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TRANSLATION EQUATION AND SINCOV'S EQUATION – A HISTORICAL REMARK

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Abstract. Gottlob Frege (1848 – 1925), the world famous logician was also a pioneer in iteration theory. His habilitation thesis *"Rechnungsmethoden, die sich auf eine Erweiterung des Grössenbegriffes gründen"* ("Methods of Calculation based on an Extension of the Concept of Quantity") published 1874 yields a foundation of iteration theory and dynamical systems in one and also in several variables. He considers there the translation equation

$$f(s, f(t, x)) = f(s + t, x)$$

and all the three so-called Aczél-Jabotinsky equations connected with the differentiable solutions of it. By this way Frege e.g. recognized also the importance of the *infinitesimal generator* of a dynamical system. A comprehensive presentation of this matter may be found in Gronau [4]. Frege treated in this connection also Sincov's equation

$$\Psi(z, x) = \Psi(z, y) + \Psi(y, x)$$

and gave its general solution almost 30 years before Sincov. The history and background of Sincov's equation is described in Gronau [5].

Here we give a detailed description of the connection between the translation equation and the Sincov equation.
Keywords. Iteration Theory, Gottlob Frege, history of mathematics.

Résumé. Gottlob Frege (1848 - 1925), le logicien mondialement connu était aussi un pionnier en la théorie d'itération. Sa thèse d'un doctorat d'État *"Rechnungsmethoden, die sich auf eine Erweiterung des Grössenbegriffes gründen"* ("Méthodes de Calcul basé sur une Extension du Concept de Quantité") publié en 1874 rapporte une fondation de la théorie d'itération et des systèmes dynamiques dans une et aussi dans plusieures variables. Il considère l'équation de translation $f(s, f(t, x)) = f(s + t, x)$ et aussi toutes les trois équations d'Aczél-Jabotinsky en connexion avec les solutions differentiables de l'équation de translation. Frege par exemple a reconnu aussi l'importance du générateur infinitésimal d'un système dynamique. Une présentation compréhensive de cette matière peut tre trouvée dans Gronau [4]. Frege a traité dans ce connexion l'équation de Sincov $\Psi(z, x) = \Psi(z, y) + \Psi(y, x)$ et a donné sa solution générale près de 30 ans avant Sincov. L'histoire et le contexte de l'équation de Sincov sont écrits dans Gronau [5].

Ici, on donne une description détaillée de la connexion entre l'équation de translation et l'équation de Sincov.
Mots-clefs. Théorie d'itération, Gottlob Frege, Histoire de mathématiques.

Introduction

The aim of Frege in his "Rechnungsmethoden" from 1874 (Frege [3]) was to generalize the iterates of a function, that means to enlarge the value of iterative powers from natural numbers to a larger domain. One can

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say that Frege with this question at this time was a pioneer of iteration theory. For a function f and naturals n the iterative power f^n of f is defined recursively by

$$f^1 = f \quad \text{and} \quad f^{n+1} = f^n \circ f.$$

Iterative powers satisfy the addition law

$$f^{n_0} \circ f^{n_1} = f^{n_0+n_1}, \quad n_0, n_1 \in \mathbb{N}. \tag{1}$$

From this addition law writing the iterative powers in the form $f^n(x) = f(n, x)$ Frege comes in a natural way to the functional equation (in Frege's words: "Functionalgleichung", Frege [3], p.4):

$$f(n_0, f(n_1, x)) = f(n_0 + n_1, x). \tag{2}$$

Frege requires that this functional equation should be satisfied by functions $f(n, x)$ for $n_0,\ n_1$ out of a larger domain than that of the naturals. This is the meaning of the sentence "Extension of the Concept of Quantity" in the title of his thesis [3].

1. The translation equation

1.1. The translation in one real variable and its connection to Sincov's equation

Equation (2) is the so-called *translation equation*

$$f(s, f(t, x)) = f(s + t, x). \tag{3}$$

In the sequel we proceed as Frege in [3] did. Since the aim of this note is just to give a sketch of problems connected with the translation equation we omit here and in the sequel specific definitions of the domain and co-domain of the involved functions. Let $f(t, x)$ be a solution of (3) and let t be expressible in the form

$$t = \Psi(f, x) \tag{4}$$

that is :

$$t = \Psi(y, x) \quad \text{iff} \quad f(t, x) = y.$$

Let $f(t, x) = y$ and $f(s, y) = z$, hence by (3) $f(s+t, x) = f(s, (f(t, x)) = f(s, y) = z$. So we get $\Psi(z, x) = s + t$, whence

$$\Psi(z, x) = \Psi(z, y) + \Psi(y, x) \tag{5}$$

which is valid for all $y, z \in \mathrm{Orb}(x) = \{f(t, x) \mid t \in \mathbb{R}\}$.

1.2. The Sincov equation

Equation (5) is the so-called **Sincov-equation**, named after *Dmitrii Matveevich Sincov (*Д. М. Синцовь*)* (1867 – 1946). More details about it may be found in Gronau [5]. The *general solution* of it is already given by Frege 1874 and later on by Sincov [7] in 1903:

$$\Psi(z, y) = \vartheta(z) - \vartheta(y), \tag{6}$$

where ϑ is an arbitrary function.

Sincov's proof is very simple. Write (5) in the form

$$\Psi(z, y) = \Psi(z, x) - \Psi(y, x) \tag{7}$$

Since in (7) the left hand side is independent from x we can define $\vartheta(y) = \Psi(y, x)$ for one fixed x. So any solution of (5) has to have the form (6). On the other hand for any arbitrary function ϑ the function Ψ given by (6) is a solution of the Sincov equation (5). The proof of Frege is slightly more complicated but comes to the same result.

1.3. **Application of Sincov's equation**

Suppose now that the conditions of subsection 1.1 hold. The Sincov equation is then fulfilled for a fixed x and for all $y, z \in \mathrm{Orb}(x)$. So we can use (6) for all y in $\mathrm{Orb}(x)$. With $f(s, y) = z$ whence $s = \Psi(z, y)$ we get the identity

$$s = \vartheta(f(s, y)) - \vartheta(y) \quad \text{for } y \in \mathrm{Orb}(x) \tag{8}$$

for an arbitrary function ϑ. If we assume ϑ to be invertible then

$$f(s, y) = \vartheta^{-1}(s + \vartheta(y)) \quad \text{for } y \in \mathrm{Orb}(x). \tag{9}$$

As it turns out that every function defined in such a way is a solution of the translation equation (3).

Frege used formula (8) to construct examples of solutions of the translation equation. So, for example, he used this method to derive the Binet formula for the Fibonacci numbers (without mentioning the name Fibonacci).

After them by differentiation Frege derived from the translation equation (3) differential equations, the so-called *Aczél-Jabotinsky differential equations* where he also considered functions in several real variables, see [2] and [4].

2. Conclusion

Frege's thesis was published in 1874 but, unfortunately, the results of this paper were ignored from the mathematical community regardless of the fact that they were full of new ideas. This has several reasons. The small twenty seven pages booklet was published in a publishing house more renown for publications in humanities. The style of the paper is rather academic and very complicate to understand. So it was almost unknown to the mathematical community. Frege himself got after his habilitation a professorship in Leipzig. He switched then to mathematical logic where he became famous as one of the founders of the modern logic.

At almost the same time in 1871 Ernst Schröder (1841 – 1902) published his paper [6] *Ueber iterirte Funktionen* (On iterated functions) in the mostly renowned journal Mathematische Annalen. The topic of this paper was also the generalization of iterated functions. He treated the translation equation together with the so-called *embedding problem*. Schröder's paper gave more tools to handle this problem e.g. with the *Schröder functional equation* and inspired the following generation essentially (see e.g. [1], [8] and [9]).

As a curiosity it is to mention that Schröder published only three purely mathematical papers and then became also a famous logician.

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