

OPTIMIZATION AND STATISTICAL METHODS FOR HIGH FREQUENCY FINANCE*

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Abstract. High Frequency finance has recently evolved from statistical modeling and analysis of financial data – where the initial goal was to reproduce stylized facts and develop appropriate inference tools – toward trading optimization, where an agent seeks to execute an order (or a series of orders) in a stochastic environment that may react to the trading algorithm of the agent (market impact, inventory). This context poses new scientific challenges addressed by the minisymposium OPSTAHF.

INTRODUCTION

High Frequency Finance requires increasingly sophisticated algorithms in order to manage a series of orders from the point of view of an agent (a trader) that wishes to execute his or her orders in a limited amount of time, under various trading objectives and market constraints, possibly at very fine temporal scales. Section 1 (contributed by M. Labadie) gives a general framework for operating such a task, by means of a stochastic control program (using HJB equations) in order to deal with inventory constraints. Section 2 (contributed by Gilles Pagès) proposes a new on-line optimisation algorithm for posting passive orders in a limit order book. Section 3 (contributed by H. Pham) proposes a tick-by-tick price model that incorporates microstructure effects at fine scales and that diffuses at coarse scales and for which a stochastic control algorithm for executing orders is developed. Finally, Section 4 (contributed by C.A. Lehalle) offers a wider view for order book dynamics reviewing some recent approaches and suggesting an opening in the direction of the recent and attractive mean-field game theory.

1. HF TRADING UNDER INVENTORY AND DIRECTIONAL RISK CONSTRAINTS

1.1. High-frequency trading and market-makers.

The high-frequency traders (HFTs) are trading algorithms, with the fastest response or lowest latency, that buys and sells assets in electronic and listed markets, trying to capitalise ephemeral arbitrage opportunities.

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One special kind of HFTs are market-makers. HFTs offer liquidity to the market, i.e. they place both a buying and a selling limit order on the Limit Order Book (LOB). Since they are offering liquidity, their buying price (bid price) is smaller than their selling price (ask price); therefore, each time they buy and sell they earn the spread (difference between ask and bid prices). However, market-makers suffer execution risks since they cannot control when and how their orders will be executed.

1.2. Stochastic model and Hamilton-Jacobi-Bellman equation (HJB)

The optimisation programme for a market-maker is thus to maximise the profit by trading the spread over and over throughout the day whilst minimising both the execution and inventory risks. We extend the model of Avellaneda and Stoikov [3], where the only state variable is the mid-price $S(t)$, to a general Markovian setting with several continuous state variables $Y(t) = (S(t), \Sigma(t), Z(t), \dots)$. In our case $Z(t)$ is the half-spread of the market and $\Sigma(t)$ the volatility of the mid-price. We use the convention of small letters for the values of the state variables at time t , e.g. $y = Y(t)$. The market-maker places two limit orders of one share on each side of the LOB: a selling limit order at $S(t) + \delta^+$ and buying limit order at $S(t) - \delta^-$. The arrival of buying and selling market orders to the LOB is assumed to be Poisson and independent; the distribution of how many levels in the LOB are consumed by a single market order is exponential (see Avellaneda and Stoikov [3] for details).

Assume the utility function is the PNL (profit and loss) $X(t) + S(t)Q(t)$. Since the only variables the market-maker can control is where to place her quotes, i.e. (δ^+, δ^-) , then her value function is (see Fodra and Labadie [7, 8])

$$u(t, y, q, x) = \max_{(\delta^+, \delta^-)} \mathbb{E}_{t, y, q, x} [X(T) + Q(T)S(T)], \quad (1)$$

where $\mathbb{E}_{t, y}[\cdot] := \mathbb{E}[\cdot | Y(t) = y]$. The associated HJB equation of (1) is

$$\begin{aligned} (\partial_t + \mathcal{L})u + \max_{\delta^+ \in \mathcal{A}} Ae^{-k[z+\delta^+]} [u(t, y, q-1, x+(s+\delta^+)) - u(t, y, q, x)] \\ + \max_{\delta^- \in \mathcal{A}} Ae^{-k[z+\delta^-]} [u(t, y, q+1, x-(s-\delta^-)) - u(t, y, q, x)] &= 0, \\ u(T, y, q, x) &= x + qs, \end{aligned} \quad (2)$$

where \mathcal{L} is the infinitesimal operator associated to the continuous state variables $Y(t)$. The *jump* part of the equation (i.e. the max terms) corresponds to the infinitesimal operator for the discrete state variable $Q(t)$. In the presence of inventory-risk aversion ε and transaction costs α , the HJB equation takes the form (see Fodra and Labadie [7, 8])

$$\begin{aligned} (\partial_t + \mathcal{L})u + \max_{\delta^+ \in \mathcal{A}} Ae^{-k[z+\delta^+]} [u(t, y, q-1, x+(s+\delta^+) - \alpha) - u(t, y, q, x)] \\ + \max_{\delta^- \in \mathcal{A}} Ae^{-k[z+\delta^-]} [u(t, y, q+1, x-(s-\delta^-) - \alpha) - u(t, y, q, x)] &= \varepsilon\nu\sigma^2q^2, \\ u(T, y, q, x) &= x + sq - \varepsilon\eta zq^2. \end{aligned} \quad (3)$$

There are similar models in the literature, where the controls are formally explicit in terms of a certain ODE system, but they can only be computed numerically. For more details see e.g. Cartea and Jaimungal [5] and Géant *et al* [11].

1.3. Exact solution without inventory penalty

When there is no inventory penalty, the exact solution of the HJB equation (2) is (see Fodra and Labadie [7,8])

$$\begin{aligned}
 u(t, y, q, x) &= u_{\text{hold}}(t, y, q, x) + u_{\text{mm}}(t, y), \\
 u_{\text{hold}} &:= x + q(s + \Delta), \quad \Delta := \mathbb{E}_{t,y}[S(T)] - s \\
 u_{\text{mm}} &:= \frac{2}{e} \mathbb{E}_{t,y} \left[\int_t^T \frac{A}{K} e^{-KZ} \cosh(K\Delta) d\xi \right], \\
 \psi_* &:= \delta_*^+ + \delta_*^- = \frac{2}{k} \quad (\text{market-maker's spread}), \\
 r_* &:= \frac{p^+ + p^-}{2} = s + \Delta \quad (\text{centre of spread}).
 \end{aligned}
 \tag{4}$$

From (4) we get the following observations. (i) The solution u has two strategies: one *buy-and-hold* u_{hold} , which is the expected value of the portfolio if the trader keeps the position until the end, and one *market-maker* u_{mm} , which is the gain from trading the spread very fast and frequently (hence the integral). (ii) Since $u_{\text{mm}} > 0$, the market-maker gets a better PNL by trading the spread than a buy-and-hold investor. (iii) The market-maker's spread ψ_* is constant and inversely proportional to k : when k increases, the market orders are less likely to consume several levels of the LOB, hence the market-maker needs to offer better prices to get executed, which means a smaller spread. (iv) The centre r_* of the spread is not the current mid-price s but $\mathbb{E}_{t,y}[S(T)]$, i.e. the conditional expectation of the value of the mid-price. This means that the market-maker plays a price arbitrage: she tilts her quotes, accepting an inventory risk, in order to capitalise the current mispricing in the long run.

1.4. Asymptotic solution with inventory penalty

In the case of inventory penalty ε and transaction costs α , the solution and controls are not explicit. However, using *perturbation methods* on ε we can find the explicit expression of the controls up to first-order (see Fodra and Labadie [8]):

$$\begin{aligned}
 \psi_{\alpha*} &= \frac{2}{k} + 2\alpha + 2\varepsilon\tilde{\pi} + O(\varepsilon^2) \quad (\text{market-maker's spread}), \\
 r_{\alpha*} &= s + \Delta - \varepsilon \left\{ \mathbb{E}_{t,y} \left[\int_t^T H(\alpha)\tilde{\pi}d\xi \right] + 2q\tilde{\pi} \right\} + O(\varepsilon^2) \quad (\text{centre of spread}), \\
 \Delta &:= \mathbb{E}_{t,y}[S(T)] - s, \\
 H(\alpha) &:= \frac{4}{e} A e^{-K(Z+\alpha)} \sinh(K\Delta), \\
 \tilde{\pi} &:= \eta \mathbb{E}_{t,y}[Z(T)] + \nu \mathbb{E}_{t,y} \left[\int_t^T \Sigma^2 d\xi \right].
 \end{aligned}
 \tag{5}$$

From (5) we get the following observations. (i) The spread $\psi_{\alpha*}$ has a linear correction via the unitary transaction cost α and the unitary inventory risk $\tilde{\pi}$. (ii) The current inventory q affects the centre $r_{\alpha*}$ of the spread, compensating the directional bet via Δ . (iii) The transaction cost α makes the spread wider, but the total gain for trading the spread is constant and independent of α . (iv) If $\alpha > 0$ then the market-maker will be less executed due to a wider spread. If everyone in the market have the same behaviour then the market spread widens, i.e. the available liquidity diminishes and the trading costs for market orders increases. (v) If $\alpha < 0$ (as in the case of rebates in certain alternative stock markets) then the market-maker reduces her spread. If the rebate is big enough then the market-maker can have a zero-width spread, i.e. she trades one leg with a market order. With this strategy, called *rebate arbitrage*, all the profit comes from the rebate $-\alpha$.

2. OPTIMAL POSTING PRICE OF LIMIT ORDERS : LEARNING BY TRADING

2.1. Introduction

Our aim is to propose an on-line optimization posting method of passive orders in a limit order book, without specifying the limit order book dynamics. We consider an agent who wants to buy during a short period $[0, T]$ a quantity Q_T of traded assets and we look for the optimal distance to the “fair” price (usually mid-, possibly best bid) where he has to post his limit order to minimize the execution cost (the case of a sell limit order may be obtained by symmetry). We assume the agent is a *learning trader*: he searches for the optimal posting distance (the price that balances adverse selection and non execution risk) by successive trials, errors and corrections. The derived recursive optimization procedure yields the best price adjustment to apply to an order for a given stopping time (reassessment dates) given past market observations.

We model the execution process of orders by a Poisson process $(N_t^{(\delta)})_{0 \leq t \leq T}$ with intensity $\Lambda_T(\delta, S)$ depending on the fair price $(S_t)_{t \geq 0}$ and the distance of order submission δ . The execution cost results from the sum of the price of the executed quantity and a penalty function depending on the remaining quantity to be executed at the end of the period $[0, T]$. This penalty $\kappa \Phi(Q)$ models the excess cost induced by crossing the spread and the resulting market impact of this execution. An optimal distance δ^* is defined as a minimizer of the execution cost. This leads to an optimization problem under constraints which we solve by using a recursive stochastic procedure with projection (see [14] and [15]). We prove the *a.s.* convergence of the constrained algorithm under additional assumptions on the execution cost function (and light assumptions on the fair price dynamics $(S_t)_{t \in [0, T]}$). From a practical viewpoint, we give criteria on the model parameters that ensure the viability of the algorithm and the existence of a cost minimizer) relying on a functional co-monotony principle presumably satisfied by the fair price process.

2.2. Modeling and design of the algorithm

We focus our work on the problem of optimal trading with limit buy orders on one security. We only model the execution flow which reaches the price where the limit order is posted on a short period T with a Poisson process, namely

$$N_T^{(\delta)} \quad \text{with intensity} \quad \Lambda_T(\delta, S) := \int_0^T \lambda(S_t - (S_0 - \delta)) dt, \quad (6)$$

where $0 \leq \delta \leq \delta_{\max}$ ($\delta_{\max} \in (0, S_0)$ is the depth of the order book), $\lambda : [-S_0, +\infty) \rightarrow \mathbb{R}_+$ is a finite non-increasing convex function and $(S_t)_{t \geq 0}$ is a càdlàg stochastic process modeling the dynamics of the the best opposite price of a security stock.

Over the period $[0, T]$, we aim to execute a portfolio of size $Q_T \in \mathbb{N}$ invested in the asset S . The execution cost for a distance δ is $\mathbb{E}[(S_0 - \delta)(Q_T \wedge N_T^{(\delta)})]$. We add to this execution cost a *market impact penalty function* $\Phi : \mathbb{R} \mapsto \mathbb{R}_+$, non-decreasing and convex, with $\Phi(0) = 0$, to model the *impact effect* (additional cost) induced by the execution of the remaining quantity $(Q_T - N_T^{(\delta)})_+$. The resulting cost function reads

$$C(\delta) := \mathbb{E} \left[(S_0 - \delta) \left(Q_T \wedge N_T^{(\delta)} \right) + \kappa S_T \Phi \left((Q_T - N_T^{(\delta)})_+ \right) \right], \quad \delta \in [0, \delta_{\max}] \quad (7)$$

(where $\kappa > 0$ is a free tuning parameter). Prior to solving $\min_{0 \leq \delta \leq \delta_{\max}} C(\delta)$ by implementing a stochastic gradient descent numerically, we show that under natural assumptions on Q_T and κ , the function C is twice differentiable and *strictly convex on* $[0, \delta_{\max}]$ with $C'(0) < 0$ and that the derivatives $C^{(k)}$, $k = 0, 1, 2$, have a representation as an expectation. Consequently,

$$\operatorname{argmin}_{\delta \in [0, \delta_{\max}]} C(\delta) = \{\delta^*\}, \quad \delta^* \in (0, \delta_{\max}] \quad \text{and} \quad \delta^* = \delta_{\max} \quad \text{iff} \quad C \text{ is non-increasing on } [0, \delta_{\max}].$$

The representation of C' as an expectation means that there exists a Borel functional $H : [0, \delta_{\max}] \times \mathbb{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$ such that $C'(\delta) = \mathbb{E} \left[H(\delta, (S_t)_{t \in [0, T]}) \right]$, $\delta \in [0, \delta_{\max}]$ where $\mathbb{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$ denotes the set of càdlàg functions from $[0, T]$ to \mathbb{R} (see [18] for an explicit form for H). In practice to implement numerically the recursive procedure, we have to replace the “copies” $S^{(n)}$ by copies $\bar{S}^{(n)}$ of a time discretization $\bar{S} = (\bar{S}_{t_i})_{0 \leq i \leq m}$, typically an Euler scheme of a Brownian diffusion process. Then, with an obvious abuse of notation for the function H , we can write the *implementable procedure* as follows:

$$\delta_{n+1} = \text{Proj}_{[0, \delta_{\max}]} \left(\delta_n - \gamma_{n+1} H \left(\delta_n, (\bar{S}_{t_i}^{(n+1)})_{0 \leq i \leq m} \right) \right), \quad n \geq 0, \quad \delta_0 \in [0, \delta_{\max}], \quad (8)$$

where $(\bar{S}_{t_i}^{(n)})_{0 \leq i \leq m}$ are copies of $(\bar{S}_{t_i})_{0 \leq i \leq m}$ either independent or sharing “ergodic” properties, namely some averaging properties in the sense of [19]. In the first case, one will think about simulated data after a calibration process and in the second case to a direct implementation using a historical high frequency database of best opposite prices of the asset S .

Several *a.s.* convergence results are established in [18] under various types of assumptions on the sequences $(\bar{S}_{t_i}^{(n)})_{0 \leq i \leq m}$: a first setting is to assume that the $(\bar{S}^{(n)})$ are *i.i.d.* which correspond in the real world to the Monte Carlo simulation of a formerly calibrated model for the dynamics of S ; in a second setting we assume that we work with a historical high frequency dataset of prices of the asset assumed to share rather light averaging properties. In both cases we show that

$$\delta_n \xrightarrow{a.s.} \delta^*.$$

Several numerical experiments on a true dataset are reproduced and analyzed in [18].

2.3. Criteria for the convexity and monotony at the origin

In this section, we look for simple criteria involving the parameter κ , that imply the requested assumption on the execution cost function C (or \bar{C}) without ever needing to really specify the process S . The original form of the criteria implying that $C'(0) < 0$ and $C'' \geq 0$ cannot really be used in practice since they involve ratios of expectations of functionals combining both the dynamics of the asset S and the execution parameters in a highly nonlinear way. The key establishing the criteria is the *functional co-monotony principle* established in [18, 29] for a wide class of diffusions and their associated time discretization schemes.

Definition 2.1 (Co-monotony). A Borel functional $F : \mathbb{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$ is non-decreasing (resp. non-increasing) if

$$\forall \alpha, \beta \in \mathbb{D}([0, T], \mathbb{R}), \quad (\forall t \in [0, T], \alpha(t) \leq \beta(t)) \implies F(\alpha) \leq F(\beta) \quad (\text{resp. } F(\alpha) \geq F(\beta)).$$

Two functionals F and G on $\mathbb{D}([0, T], \mathbb{R})$ are co-monotonic if they have the same monotony. A functional F has polynomial growth if

$$\exists r > 0 \quad \text{s.t.} \quad \forall \alpha \in \mathbb{D}([0, T], \mathbb{R}), \quad |F(\alpha)| \leq K (1 + \|\alpha\|_\infty^r).$$

Definition 2.2 (Functional co-monotony principle). A càdlàg (resp. continuous) process $(S_t)_{t \in [0, T]}$ satisfies a functional co-monotony principle if for every pair $F, G : \mathbb{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$ of co-monotonic \mathbb{P}_S -*a.s.* continuous Borel functionals (for the sup-norm over $[0, T]$) with polynomial growth such that $F(S), G(S)$ and $F(S)G(S) \in L^1$, one has

$$\mathbb{E} \left[F((S_t)_{t \in [0, T]}) G((S_t)_{t \in [0, T]}) \right] \geq \mathbb{E} \left[F((S_t)_{t \in [0, T]}) \right] \mathbb{E} \left[G((S_t)_{t \in [0, T]}) \right]. \quad (9)$$

Among processes satisfying such a functional monotony principle, the (fractional) Brownian motion, Brownian one-dimensional diffusions, Liouville processes Lévy processes, etc (see [29]).

Theorem 2.3. Assume that $(S_t)_{t \in [0, T]}$ satisfies a functional co-monotony principle. Assume that $\lambda(x) = Ae^{-kx}$, $x \in \mathbb{R}$, $A, k \in (0, +\infty)$. Then the following monotony and convexity criteria hold true.

(a) MONOTONY AT THE ORIGIN:

$$C'(0) < 0 \quad \text{as soon as} \quad Q_T \geq 2T\lambda(-S_0) \quad \text{and} \quad \kappa \leq \frac{1 + kS_0}{k\mathbb{E}[S_T] (\Phi(Q_T) - \Phi(Q_T - 1))}.$$

(b) CONVEXITY. Let $\rho_Q \in \left(0, 1 - \frac{\mathbb{P}(N^\mu = Q_T - 1)}{\mathbb{P}(N^\mu \leq Q_T - 1)} \Big|_{\mu = T\lambda(-S_0)}\right)$. If $\Phi \neq \text{id}$, assume that Φ satisfies

$$\forall x \in [1, Q_T - 1], \quad \Phi(x) - \Phi(x - 1) \leq \rho_Q (\Phi(x + 1) - \Phi(x)).$$

If $Q_T \geq 2T\lambda(-S_0)$ and $\kappa \leq \frac{2}{k\mathbb{E}[S_T] \Phi'_\ell(Q_T)}$, then $C''(\delta) \geq 0$, $\delta \in [0, \delta_{\max}]$, so that C is convex on $[0, \delta_{\max}]$ or for the Euler scheme of $(S_t)_{t \in [0, T]}$ with step $\frac{T}{m}$, for $m \geq m_{b, \sigma}$ of a diffusion with constant diffusion coefficient.

(c) The same inequalities hold for tany \mathbb{R}^{m+1} -time discretization sequence $(S_t)_{0 \leq i \leq m}$ which satisfies a co-monotony principle.

3. SEMI MARKOV MODEL FOR MARKET MICROSTRUCTURE AND HF TRADING

3.1. Markov renewal model for stock price

The dynamics of the mid-price are given by the càd-làg piecewise constant process:

$$P_t = P_0 + 2\delta \sum_{n=1}^{N_t} J_n,$$

where P_0 is the opening mid-price, $2\delta > 0$ is the tick size, N_t is the point process associated to the tick times $(T_n)_n$ (i.e. the price jump times), and $(J_n)_n$ is the marks sequence valued in $\{+1, -1\}$ indicating whether the price jumped upwards ($J_n = +1$) or downwards ($J_n = -1$) at time T_n , called price direction. We use a Markov renewal approach to model the marked point process (T_n, J_n) . (J_n) is an irreducible Markov chain with symmetric transition matrix of diagonal terms $\frac{1+\alpha}{2}$, and antidiagonal terms $\frac{1-\alpha}{2}$, where $\alpha \in [-1, 1)$ represents the correlation between two consecutive price returns. Estimation on real data leads to a negative value of α , meaning that the stock price exhibits a short-term mean-reversion, which is a well-known stylized fact about high-frequency data, also called microstructure noise. Denoting by $S_n = T_n - T_{n-1}$ the sequence of inter-arrival times associated to (N_t) , we assume that conditionally on the sign $J_n J_{n+1} = \pm 1$ of two consecutive price directions, $(S_n)_n$ is an independent sequence with distribution: $F_\pm(s) = \mathbb{P}[S_{n+1} \leq s | J_n J_{n+1} = \pm 1]$, and density $f_\pm(s)$. The distinction between F_+ and F_- models the clustering of trading activity, that is the irregular spacing of tick times according to the mean-reverting or trend period of price jumps. We denote by $F = \frac{1+\alpha}{2} F_+ + \frac{1-\alpha}{2} F_-$ the unconditional distribution of (S_n) . The intensity function of price jump in the same (resp. opposite) direction is then given by: $h_+ = \frac{1+\alpha}{2} \frac{f_+}{1-F}$ (resp. $h_- = \frac{1-\alpha}{2} \frac{f_-}{1-F}$).

Markov embedding. Let us define the last price jump direction: $I_t = J_{N_t}$, and the elapsed time since the last jump: $S_t = t - \sup\{T_n : T_n \leq t\}$. Then, the price process (P_t) is embedded into a Markov system (P_t, I_t, S_t) with three observable state variables.

Diffusive behavior at macroscopic scale. The scaled price process $P_t^{(T)} = \frac{P_{tT}}{\sqrt{T}}$, $t \in [0, 1]$, exhibits a diffusive behavior given by the following functional central limit theorem:

$$\lim_{T \rightarrow \infty} P^{(T)} \stackrel{(d)}{=} \sigma_\infty W,$$

where W is a Brownian motion, and $\sigma_\infty^2 = \frac{1}{\int_0^\infty t dF(t)} \left(\frac{1+\alpha}{1-\alpha} \right)$ is the macroscopic variance. Estimation, simulation and more properties of this Markov renewal model are studied in [28].

3.2. Market order flow modeling

We model the market order flow of the Limit Order Book (LOB) by a marked point process $(\theta_k, Z_k)_k$, where $(\theta_k)_k$ represents the arrival times of the market order, and $(Z_k)_k$, valued in $\{\pm 1\}$ is the side of the exchange in the LOB: $Z_k = +1$ (resp. -1) means that the trade is exchanged at the best bid (resp. ask) price, thus associated to a market sell (resp. buy) order. The existing dependence between the market order flow and the stock price is modeled as follows. The counting process $(M_t)_t$ associated to $(\theta_k)_k$ is a Cox process with conditional intensity $\lambda(S_t)$, where $\lambda(s)$ is a decreasing function, representing the fact that trade arrivals should occur less often in a period of price stability, i.e. when S_t is large. The dependency between market order trade and stock price is modeled through the relation: $Z_k = \Gamma_k I_{\theta_k^-}$, where $(\Gamma_k)_k$ is an i.i.d. sequence on $\{\pm 1\}$ with mean $\rho \in [-1, 1]$. This means that $\rho = Corr(Z_k, I_{\theta_k^-})$, with the interpretation that when $\rho > 0$ (resp $\rho < 0$) most of the trades arrive on the strong (resp. weak) side $+I_{\theta_k^-}$ (resp. $-I_{\theta_k^-}$) of the LOB. Estimation on real data usually leads to negative value of ρ , which formalizes the observed fact that most of the trades arrive in the unfavorable side of the LOB for the market maker, inducing the so-called adverse selection risk.

3.3. The market making application

The agent strategy consists in placing continuously limit orders of small size L at the best available price, while respecting an inventory constraint: her control is described by a process $(\ell_t^\pm)_t$ valued in $\{0, 1\}$ with the convention that when $\ell_t^\pm = 1$, a limit order of size L is posted at time t on the side $\pm I_{t-}$. When a market order arrives, she is executed on the side $\pm I_{t-}$ of the LOB, of ℓ_t^\pm times a random variable distributed as $\vartheta_L^\pm(dk)$, with support on $\{0, \dots, L\}$. The objective of the market maker is to maximize over her limit orders strategies the expected gain:

$$\mathbb{E}[PNL_T - CLOSE(Y_T) - \eta \cdot RISK_{0,T}], \tag{10}$$

where $PNL_T = X_T + Y_T P_T$ is the portfolio value at terminal date T , evaluated at the mid price P , with X the cash holdings, and Y the inventory constrained to lie in a given bounded set \mathbb{Y} , $CLOSE(Y_T) = -(\delta + \varepsilon)|Y_T|$, is the liquidated value of the inventory at terminal date with $\varepsilon \geq 0$ a fixed fee paid at each transaction, and $RISK_{0,T} = \int_0^T Y_t^2 \cdot d[P]_t$ represents the market risk by holding the inventory during the whole trading period, weighted by the penalization parameter $\eta \geq 0$. The value function v to this Markovian stochastic control on finite horizon (10), involving the state variables $(t, P_t, I_t, S_t, X_t, Y_t)$, can be written in the reduced-form:

$$v(t, p, i, s, x, y) = x + yp + \omega(t, s, yi),$$

where $\omega(t, s, q)$ is the unique bounded viscosity solution to:

$$\begin{aligned} & [\partial_t + \partial_s] \omega + 2\delta(h_+ - h_-)q - 4\delta^2 \eta (h_+ + h_-)q^2 \\ & + \max_{q-L \in \mathbb{Y}} (\mathcal{N}_{trd}^+[\ell] + \mathcal{N}_{jmp}^+[\ell]) \omega + \max_{q+L \in \mathbb{Y}} (\mathcal{N}_{trd}^-[\ell] + \mathcal{N}_{jmp}^-[\ell]) \omega = 0, \\ & \omega(T, s, q) = -|q|(\delta + \varepsilon), \end{aligned}$$

and the operators \mathcal{N} 's are given by:

$$\begin{aligned} \mathcal{N}_{trd}^\pm[\ell] \omega & := \frac{1 \pm \rho}{2} \lambda(s) \int [\omega(t, s, q \mp k\ell) - \omega(t, s, q) + k\ell(+\delta - \varepsilon)] \vartheta_L^\pm(dk) \\ \mathcal{N}_{jmp}^\pm[\ell] \omega & := h_\pm(s) [\omega(t, 0, \pm q - L\ell) - \omega(t, s, q) + L\ell(-\delta - \varepsilon)]. \end{aligned}$$

The solution (value function and optimal control) of the market making problem is then completely characterized in terms of an integro-ordinary differential equation involving three observable state variables; current time t , elapsed time s from last jump, and strong inventory $q = yi$. We provide a convergent numerical scheme for

solving this IODE, and illustrate our results with some computational tests displaying the shape of the optimal policies, see [27].

4. DYNAMICAL MODELS FOR ORDERBOOK DYNAMICS: WHICH ONES AND WHAT FOR

This work is mainly based on the paper *Efficiency of the Price Formation Process in Presence of High Frequency Participants: a Mean Field Game analysis*, a joint work with PL Lions, JM Lasry and A Lachapelle [16].

4.1. The emergence of a need in orderbook dynamics modelling

To manage the increase of complexity following the multiplication of venues and dispersion of liquidity across all trading venues available, market participants more intensively used electronic trading and quantitative (i.e. optimal) slicing of large orders [1]. Besides, the need for cross-venue arbitrage of latency and liquidity increased, giving birth to a new kind of *middle-men*: the High Frequency traders [26]. A better understanding of the orderbook dynamics improves: the understanding by regulators and policy makers of the contribution of each market participant to the efficiency of the PFP (Price Formation Process); the existing market impact models ([10] [2] [21] [24]), a key component of the PFP dynamics; and the capability to use more sophisticated optimal execution schemes ([4] [30] [13] [12]).

Recently, some papers focused on this topic: a Dynamic Model of the Limit Order Book [31]; a Fokker-Planck description for the queue dynamics of large tick stocks [9]; Price Dynamics in a Markovian Limit Order Book Market [6]; Measuring the resiliency of an electronic limit order book – [17]; State-Dependent Model of Order Book Dynamics (forthcoming, by Huang, Rosenbaum, Lehalle).

4.2. The Mean Field Game approach as a synthesis

MFG (Mean Field Games) are designed to model this type of dynamics [20]. An MFG model of endogenous volatility has been proposed in “Large investor trading impacts on volatility” [25]. An MFG-based model has already been designed at a larger time scale in “High Frequency Simulations of an Order Book: a Two-Scales Approach” [22]. We wanted to provide a framework at the smallest one. In this paper we model participant behaviours in the orderbook; obtain “master equations” describing the emerging dynamics; use them to describe the outcome of the mixing of HFT and Institutional Investors in the same orderbook.

4.2.1. One queue modelling

Sellers only, they arrive in the “game” according to a Poisson process of intensity λ , they compare their value function $u(x)$ (where x is the size of the queue) to zero to choose to wait in the queue (when $u(x) > 0$) or not, the queue is consumed by a Poisson process of intensity $\mu(x)$, in case of transaction, a “pro-rata” scheme is used (“equivalent” to infinitesimal possibility to modify orders): q/x of the order is part of it; relaxed for FIFO in few slides.

Dynamics. The value function evolves following these four main possible events (we have price impact $P(x)$ and waiting cost proportional to cq):

$$\begin{aligned}
 u(x) = & (1 - \lambda \mathbf{1}_{u(x) > 0} dt - \mu(x) dt) \cdot u(x) && \leftarrow \text{nothing happens} \\
 & + \lambda \mathbf{1}_{\{u(x) > 0\}} dt \cdot u(x + q) && \leftarrow \text{new entrance} \\
 & + \mu(x) dt \cdot \left(\frac{q}{x} P(x) + (1 - \frac{q}{x}) u(x - q) \right) && \leftarrow \text{transaction} \\
 & - c q dt && \leftarrow \text{waiting cost}
 \end{aligned}$$

To solve it at the k th order, we will perform a Taylor expansion of $u(x + q)$ and $u(x - q)$ for small q at the k th order). At the second order, we obtain:

$$0 = \frac{\mu(x)}{x}(P(x) - u) - c + (\lambda \mathbf{1}_{\{u>0\}} - \mu(x))u' + q\left(\frac{1}{2}(\lambda \mathbf{1}_{\{u>0\}} - \mu(x))u'' + \frac{\mu(x)}{x}u'\right).$$

This corresponds to a (trivial) shared risk Mean Field Game monotone system with $N = 1$. The *mean field* aspect does not come from the continuum of agents (for every instant, the number of players is finite), but rather from the stochastic continuous structure of entries and exits of players.

Solution for a specific form of $\mu(x)$. At queue sizes x^* such that $x^* = \mu(x^*)P(x^*)/c$, u sign changes. Moreover, for the specific case $\mu(x) = \mu_1 \mathbf{1}_{x < S} + \mu_2 \mathbf{1}_{x \geq S}$: There is a point strictly before S where u switches from negative to positive. It means that participants anticipate service improvement.

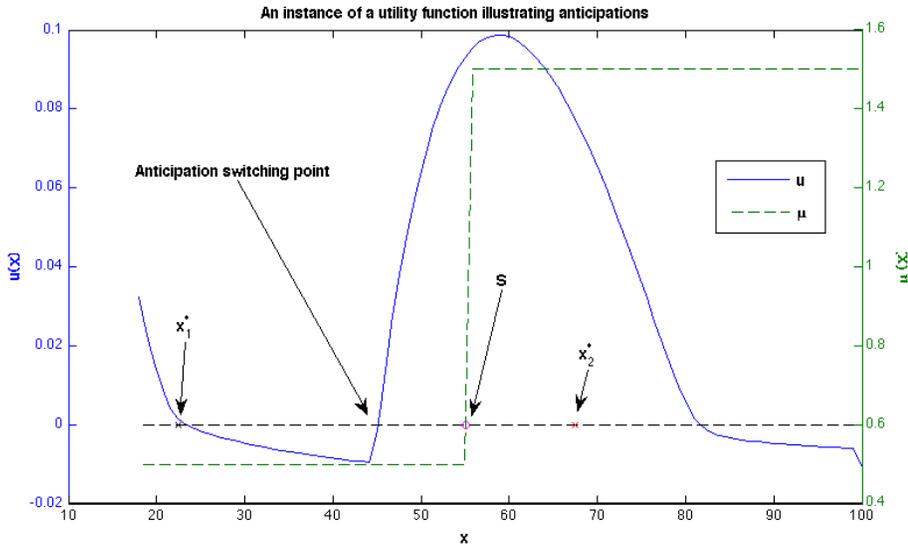


FIGURE 1. All the components of the one queue model.

4.2.2. Going further

In the paper [16], a model with two coupled queues is developed and analysis in a first step, and then extended to more than one class of trading agents. We then use this model to understand the mechanism of liquidity dynamics in a limit orderbook when traditional investors are mixed with high frequency traders. Such results can help regulators and policy makers to understand the stakes and the potential consequences of their decisions to adjust the market microstructure (see [23] for more details about practical aspects).

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