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THE PILGERSCHRITT (LIEDL) TRANSFORM

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Abstract. The Pilgerschritt transform was introduced by Roman Liedl, Innsbruck, in the late seventies of the last century. He came up with the idea for this transform after studies of the Volterra product integral and dealing with questions in iteration theory. This paper will show up the main ideas, the achieved results, a lot of open problems and some closely related ideas which followed later on.

Keywords: product integral, Pilgerschritt transform, one parameter groups, homomorphism, Lie groups, geodesic lines.

La "Pilgerschritt Transformation" ("transformation pas du pèlerin") a été introduite par Roman Liedl, Innsbruck, à la fin des années 70 du siècle passé. Il avait l' idée après avoir etudié l' intégrale multiplicative de Volterra et avoir traité des questions en théorie de l' itération. Cet article montre les idées principales, les résultats, une multitude de questions ouvertes et quelques idées étroitement liées apparues ultérieurement.

Mots clefs: intégrale multiplicative, transformation Pilgerschritt, sous-groupe à un paramètre, homomorphisme, groupe de Lie, géodsique.

AMS (2000) classification: 22E30, 37C10, 53B20

Introduction

This paper gives an overview about the Pilgerschritt (Liedl) transform. Main topics are:

- The main question dealt with
- The ideas at the beginning
- First results
- further ideas for using this method
- generalisations and new attempts
- recent research and results

Names to be mentioned in connection with this topic:

R. Liedl, G. Targonski, H. Reitberger, N. Netzer, H. Herdlinger, W.F.-R.;

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P. Stadler, Ch. Winklmair, W. Unterwelz, M. diLuca.

1. Preparatory Studies

In Innsbruck a group around Roman Liedl studied the so called "product integral" going back to Volterra [37]. By short words, this integral can be described as follows.
Let $P : [a, b] \to M_n(\mathbb{R})$ be a piecewise constant function (step function) into the set of $n \times n$ - matrices with according partition $a = t_0 < t_1 < \ldots < t_k = b$. Then one can define the product integral by

$$\prod_{l=1}^{k} \mathrm{e}^{P(\xi_l)(t_l - t_{l-1})} = \mathrm{e}^{P(\xi_k)(t_k - t_{k-1})} \cdot \ldots \cdot \mathrm{e}^{P(\xi_1)(t_1 - t_0)},$$

where ξ_l ist arbitrarily chosen, $t_{l-1} < \xi_l < t_l$. Take care that the product is non-commutative in general!
For continuous functions $P : [a, b] \to M_n(\mathbb{R})$ the product integral can be defined as the limit when P is uniformly approximated by step functions.
On the other hand, this product integral is nothing else but the fundamental solution (by some authors known as resolvent, matrizant, ...) of the linear differential equation $y' = P \cdot y$ in $\mathbb{R}^n$, and the Euler method to find approximate solutions numerically leads to the product

$$y(b) \approx (E + (t_k - t_{k-1})P(t_{k-1})) \cdot \ldots \cdot (E + (t_1 - t_0)P(t_0)) \cdot y(a),$$

where E denotes the unit matrix—in other words, we replace each factor $\mathrm{e}^{P(\xi_l)(t_l - t_{l-1})}$ in the formula of the product integral by its Taylor expansion of order 1.
For a detailed description see the original paper of Volterra [37], the paper of Birkhoff [1], the monograph [3], a summary article like [20] or [21] or some thesis' in Innsbruck [34], [32]. As a consequence in the other direction, H. Herdlinger wrote a dissertation on the opposite way: non-commutative differentiation. See [13].

2. The Main Idea

Starting point was the question of finding iterative roots respectively iteration groups. Therefore, let X be an arbitrary set and $f : X \to X$ a bijective mapping. The problem is to find an iteration group $(f_t : X \to X)_{t \in \mathbb{R}}$ such that $f_0 = \mathrm{id}_X$ and $f_{t+s} = f_t \circ f_s$ for $t, s \in \mathbb{R}$. As it is (and was) well known, in general this problem has no solution. Thus some of the assumptions have to be specialized.
We may give another description of the problem mentioned above: Let X be a set and denote by $B = B(X, X)$ the group of bijections from X into X (with composition of functions as the group operation). Finding an iteration group for a bijection f is nothing else than finding a group homomorphism $h : \mathbb{R} \to B(X, X)$ with the property $h(1) = f$. Thus we may give an abstract version of the problem: Let G be a group (for example $B(X, X)$) and $g \in G$ an element. Find a group homomorphism $h : \mathbb{R} \to G$ such that $h(1) = g$.
Using ideas from (elementary) analysis we have to deal with limits and the notion of convergence. Therefore some (nondiscrete) topology should be imposed on the group G. The best known groups with a topology are the topological vector spaces, especially $\mathbb{R}^n$, and the matrix groups respectively Lie groups. In these groups the restriction of a homomorphism $h : \mathbb{R} \to G$ to the interval $[0, 1]$ denotes a 'shortest line', a geodesic line (in the language of differential geometry) between the unit element $e = h(0)$ and the endpoint $g = h(1)$.
Usual methods in differential geometry for finding a geodesic line connecting two points on a manifild are shooting methods: You choose a tangent vector at the starting point, evaluate (differential equation of order 1) the endpoint of the geodesic line according to these initial conditions, and modify the tangent vector according to the errors. Of course, this method can only approximate the 'wanted' geodesic.
In Lie groups for endpoints not 'far away' from the unit, there is another tool available: The logarithm as the inverse of the exponential. On the other hand, the logarithm is given by a (convergent) power series, and using

this series up to a given power also just gives an approximation of the necessary tangent at the starting point, the unit e of the group.
At this point R. Liedl had a new idea: Choose an arbitrary path $\varphi : [0,1] \to G$ connecting the unit e with the given element g, and transform the path in a deterministic way to a path $\widetilde{\varphi}$, such that $\widetilde{\varphi}$ is the restriction of the achieved homomorphism $h : \mathbb{R} \to G$, or at least the sequence φ, $\widetilde{\varphi}$, $\widetilde{\widetilde{\varphi}}$, ... converges to this restriction.

2.1. The Pilgerschritt Transform

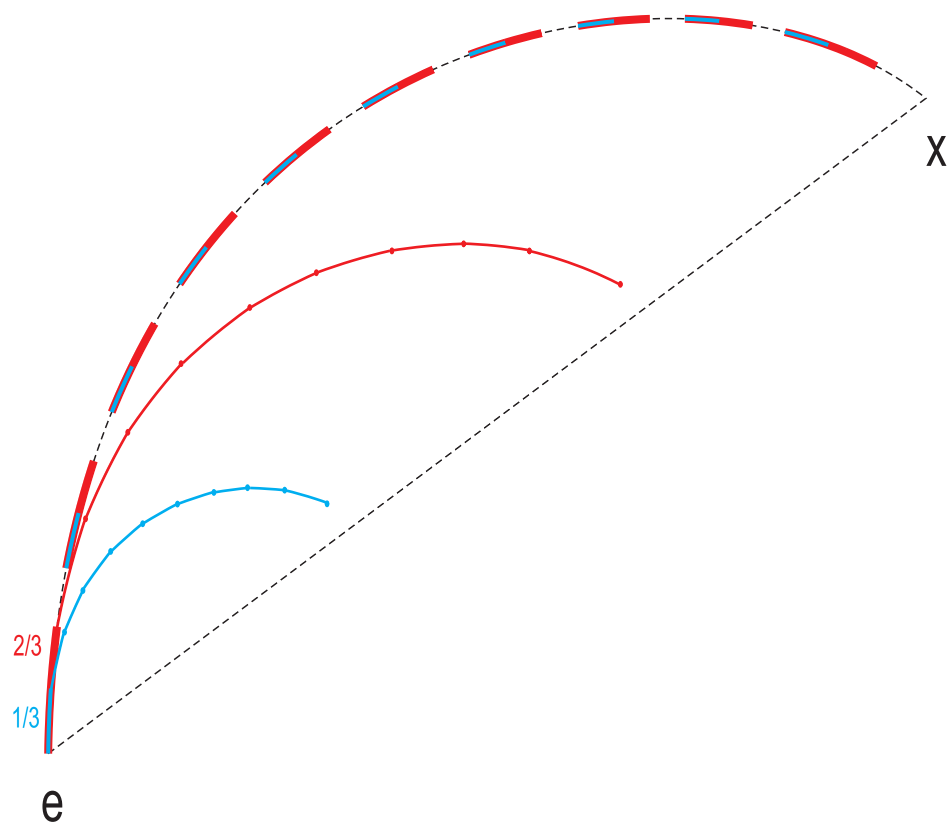
Let G be a topological group, $g \in G$ and $\varphi : [0,1] \to G$ a (continuous) path connecting the unit element e with the given element g, i.e., $\varphi(0) = e$ and $\varphi(1) = g$. Then we define a new path $\widetilde{\varphi}$ by the following process (compare it with a similarity deformation!).
Let $\mathcal{Z} = (0 = t_0 < t_1 < \ldots < t_m = 1)$ be a partition of the interval $[0,1]$, and let $\tau \in [0,1]$ be a real number. The Pilgerschritt product with respect to $\mathcal{Z}$ and τ is given as the product

$$\pi(\varphi, \mathcal{Z}, \tau) = \left(\varphi(t_{m-1} + \tau(t_m - t_{m-1})) \cdot \varphi(t_{m-1})^{-1}\right) \cdot \ldots \cdot \left(\varphi(t_1 + \tau(t_1 - t_0)) \cdot \varphi(t_0)^{-1}\right).$$

If the limit of this expression $\pi(\varphi, \mathcal{Z}, \tau)$, when the mesh size of $\mathcal{Z}$ tends to 0, exists, this limit will be called the Pilgerschritt transform of φ, i.e.

$$\widetilde{\varphi}(\tau) = \lim_{|\mathcal{Z}| \to 0} \pi(\varphi, \mathcal{Z}, \tau).$$

The following figure illustrates this definition. We take a partition $\mathcal{Z}$ as above, the restrictions of the path φ to the intervals $[t_{j-1}, t_{j-1} + \tau(t_j - t_{j-1})]$ are marked in bold (for one value τ, e.g. approximately $\tau = \frac{2}{3}$). Multiplication of the factors $\varphi(t_{j-1} + \tau(t_j - t_{j-1})) \cdot \varphi(t_{j-1})^{-1}$ is just putting togehter these marked pieces. According to the non-commutativity of multiplication in the group this process need not lead to an endpoint on the straight line connecting $\varphi(0)$ and $\varphi(1)$. Another line indicates the construction for $\tau = \frac{1}{3}$.



For a detailed description see for example the original papers [17], [18], [19].
Of course, the question arises when this limit does exist. Suppose that the group G has a differentiable structure (Lie group or Banach Lie group), and the path φ is continuously differentiable. A Taylor expansion of the product terms in π gives

$$\begin{aligned}&\big(\varphi(t_{k-1}+\tau(t_k-t_{k-1}))\cdot\varphi(t_{k-1})^{-1}\big)=\\&(\varphi(t_{k-1})+\tau(t_k-t_{k-1})\;\varphi'(t_{k-1})+R)\cdot\varphi(t_{k-1})^{-1}=\\&e+\tau(t_k-t_{k-1})\;\varphi'(t_{k-1})\;\varphi(t_{k-1})^{-1}+R_1\end{aligned}$$

where R respectively R_1 denote remainder terms. An easy calculation shows that for taking the limit $|\mathcal{Z}|\to 0$ these remainder terms are negligable, and we end up with the product integral in the sense of Volterra and with coefficient function $t\mapsto\tau\varphi'(t)\;\varphi(t)^{-1}$.
Another interpretation can be made by looking at the differential equation

$$y'=\tau\varphi'(t)\;\varphi(t)^{-1}\cdot y.$$

A usual Euler method gives rise to the product above (without remainder terms). Thus in Lie groups and Banach Lie groups the Pilgerschritt transform of a $\mathcal{C}^1$-path φ may be defined equivalently by the following process:

(1) Solve the differential equation $y' = \tau\varphi'(t)\ \varphi(t)^{-1} \cdot y$ for $\tau \in [0,1]$ and the initial condition $y(0) = e$.
(2) Denote this solution by $\widehat{\varphi}(t,\tau)$.
(3) Put $\widetilde{\varphi}(\tau) = \widehat{\varphi}(1,\tau)$.

In order to give an answer to the question: Whenever φ is a $\mathcal{C}^1$-path in a Lie group or a Banach Lie group, the Pilgerschritt transform exists.

First results on properties of the transformed path $\widetilde{\varphi}$ can be given as follows.
(1) $\widetilde{\varphi}(0) = \varphi(0)$, $\widetilde{\varphi}(1) = \varphi(1)$.
(2) If φ is the restriction of a homomorphism $h : \mathbb{R} \to G$ to the interval $[0,1]$, then $\widetilde{\varphi} = \varphi$.
(3) If φ is the restriction of a homomorphism $h : \mathbb{R} \to G$ to the interval $[0,1]$ up to a transform of the parameter, then $\widetilde{\varphi} = h|[0,1]$.
(4) $\widetilde{\varphi}$ is homotopic to φ in the group G.
(5) $\widetilde{\varphi}$ is a $\mathcal{C}^\infty$-function.

Remark 1: As it is well known, in Lie groups one can use the logarithm (given for example by its power series) to compute a homomorphism $h : \mathbb{R} \to G$ passing through a given element g. However, this power series has a finite radius of convergence, and the idea of the method of Pilgerschritt transform is to have a tool <u>without</u> using the logarithm.

Remark 2: The product description of the Pilgerschritt transform

$$\left(\varphi(t_{m-1} + \tau(t_m - t_{m-1})) \cdot \varphi(t_{m-1})^{-1}\right) \cdot \ldots \cdot \left(\varphi(t_1 + \tau(t_1 - t_0)) \cdot \varphi(t_0)^{-1}\right)$$

gave rise to the name “Pilgerschritt transform”: In medieval ages christian pilgrims had a method to enlarge the number of steps: After 2 or 3 steps forward they went 1 step back—like in this product.

2.2. The Pilgerschritt sequence

Now suppose that the group G is a Lie group or a Banach Lie group and the path φ is $\mathcal{C}^1$. Then the Pilgerschritt transform $\widetilde{\varphi}$ exists, and also the subsequent transforms $\widetilde{\widetilde{\varphi}}$, $\widetilde{\widetilde{\widetilde{\varphi}}}$, …
What can be said about this sequence?
(1) If the group G is commutative, the $\widetilde{\varphi}$ is the restriction of a homomorphism $h : \mathbb{R} \to G$.
(2) If the group G is nilpotent, the sequence φ, $\widetilde{\varphi}$, $\widetilde{\widetilde{\varphi}}$, $\widetilde{\widetilde{\widetilde{\varphi}}}$, … gives the restriction of a homomorphism $h : \mathbb{R} \to G$ after <u>finitely many</u> steps.
(3) If the group G is solvable, the sequence φ, $\widetilde{\varphi}$, $\widetilde{\widetilde{\varphi}}$, $\widetilde{\widetilde{\widetilde{\varphi}}}$, … converges to the restriction of a homomorphism $h : \mathbb{R} \to G$ in the case that the endpoint $\varphi(1)$ is close to the unit e.
(4) If the group G is ‘arbitrary’, the sequence φ, $\widetilde{\varphi}$, $\widetilde{\widetilde{\varphi}}$, $\widetilde{\widetilde{\widetilde{\varphi}}}$, … converges to the restriction of a homomorphism $h : \mathbb{R} \to G$ in the case that $\varphi'(t)$ is ‘small enough’ uniformly on the interval $[0,1]$.

Details can be found in the papers [4], [5], [6], [7], [8], [16], [24], [27], [28], [30], [33].

2.3. The Pilgerschritt transform in other groups

There were also attempts to study the Pilgerschritt transform in groups of diffeomorphisms and in the group of invertible formal power series’.
Within the group of diffeomorphisms $\mathcal{D}(O,O)$ (O is an open subset in the $\mathbb{R}^n$) the Pilgerschritt transform can be described by:
(1) Let $\mathcal{D}(O,O)$ be endowed with the compact-open topology.
(2) Let $\varphi : [0,1] \to \mathcal{D}(O,O)$ be a $\mathcal{C}^1$-path.
(3) Denote $F_1(x,t) = \varphi(t)(x)$ and $F_{-1}(x,t) = (\varphi(t))^{-1}(x)$ for $x \in O$ and $t \in [0,1]$.

(4) Solve the differential equation $D_tF(x,t,\tau) = \tau(D_tF_1)(F_{-1}(F(x,t,\tau),t),t)$ for $\tau \in [0,1]$ and the initial condition $F(x,0,\tau) = x$.
(5) Denote this solution by $\widehat{\varphi}(t,\tau)(x) = F(x,t,\tau)$.
(6) Put $\widetilde{\varphi}(\tau) = \widehat{\varphi}(1,\tau)$.

Results can be found in the papers [13], [26], [5].

2.4. Numerical treatment

Besides the papers mentioned above a numerical attempt to the Pilgerschritt transform (in order to approximate $\widetilde{\varphi}$ for a given φ) were done by a lot of students within their diploma thesis or dissertation. A short list of such attempts is given by [2], [10], [11], [14], [25], [36].

3. The Pilgerschritt transform - problems of convergence

As stated above, the Pilgerschritt sequence is an elegant method to find homomorphisms in Lie groups within a finite number of steps if the group G is commutative or nilpotent. On the other hand, for solvable and 'arbitrary' Lie groups the method converges "if the original path ist close enough to the unit e". But 'how far away' from the unit is it allowed to go? And are there methods to accelerate the convergence?

3.1. Fast Pilgerschritt transform

It was an algebraic manipulation of the coefficients in the differential equation to simulate several Pilgerschritt transformations within one step (= one differential equation approximates n steps of Pilgerschritt transform). A detailed description of this method can be found in [15], [23], [29], [31].

3.2. The region of convergence

First attempts to this question have been done within a solvable group which is very easy to handle, the group of affine transformations of the real line or of the complex plane (or, equivalently, the group of upper triangular 2×2 - matrices). For this group there exist analytic results on a large region of convergence (which allows the endpoint of the path even to be 'far away' from the unit), and the results also give a (numeric) border between the region of convergence and the region of divergence in an neighbourhood of the unit. However, a detailed description of these regions is still far away from being known.
See [8], [9], [38].

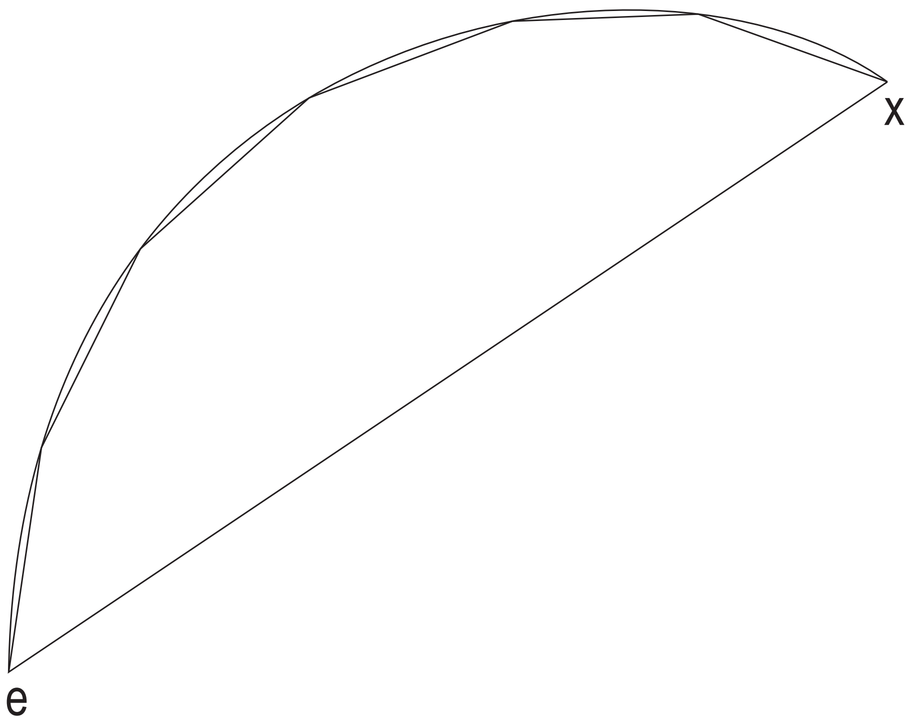
4. New ideas

When the problem of convergence came up, R. Liedl had new ideas coming up from differential geometry: Restrictions of homomorphisms from $\mathbb{R}$ to the Lie group G are geodesic lines according to the right or left invariant connection. But how to adopt this idea?

4.1. The short ruler

Suppose that you have a short ruler and you want to connect on the paper two points 'far away' from each other. Then you just connect these points by an arbitrary path and try to shorten this path by use of a short ruler. Does this method (used iteratively) converge to a geodesic line?
The problem was posed in ECIT87 by R. Liedl [22].
The following figure shows this process. Take the path φ connecting to points in the plane. Then you take a short ruler and draw a straight line from the starting point to another point on the curve described by φ. From this point you draw the next straight line to another point, and so on. By compactness of the image of φ a finite number of steps gives a polygonal line connecting starting point and endpoint. Repeating this process with this polygonal line, the question arises wheather this process (repeated infinitely many times) leads to the geodesic,

the straight line between starting point and endpoint.



A positve answer for normed vector spaces (as a first attempt) was given by P. Stadler [35].

4.2. **Parallel transport**

Another idea was to use the method of Pilgerschritt transform on manifolds with a linear connection. However, we have no parallel displacement independent from the path. Therefore, the idea (see [4]) was to transport all the tangent vectors along φ to the starting point $\varphi(0)$, then multiply them by a constant $\tau \in [0,1]$, and find a path such that these new vectors are the parallel displaced tangent vectors. Locally (using a parametrization of the manifold) this method can be done – the only need are explicite ordinary differential equations of first order. There exists also a convergence theorem on the Pilgerschritt sequence for the sphere S^2.
Further investigations were done via diploma thesis' by [12], [39].

5. Summary

Some results on the convergence of the sequence of Pilgerschritt transforms could be proved. However, there are several open problems, mainly: What can be said about the region of convergence?

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