the remaining allowances. This can be written as follows.

Each producers with $A_j(p^{\mathrm{CO}_2}(\mathbf{A})) > 0$ obtains the following quantity $\delta_j(\mathbf{A})$ of allowances

$$\delta_j(\mathbf{A}) := \begin{cases} A_j(p^{\mathrm{CO}_2}(\mathbf{A})), & \text{if } \Delta(A_j) \geq 0, \\ \\ A_j(p^{\mathrm{CO}_2}(\mathbf{A})^+) + \dfrac{(-\Delta(A_j))^+}{\displaystyle\sum_{i=1}^{J}(-\Delta(A_i))^+} \left(\Omega - \displaystyle\sum_{i=1}^{J} A_i(p^{\mathrm{CO}_2}(\mathbf{A})) \mathbb{1}_{\{\Delta(A_i)\geq 0\}}\right), & \text{otherwise.} \end{cases} \tag{10}$$

2.3. Carbon and electricity market coupling

In the following, we formalize the coordination of a producer's strategy on the CO_2 and electricity markets.

As mentioned earlier, for each producer, the marginal cost function is parametrized by the positions $\mathbf{A}$ of the producers on the carbon market. Indeed, producer j can obtain CO_2 emission allowances on the market to avoid penalization for (some of) its emissions. Those emissions that are not covered by allowances are penalized at a unit rate $\mathfrak{p}$.

A profile of an offer to buy from the producers $\mathbf{A} = (A_1, \dots, A_J)$, through the CO_2 market clearing, corresponds to a unit price of $p^{\mathrm{CO_2}}(\mathbf{A})$ of the allowance and quantities $\delta_j(\mathbf{A})$ of allowances bought by each producer (defined by the market rules (9),(10)).

The following minimal assumption on the CO_2 emission related to the electricity production will be restricted in Assumption 3.3.

Assumption 2.6. *We assume that for all producers $\{j = 1, \dots, J\}$, the emission rate (originally in CO_2 t/Mwh) $q \mapsto e_j(q)$ is positive.*

For each producer, we fix the maximal amount $\mathcal{E}_j$ of allowances to buy to $\int_0^{\kappa_j} e_j(z)dz$.

Then, the marginal production cost function $c_j^{\mathbf{A}}(\cdot)$, parametrized by the emission regulations, comes out as

$$q \mapsto c_j^{\mathbf{A}}(q) = \begin{cases} c_j(q) + e_j(q)p^{\mathrm{CO_2}}(\mathbf{A}), & \text{for } q \in [0, \kappa_j^{\mathrm{CO_2}} \wedge \kappa_j] \\ c_j(q) + e_j(q)\mathfrak{p}, & \text{for } q \in [\kappa_j^{\mathrm{CO_2}} \wedge \kappa_j, \kappa_j] \end{cases} \tag{11}$$

where $\kappa_j^{\mathrm{CO_2}}$ is such that $\int_0^{\kappa_j^{\mathrm{CO_2}}} e_j(z)dz = \delta_j(\mathbf{A})$, and where $c_j(\cdot)$ stands for the marginal production cost without any emission regulation.

In this coupled market setting, the strategy of producer j thus makes a pair (A_j, s_j). The set of admissible strategy profile is defined as

$$\mathbf{\Sigma} = \{(\mathbf{A}, \mathbf{s});\ \mathbf{A} \in \mathscr{A},\ \mathbf{s} \in \mathcal{S}\}, \tag{12}$$

where in the definition of $\mathcal{S}$ in (2), we use $\mathscr{C}_j = \left\{c_j^{\mathbf{A}};\ \mathbf{A} \in \mathscr{A}\right\}$. Prices for allowances and electricity, $p^{\mathrm{CO_2}}((\mathbf{A}, \mathbf{s}))$ and $p^{\mathrm{elec}}((\mathbf{A}, \mathbf{s}))$, quantities of allowances bought by each producer, $\delta_j((\mathbf{A}, \mathbf{s}))$ and market shares on electricity market $\varphi_j((\mathbf{A}, \mathbf{s}))$ of each producer corresponds to any strategy profile $(\mathbf{A}, \mathbf{s}) \in \mathbf{\Sigma}$, through the market mechanisms described.

3. Nash Equilibrium analysis

3.1. Definition

We suppose that the J producers behave non cooperatively, aiming at maximizing their individual market share on the electricity market. For a strategy profile $(\mathbf{A}, \mathbf{s}) \in \mathbf{\Sigma}$, the market share of a producer j depends upon its strategy $(A_j, s_j(\cdot))$ but also on the strategies $(\mathbf{A}_{-j}, \mathbf{s}_{-j})$ of the other producers [2]. In this set-up the natural solution is the Nash equilibrium (see e.g. [1]). More precisely we are looking for a strategy profile $(\mathbf{A}^*, \mathbf{s}^*) = ((A_1^*, s_1^*), \cdots, (A_J^*, s_J^*)) \in \mathbf{\Sigma}$ that satisfies Nash equilibrium conditions: none of the producers would strictly benefit, that is, would strictly increase its market share from a unilateral deviation. Namely, for any producer j strategy $(\mathbf{A}_j, \mathbf{s}_j)$ such that $((\mathbf{A}_{-j}^*, \mathbf{s}_{-j}^*); (A_j, s_j)) \in \mathbf{\Sigma}$, we have [3]

$$\varphi_j((\mathbf{A}^*, \mathbf{s}^*)) \geq \varphi_j((\mathbf{A}_{-j}^*, \mathbf{s}_{-j}^*); (A_j, s_j)), \tag{13}$$

[2] Here $\mathbf{v}_{-j}$ stands for the profile $(v_i, \cdots, v_{j-1}, v_{j+1}, \cdots, v_J)$.

[3] $(\mathbf{v}_{-j}; v)$ stands for $(v_1, \cdots v_{j-1}, v, v_{j+1}, \cdots v_J)$

where φ_j is the quantity of electricity sold. Note that the dependency in terms of $\mathbf{A}$ through the marginal cost $c_j^{\mathbf{A}}$ is now explicit in φ_j.

Condition (13) has to be satisfied for any unilateral deviation of any producer j. In particular (13) has to be satisfied for a producer j admissible deviation (A_j^*, s_j) such that $((\mathbf{A}_{-j}^*, \mathbf{s}_{-j}^*); (A_j^*, s_j)) \in \mathbf{\Sigma}$ where producer j would only change its behavior on the electricity market. Consequently,

Remark 3.1. The electricity strategy component $\mathbf{s}^*$ of the Nash equilibrium $(\mathbf{A}^*, \mathbf{s}^*)$ is also a Nash equilibrium for the restricted electricity game, where producers only behave on the electricity market with marginal electricity production costs $c_j^{\mathbf{A}^*}(\cdot)$, $j = 1, \cdots J$.

The next section focuses on determining Nash equilibria on the game restricted to the electricity market.

3.2. Equilibrium on the power market

In this restricted set-up, we consider that the marginal costs $\{c_j, j = 1 \dots, J\}$ are known data, possibly fixed through the position $\mathbf{A}$ on the CO_2 market. In this section, we refer to $\mathcal{S}$ as the set of admissible strategy profiles, in the particular case where $\mathscr{C}_j = \{c_j\}$ for each $j = 1, \dots, J$.

The Nash equilibrium problem is as follows: find a strategy profile $\mathbf{s}^* = (s_1^*, \dots, s_J^*) \in \mathcal{S}$ such that

$$\forall j, \forall\, s_j \neq s_j^*, \quad \varphi_j(\mathbf{s}^*) \geq \varphi_j(\mathbf{s}_{-j}^*; s_j). \tag{14}$$

The following proposition exhibits a Nash equilibrium, whereby each producer must choose the strategy denoted by C_j, and referred to as *marginal production cost strategy*. It is defined by

$$C_j(q) = \begin{cases} c_j(q), & \text{for } q \in \mathrm{Dom}(c_j) \\ p_{\mathrm{lolc}}, & \text{for } q \notin \mathrm{Dom}(c_j). \end{cases} \tag{15}$$

Proposition 3.2.
(i) *For any strategy profile $\mathbf{s} = (s_1, \dots, s_J)$, no producer $j \in \{1, \dots, J\}$ can be penalized by deviating from strategy s_j to its marginal production cost strategy C_j, namely, $\varphi_j(\mathbf{s}) \leq \varphi(\mathbf{s}_{-j}; C_j)$. In other words, for any producer j, C_j is a dominant strategy.*
(ii) *The strategy profile $\mathbf{C} = (C_1, \dots C_J)$ is a Nash equilibrium.*
(iii) *If the strategy profile $\mathbf{s} \in \mathcal{S}$ is a Nash equilibrium, then we have $p^{elec}(\mathbf{s}) = p^{elec}(\mathbf{C})$ and for any producer j, $\varphi_j(\mathbf{s}) = \varphi_j(\mathbf{C})$.*

Point (ii) of the previous proposition is a direct consequence of the dominance property (i). The proof of both (i) and (iii) can be found in [3]. Point (ii) of the proposition exhibits a Nash equilibrium strategy profile. Clearly this equilibrium is not unique since we can easily show that a producer's given supply can follow from countless different strategies. Nevertheless point (iii) shows that for any Nash equilibrium, the associated electricity prices are the same and the quantity of electricity bought by any producer j is the same for all equilibrium profiles.

3.3. Coupled market design using Nash equilibrium

From this point we restrict our attention to a particular market design. In the following, the scope of the analysis applies to a special class of producers, a specific electricity market price clearing (satisfying Definition 2.3) and a range of quantities Ω of allowances available on the CO_2 market. Although not necessary, the following restriction simplifies the development.

Assumption 3.3. On the producers. *Each producer j operates a single production unit, for which*

(i) *the marginal cost contribution that does not depend on the producer positions $\mathbf{A}$ in the CO_2 market is constant, $q \mapsto c_j(q) = c_j$. The related emission rate $q \mapsto e_j(q) = e_j$ is assumed to be a positive constant,*
(ii) *the producers are different pairwise: $\forall i, j \in \{1, \cdots J\}, (c_i, e_i) \neq (c_j, e_j)$.*

For each producer, the maximal amount $\mathcal{E}_j$ of allowances to buy is now $e_j\kappa_j$.

As a consequence of Assumption 3.3, the marginal production cost in (11) simply writes as

$$q \mapsto c_j^{\mathbf{A}}(q) = \begin{cases} c_j + e_j p^{\mathrm{CO_2}}(\mathbf{A}), & \text{for } q \in [0, \dfrac{\delta_j(\mathbf{A})}{e_j} \wedge \kappa_j], \\ c_j + e_j \mathfrak{p}, & \text{for } q \in [\dfrac{\delta_j(\mathbf{A})}{e_j} \wedge \kappa_j, \kappa_j]. \end{cases} \tag{16}$$

For a given strategy profile on the electricity market, Definition 2.3 gives a range of possible determinations for the electricity price. Previously, the analysis of the Nash Equilibrium restricted to the electricity market did not require a precise clearing price determination. Nevertheless to extend our analysis of the coupling we need to explicit this determination and assume the following:

Assumption 3.4. On the electricity market. *For a given strategy profile* $\mathbf{s}$ *of the producers, the clearing price of electricity is* $p^{elec}(\mathbf{s})$. *The market rule fixes* $p^{elec}(\cdot) = \overline{p}(\cdot)$ *or* $p^{elec}(\cdot) = \underline{p}(\cdot)$ *as defined in* (5).

Note that this choice of clearing price ensures the increasing behavior of $p^{\mathrm{elec}}(\cdot)$ in terms of the carbon price.

The quantity Ω of CO_2 allowances available on the market plays a crucial role in the market design. As a matter of fact, if this quantity is too high, its market price will drop to zero, leaving the market incapable of fulfilling its role of decreasing CO_2 emissions. Therefore we clearly need to make an assumption that restricts the number of allowances available. Capping the maximum quantity of allowances available requires information on which producers are willing to obtain allowances. This is the objective of the following paragraph where we define a *willing to buy* function that plays a central role in the analysis of Nash equilibria.

In this paragraph, we aim at guessing a Nash equilibrium candidate. We base our reasoning on the dominant strategies on the electricity market alone (see Proposition 3.2). Remark 3.1 allows us to fix the electricity market strategy as a *marginal production cost strategy*, given the marginal cost functions $\mathbf{C}^{\mathbf{A}} = \{c_j^{\mathbf{A}}, j = 1, \dots J\}$ imposed by the output of the CO_2 clearing, as in (16).

In particular, when $\mathbf{A} \in \mathscr{A}$, the strategies $(\mathbf{A}, \{c_j^{\mathbf{A}}, j = 1, \dots J\})$ are admissible strategies as defined in (12).

From now on, all the strategy profiles that we consider on the carbon market are assumed to be admissible.

In the following, as the discussion will mainly focus on the impact of strategies $\mathbf{A}$ through the carbon market, we denote the electricity market output as:

$$p^{\mathrm{elec}}(\mathbf{A}) \text{ instead of } p^{\mathrm{elec}}(\mathbf{C}^{\mathbf{A}}) \quad \text{and} \quad (\varphi_1(\mathbf{A}), \dots, \varphi_J(\mathbf{A})) \text{ instead of } (\varphi_1(\mathbf{C}^{\mathbf{A}}), \dots, \varphi_J(\mathbf{C}^{\mathbf{A}})).$$

To begin with, we consider an exogenous CO_2 cost τ similar to a CO_2 tax: the producers' marginal cost becomes for any $\tau \in [0, \mathfrak{p}]$, $c_j^\tau(\cdot)$,

$$c_j^\tau(q) = c_j + \tau e_j, \text{ for } q \in [0, \kappa_j], \ j = 1, \dots, J.$$

In this *tax* framework, the dominant strategy on the electricity market is also parametrized by τ as $\mathbf{C}^\tau = \{c_j^\tau, j = 1, \dots J\}$ defined in (15). The clearing electricity price and quantities derive as

$$p^{\mathrm{elec}}(\tau) = p^{\mathrm{elec}}(\mathbf{C}^\tau) \quad \text{and} \quad (\varphi_1(\tau), \dots, \varphi_J(\tau)) = (\varphi_1(\mathbf{C}^\tau), \dots, \varphi_J(\mathbf{C}^\tau)). \tag{17}$$

We determine the *willing-to-buy-allowances functions* $\mathcal{W}_j(\cdot)$ and $\mathcal{W}(\cdot)$, as follows:

$$\mathcal{W}_j(\tau) = e_j \varphi_j(\tau) \quad \text{and} \quad \mathcal{W}(\tau) = \sum_{j=1}^{J} \mathcal{W}_j(\tau) \tag{18}$$

For producer j, $\mathcal{W}_j$ is the quantity of emissions it would produce under the penalization τ, and consequently the quantity of allowances it would be ready to buy at price τ. Given the CO_2 value τ, the total amount $\mathcal{W}(\tau)$

represents the allowances needed to cover the global emissions generated by the players who have won electricity market shares. We also define the functions

$$\overline{\mathcal{W}}_j(\tau) = e_j \kappa_j \mathbb{1}_{\{\varphi_j(\tau)>0\}} \quad \text{and} \quad \overline{\mathcal{W}}(\tau) = \sum_{j=1}^{J} \overline{\mathcal{W}}_j(\tau) \tag{19}$$

Given that the CO_2 value τ, $\overline{\mathcal{W}}(\tau)$ is the amount of allowances needed by the producers who have won electricity market shares and want to cover their overall production capacity κ_j. Obviously we have

$$\mathcal{W}(\tau) \le \overline{\mathcal{W}}(\tau), \ \forall \tau \in [0, \mathfrak{p}].$$

We now can state our last design assumption

Assumption 3.5. On the carbon market design. *The available allowances* Ω *satisfy* $\overline{\mathcal{W}}(\mathfrak{p}) < \Omega < \mathcal{W}(0)$.

Assumption 3.5 allows us to define two prices of particular interest for the construction of the equilibrium strategy:

$$\tau^{\text{guess}} = \sup\{\tau \in [0, \mathfrak{p}] \text{ s.t. } \mathcal{W}(\tau) > \Omega\}, \text{ and } \overline{\tau}^{\text{guess}} = \sup\{\tau \in [0, \mathfrak{p}] \text{ s.t. } \overline{\mathcal{W}}(\tau) > \Omega\}. \tag{20}$$

Observe that we always have $\tau^{\text{guess}} \le \overline{\tau}^{\text{guess}}$.

Lemma 3.6. *The function* $\tau \mapsto \mathcal{W}(\tau)$ *is non increasing:* $\mathcal{W}(t') \le \mathcal{W}(t), \ \forall \, 0 \le t < t' \le \mathfrak{p}$.

3.3.1. *Towards an equilibrium strategy*

In the following we do not explicit a Nash equilibrium. Instead we establish (Lemmas 3.7 and 3.7) that for all Nash coupled equilibrium, the associated CO_2 price is greater that τ^{guess} while the CO_2 price associated to all strong Nash equilibrium lies in the intervall $[\tau^{\text{guess}}, \overline{\tau}^{\text{guess}}]$. Lemma 3.9 and Corollary 3.10 partially explicit a Nash equilibrium class. The proofs of these results can be found in [3].

Lower price strategy.

Consider any strategy $\mathbf{A}^{\mathcal{W}} = (A_1^{\mathcal{W}}, \dots, A_J^{\mathcal{W}})$ such that

$$A_j^{\mathcal{W}}(\tau) = \begin{cases} \mathcal{W}_j(\tau^{\text{guess}}), & \text{for } 0 \le \tau \le \tau^{\text{guess}} \\ \text{anything admissible}, & \text{for } \tau > \tau^{\text{guess}}. \end{cases} \tag{21}$$

Lemma 3.7.
(i) $p^{CO_2}(\mathbf{A}^{\mathcal{W}}) \ge \tau^{guess}$.
(ii) In the case where $p^{CO_2}(\mathbf{A}^{\mathcal{W}}) = \tau^{guess}$*, there is no unilateral favorable deviation that clears the market at a* CO_2 *price lower than* τ^{guess}.
(iii) Suppose $\mathbf{A}$ *is such that* $p^{CO_2}(\mathbf{A}) < \tau^{guess}$*. Then* $\mathbf{A}$ *is not a Nash equilibrium.*

High price strategy.

Consider any strategy $\mathbf{A}^{\overline{\mathcal{W}}} = (A_1^{\overline{\mathcal{W}}}, \cdots, A_J^{\overline{\mathcal{W}}})$ such that

$$A_j^{\overline{\mathcal{W}}}(\tau) = \begin{cases} \text{anything admissible}, & \text{for } \tau \le \overline{\tau}^{\text{guess}} \\ \overline{\mathcal{W}}_j(\tau), & \text{for } \tau > \overline{\tau}^{\text{guess}}. \end{cases} \tag{22}$$

Lemma 3.8.
(i) $p^{CO_2}(\mathbf{A}^{\overline{\mathcal{W}}}) \le \overline{\tau}^{guess}$.
(ii) In the case where $p^{CO_2}(\mathbf{A}^{\overline{\mathcal{W}}}) = \overline{\tau}^{guess}$*, there is no unilateral favorable deviation that clears the market at a* CO_2 *price higher than* $\overline{\tau}^{guess}$.

(iii) Suppose $\mathbf{A}$ *is such that* $p^{CO_2}(\mathbf{A}) > \overline{\tau}^{guess}$*. Then* $\mathbf{A}$ *is not a strong Nash equilibrium.*

Intermediate strategy.

Consider any strategy profile $\mathbf{B} = (B_1, \cdots, B_J)$ satisfying the following:

$$B_j(\tau) = \begin{cases} \overline{\mathcal{W}}_j(\tau), & \text{for } \tau > \overline{\tau}^{\text{guess}} \\ \text{anything admissible}, & \text{for } \tau^{\text{guess}} < \tau \leq \overline{\tau}^{\text{guess}} \\ \mathcal{W}_j(\tau^{\text{guess}}), & \text{for } \tau \leq \tau^{\text{guess}}. \end{cases} \tag{23}$$

This is not in general an equilibrium, nevertheless we have the following properties :

Lemma 3.9.
(i) $p^{CO_2}(\mathbf{B}) \in [\tau^{guess}, \overline{\tau}^{guess}]$.
(ii) If there exists a favorable deviation from a producer, say Producer 1, that chooses $\widetilde{B}_1$ *instead of* B_1*, such that* $p^{CO_2}(\mathbf{B}_{-1}; \widetilde{B}_1) < \tau^{guess}$*, then there exists another favorable deviation* $\widehat{B}_1$ *such that* $p^{CO_2}(\mathbf{B}_{-1}; \widehat{B}_1) = \tau^{guess}$*, and such that* $\varphi_1(\mathbf{B}_{-1}; \widehat{B}_1) \geq \varphi_1(\mathbf{B}_{-1}; \widetilde{B}_1)$.

The following corollary is a direct consequence of Lemmas 3.7, 3.8, and 3.9.

Corollary 3.10. *Let* $\mathbf{E}$ *be a (strong) Nash equilibrium. Then the following* $\mathbf{E}'$ *is also a (strong) Nash equilibrium:*

$$E'_j(\tau) = \begin{cases} \overline{\mathcal{W}}_j(\tau), & \textit{for } \tau > \overline{\tau}^{guess} \\ E_j(\tau), & \textit{for } \tau^{guess} < \tau \leq \overline{\tau}^{guess} \\ \mathcal{W}_j(\tau^{guess}), & \textit{for } \tau \leq \tau^{guess} \end{cases} \tag{24}$$

4. Conclusion

Once emitted into the atmosphere, CO_2 will remain there for more than a century. Estimating its value is an essential indicator for efficiently defining policy. Therefore, carbon valuation remains a main issue in order to design markets fostering emission reductions. In this paper, we established the links between an electricity market and a carbon auction market through the analysis of electricity producers strategies. They have been proven to lead to a Nash equilibrium enabling the computation of equilibrium prices on both markets. This equilibrium derives, for each producer, in a level electricity produced and CO_2 emissions covered. Beyond the analysis of the Nash equilibrium, we envisage the analysis of the electricity production mix, with a particular focus on renewable shares which do not participate to emissions.

References

[1] Tamer Başar and Geert Jan Olsder. *Dynamic Noncooperative Game Theory, 2nd Edition*. Society for Industrial and Applied Mathematics, 1998.

[2] Mireile Bossy, Nadia Maïzi, Geert Jan Olsder, Odile Pourtallier, and Etienne Tanré. Electricity prices in a game theory context. In *Dynamic games: theory and applications*, volume 10 of *GERAD 25th Anniv. Ser.*, pages 135–159. Springer, New York, 2005.

[3] Mireille Bossy, and Nadia Maïzi and Odile Pourtallier. Game theory analysis for carbon auction market through electricity market coupling *Commodities, Energy, and Environmental Finance*, Fields Institute for Research in Mathematical Sciences, Toronto, *(To appear)*.

[4] René Carmona, Michael Coulon, and Daniel Schwarz. The valuation of clean spread options: Linking electricity, emissions and fuels. *Quantitative Finance*, 12(12), 2012. Special Issue: Commodities.

[5] René Carmona, Michael Coulon, and Daniel Schwarz. Electricity price modeling and asset valuation: a multi-fuel structural approach. *Mathematics and Financial Economics*, 7(2):167–202, 2013.

[6] René Carmona, François Delarue, Gilles-Edouard Espinosa, and Nizar Touzi. Singular forward-backward stochastic differential equations and emissions derivatives. *Ann. Appl. Probab.*, 23(3):1086–1128, 2013.