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INVARIANT CURVES OF QUADRATIC MAPS OF THE PLANE FROM THE ONE-PARAMETER FAMILY CONTAINING THE TRACE MAP *

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Abstract. The rigorous proofs are given: (1) for the existence of the unbounded invariant curves, containing the fixed point – source $(\mu+1;\,1)$, of the maps from the one-parameter family $F_\mu(x,\,y) = (xy,\,(x-\mu)^2)$, $\mu \in [0,\,2]$; (2) for the birth of the closed invariant curve from the elliptic fixed point $(\mu-1;\,1)$ for $\mu = 3/2$. Numerical results are presented for the main steps of the evolution of this invariant curve, when μ changes in the interval $(3/2,\,2)$.

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Keywords. Quadratic map, fixed point, invariant curve.

Résumé. Les demonstrations rigoureuses sont donnes: (1) pour l'existence des courbes invariantes non bornes contenant le point fixe - source $(\mu+1;\,1)$ des cartes de la famille a un parametre $F_\mu(x,\,y) = (xy,\,(x-\mu)^2)$, $\mu \in [0,\,2]$; (2) pour la naissance de la courbe fermee invariante du point fixe elliptique $(\mu-1;\,1)$ pour $\mu = 3/2$. Numerical resultats sont presentes pour les principales etapes de l'evolution de cette courbe invariante, quand μ des changements dans l'intervalle $(3/2,\,2)$.

Mots clefs. Carte quadratique, point fixe, courbe invariante.

1. Introduction

There exists an extensive bibliography devoted to the investigation of polynomial, and in particular, of quadratic maps (see, e.g., [23], [26], [1], [28], [29], [9] – [14], [20], [21], [25] and others).
In this paper we investigate quadratic maps from the one-parameter family

$$F_\mu(x,y) = (xy,\ (x-\mu)^2), \tag{1}$$

where $\mu \in [0,\,2]$, $(x,\,y)$ is a point of the plane $\mathbf{R^2}$.
Denote by $f_{\mu,\,n}$ and $g_{\mu,\,n}$ the first and the second coordinate functions of the n-th $(n \geq 1)$ iteration F^n_μ of the map F_μ respectively.
As it follows from (1), $F_\mu : \mathbf{R^2} \to \mathbf{R^2}$ is endomorphism. There are critical straight lines $C_0 : x = 0$ and $C_\mu : x = \mu$ here. In every point of these straight lines Jacobian of F_μ is vanishing[1]. Note that $C_0 = C_\mu$ iff $\mu = 0$.

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[1]By the critical straight line we mean the straight line, which consists of F_μ-critical points, i. e. points where rank of Jacobian matrix < 2 (in considered case) (for details see [32]).

Let us call C_0' the line $y = 0$. If $\mu \neq 0$ we have $F_\mu(C_\mu) = C_0'$ and $F_\mu(C_0') = C_0^+$, where C_0^+ is the part of the straight line C_0 with $y \geq 0$. Then for any $\mu \in [0, 2]$ and $n \geq 1$ the equality $F_\mu^n(C_0) = (0; \mu^2)$ holds; and the equality $F_\mu^n(C_\mu) = (0; \mu^2)$ is correct for any $\mu \in (0, 2]$ and $n \geq 3$, where $(0; \mu^2)$ is the fixed point[2]. The point $A_{1,\mu}(0; \mu^2)$ is a sink for $0 \leq \mu < 1$ and a saddle for $1 < \mu \leq 2$. Let us note also that the equality $DF^n(A_{1,\mu}) = 0$ holds for the differential DF^n of every map F^n, $n \geq 1$. It means that the fixed point $A_{1,\mu}(0; \mu^2)$ is the flat singularity of F. This property makes difficult investigations of dynamical features of F in a neighborhood of the fixed point A_1 [3].

The family (1) is introduced in [10] in the framework of the study of the trace map F_2. The investigation of trace maps is an actual mathematical problem of quasicrystals physics (see, e.g., [7], [8], [15], [16]). The asymptotic behavior of trajectories of trace maps on different invariant sets is closely related with the spectral properties of discrete Schrödinger operator arising in quasicrystals models (for details see, e.g., [16]).

This paper is the continuation of [11]. Its goal is to describe those invariant curves of maps (1), which play the most important role in dynamics of maps (1).

The paper is organized as follows. In Sec.2 we formulate and give the proof of Theorem 2.1 for the existence of two unbounded invariant curves, containing the fixed point – source $A_{2,\mu}(\mu + 1; 1)$ $(0 < \mu \leq 2)$.

In Sec.3 we formulate and give the sketch of the proof of Theorem 3.1 for the birth of the closed invariant curve (CIC) from the elliptic fixed point $A_{3,\mu}(\mu - 1; 1)$ for $\mu = 3/2$. Note also that the fixed point $A_{3,\mu}(\mu - 1; 1)$ is a saddle for $0 < \mu < 1$, a sink for $1 < \mu < \frac{3}{2}$, a source for $\frac{3}{2} < \mu \leq 2$.

In Sec.4 we study the main steps of the evolution of the CIC for parameter values from the interval (3/2, 2) by considering numerical simulations.

2. Unbounded Invariant Curves.

Theorem 2.1, formulated below, contains the main result for the existence of some unbounded invariant curves. One of the features of Theorem 2.1 is its nonlocal character, and the second is the omission of conditions of the most close to Theorem 2.1 classical Hadamard-Perron's theorem and the central manifold theorem. Note also that the source $A_{2,\mu}(\mu + 1; 1)$ is the limit point of the preimages of unbounded rank $\{F_\mu^{-i}(C_\mu \cap K_1)\}_{i \geq 1}$ of the part of critical line C_μ in the first quadrant K_1 and $\{F_\mu^{-i}(C_0 \cap K_1)\}_{i \geq 1}$ are preimages of unbounded rank of the part of critical line C_0 in K_1 ($\{F_\mu^{-i}(C_\mu \cap K_1)\}_{i \geq 1} \subseteq \{F_\mu^{-i}(C_0 \cap K_1)\}_{i \geq 1}$ since $F^2(C_\mu) = C_0$ for $\mu \in (0, 2]$, and K_1 is invariant).

Theorem 2.1. *Let F_μ be the quadratic map (1). Then*
(2.1.1) *for any $\mu \in [0, 2]$ there exists an invariant unbounded set*

$$D_{\mu,\infty} = \{(x; y) : x \geq \mu + 1,\ y \geq 1\}$$

containing the graph of a C^1-smooth strictly increasing function $y = \Gamma_\mu^{(1)}(x)$ defined on the interval $[\mu + 1, +\infty)$ and such that

$$\Gamma_\mu^{(1)}([\mu + 1, \infty)) = [1, +\infty);$$

moreover, the graph $\Gamma_\mu^{(1)}$ of this function is a F_μ-invariant curve;
(2.1.2) *for any $\mu \in [0, 1]$ there exists a C^1-smooth strictly decreasing function $y = \Gamma_\mu^{(2)}(x)$ defined on the interval $(\mu, +\infty)$ and satisfying*

$$\Gamma_\mu^{(2)}((\mu,\ +\infty)) = (0, +\infty);$$

moreover, the graph $\Gamma_\mu^{(2)}$ of this function is a F_μ-invariant curve belonging to the unbounded noninvariant set

$$\{A_{2,\mu}(\mu + 1; 1)\} \bigcup (K_1 \setminus D_{\mu,\infty})$$

[2] By a fixed point we mean a point $(x; y)$ satisfying the equality $F_\mu(x; y) = (x; y)$ (see, e.g., [31] for non-invertible maps).

[3] For example, in [17] one can find Definition of the flat singularity for one-dimensional maps and investigation of influence of this property on existence of wandering intervals.

(K_1 is the open first quadrant of the plane xOy).
Invariant curves – graphs of functions $y = \Gamma_\mu^{(1)}(x)$ and $y = \Gamma_\mu^{(2)}(x)$ intersect each other in the fixed point – source $A_{2,\mu}(\mu+1;1)$.

We give here the new version of the proof of Theorem 2.1 in comparison of the versions of papers [11], [13]. For this goal we divide the proof of Theorem 2.1 on steps indicated in Lemmas 2.2 – 2.3, Corollary 2.4 and Lemmas 2.5 – 2.7, Propositions 2.8, 2.9.

Lemma 2.2. *Let F_μ be the map (1), $\mu \in [0, 2]$. Then*
(2.2.1) *the set $D_{\mu,\infty}$ is F_μ-invariant (i.e. the inclusion $F_\mu(D_{\mu,\infty}) \subset D_{\mu,\infty}$ holds);*
(2.2.2) *for every point $(x;\, y) \in D_{\mu,\infty} \setminus \{A_{2,\mu}(\mu+1;1)\}$ the following equalities hold:*

$$\lim_{n\to+\infty} f_{\mu,n}(x,\, y) = +\infty \qquad \lim_{n\to+\infty} g_{\mu,n}(x,\, y) = +\infty.$$

Calculating eigenvectors of the differential $DF_\mu(A_{2,\mu})$ and using Lemma 2.2, we obtain that one of eigenvectors of $DF_\mu(A_{2,\mu})$ lies in the unbounded invariant domain $D_{\mu,\infty}$, and the other unit eigenvector of $DF_\mu(A_{2,\mu})$ (for $\mu \neq 2$) lies in the unbounded noninvariant domain $\{A_{2,\mu}(\mu+1;\, 1)\} \bigcup (K_1 \setminus D_{\mu,\infty})$.

Lemma 2.3. *Let F_μ be the map (1), $\mu \in [0, 2]$. Then the restriction $F_\mu|_{D_{\mu,\infty}}$ is a diffeomorphism of $D_{\mu,\infty}$ onto $F_\mu(D_{\mu,\infty})$, where*

$$F_\mu(D_{\mu,\infty}) = \{(x;\, y) \in D_{\mu,\infty} : x \geq \mu+1,\ 1 \leq y \leq (x-\mu)^2\}.$$

Lemma 2.3 shows that the map $F_\mu|_{D_{\mu,\infty}}$ is not surjective on $D_{\mu,\infty}$. Remarks after Lemmas 2.2 and 2.3 mean that standard technique of cones can not be applied for the proof of existence of the invariant curve $\Gamma_\mu^{(1)}$ (conditions of classical Hadamard-Perron's theorem are not fulfilled [24]).

We need the result, which follows immediately from Lemma 2.3.

Corollary 2.4. *Let F_μ be the map (1), $\mu \in [0, 2]$. Then for every $n \geq 1$ the n-th image of the ray*

$$\{(x;y) \in K_1 : y = 1, x \geq \mu+1\}$$

is the graph of a strictly increasing function $y = \gamma_{\mu,n}(x)$ of the class $C^1_{(\mu+1,+\infty)}$(strictly increasing function $x = \chi_{\mu,n}(y)$ of the class $C^1_{(1,+\infty)}$)[4].

We use the presentation of the n-th image of the ray $\{(x;y) \in K_1 : y = 1, x \geq \mu+1\}$ as the graph of the function $y = \gamma_{\mu,n}(x)$.

The following claim is one of the main technical results for the proof of existence of local branches of the invariant curves containing the fixed point $A_{2,\mu}(\mu+1;1)$.

Lemma 2.5. *Let F_μ be the map (1), $\mu \in [0, 2]$. Then there are a closed θ-neighborhood*

$$U_\theta(A_{2,\mu}) = [\mu+1-\theta, \mu+1+\theta] \times [1-\theta, 1+\theta]$$

of the fixed point $A_{2,\mu}(\mu+1;1)$ and a positive constant L such that for any pair of points $z_1(x_1;\, y_1)$, $z_2(x_2;\, y_2) \in U_\theta(A_{2,\mu})$ the following inequality holds:

$$||F_\mu(z_1) - F_\mu(z_2)|| \leq L||z_1 - z_2||;$$

(we use the norm in $\mathbf{R}^2$ correlated with the product topology in $\mathbf{R}^2$). If, in addition, $\{F_\mu^j(z_i)\}_{1\leq j\leq n} \subset U_\theta(A_{2,\mu})$, $i = 1, 2$, then

$$||DF_\mu^j(z_i)(h)||_{C^0} \leq L^j||h||,$$

where $||(\cdot)||_{C^0}$ is the maximal row C^0-norm of the differential of a map, $h = z_2 - z_1$ is a vector of displacement.

[4] A function $y = \gamma_{\mu,n}(x)(x = \chi_{\mu,n}(y))$ is the polynomial from some irrational function.

Let b be an arbitrary point from the interval $(\mu+1, \mu+1+\theta]$. Using estimates of Lemma 2.5 we obtain

Lemma 2.6. *Let F_μ be the map (1), $\mu \in [0, 2]$. Then the sequence of the restrictions $\{\gamma_{\mu,n}|_{[\mu+1,b]}(x)\}_{n\geq 1}$ of C^1-smooth functions $y=\gamma_{\mu,n}(x)$ on the interval $[\mu+1, b]$ converges in C^1-norm on the interval $[\mu+1, b]$ to some C^1-smooth strictly increasing function $y=\gamma_{\mu,[\mu+1,b]}(x)$.*

Since the graph of every function $y=\gamma_{\mu,n}(x)$, $n \geq 1$, contains the unique fixed point $A_2(\mu+1,\, 1)$, which is a source, and there is no a periodic orbit with the least period 2, then the following inclusions hold:

$$F_\mu(\gamma_{\mu,[\mu+1,b]}) \subset F_\mu^2(\gamma_{\mu,[\mu+1,b]}) \subset \ldots F_\mu^n(\gamma_{\mu,[\mu+1,b]}) \subset \ldots$$

Using these inclusions, we obtain the following claim.

Lemma 2.7. *Let F_μ be the map (1), $\mu \in [0, 2]$. Then there is a unique C^1-smooth function $y=\Gamma_\mu^{(1)}(x)$, which is the extention of the function $y=\gamma_{\mu,[\mu+1,b]}(x)$ on the interval $[\mu+1, +\infty)$. The graph $\Gamma_\mu^{(1)}$ of this function is F_μ-invariant curve.*

Thus, the claim (2.1.1) of Theorem 2.1 is proved.
The invariant curve $\Gamma_\mu^{(1)}$ for $\mu = 2$ plays an important role in the investigation of the asymptotic behavior of trajectories from the exterior of F_2-invariant triangle $\Delta_2 = \{(x;\, y) : x,\, y \geq 0, x+y \leq 4\}$. There exists F_μ-invariant triangle $\Delta_\mu = \left\{(x;\, y) : x,\, y \geq 0, \frac{x}{2\mu} + \frac{y}{\mu^2} \leq 1\right\}$ for every $\mu \in (0, 2]$ (see [11]). Let $T = T(\Delta_\mu)$ be its hypothenuse.
In comparison of $D_{\mu,\,\infty}$ the unbounded set $\{A_{2,\,\mu}(\mu+1;\, 1)\} \bigcup (K_1 \setminus D_{\mu,\,\infty})$ is not invariant, and the fixed point $A_{2,\,\mu}(\mu+1;\, 1)$ is the limit point of the preimages of the unbounded ranks of the critical lines belonging to this set. For the proof of the statement (2.1.2) of Theorem 2.1 we need the sets

$$\Pi_1 = \left\{(x,y) : 0 < x \leq \mu+1, y > 1, \frac{\mu+1}{x} \leq y \leq \frac{\mu+1}{x(x-\mu)^2}\right\};$$

$$\Pi_2 = \left\{(x,y) : x \geq \mu+1, 0 < y < 1, \frac{\mu+1}{x(x-\mu)^2} \leq y \leq \frac{\mu+1}{x}\right\}; \;\; \Pi = \Pi_1 \cup \Pi_2.$$

Proposition 2.8. *Let F_μ be the map (1), $\mu \in (0, 1]$. Then*
(2.8.1.) *for any odd number $n \geq 1$ the curve $f_{\mu,n}(x,y) = \mu+1$ has in Π_1 at least one triple point $M_n^{(3)}$ (i. e. a point of the intersection of three curves $f_{\mu,n}(x,y) = \mu+1$, $f_{\mu,n+1}(x,y) = \mu$ and $f_{\mu,n+2}(x,y) = \mu(\, n \geq 1)$), at least one double point $M_n^{(2,1)}$ of the first type (i.e. a point of the intersection of two curves $f_{\mu,n}(x,y) = \mu+1$ and $f_{\mu,n-1}(x,y) = \mu$ $(n \geq 1)$) and a countable set of double points $M_{n,j}^{(2,2)}$ of the second type (i.e. the points of intersection of the curve $f_{\mu,n}(x,y) = \mu+1$ with every curve $f_{\mu,j}(x,y) = \mu$ for any $j \geq n+3$, $n \geq 1$);*
(2.8.2.) *for any even number $n \geq 1$ the curve $f_{\mu,n}(x,y) = \mu+1$ has in Π_2 at least one triple point $M_n^{(3)}$, at least one double point $M_n^{(2,1)}$ of the first type and countable set of double points $M_{n,j}^{(2,2)}$ of the second type;*
(2.8.3.) *the set Π doesn't contain points of intersection of the curves $f_{\mu,n}(x,y) = \mu+1$ and $f_{\mu,m}(x,y) = \mu$ $(m,n \geq 1)$ other than above.*

As it follows from Proposition 2.8 preimages of different orders of the straight line $x = \mu+1$ and the critical line $x = \mu$ intersect each other in double and triple points. This statement shows that there is not greater than countable set of triple points and double points of the first and the second types (see Figure 1). The sequences of these points converges to the fixed point $A_{2,\,\mu}(\mu+1;\, 1)$. It means that partial derivatives of coordinate functions of the corresponding iterations of F_μ equal zero in these points, and we must use the following version of the existence theorem for the local implicit function.

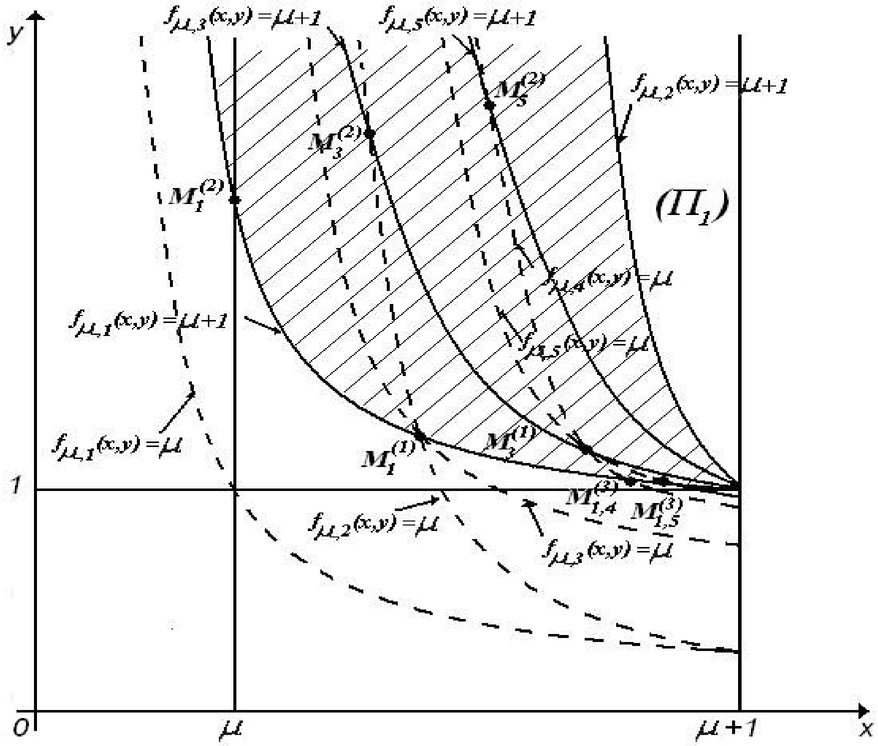

FIG. 1. *Intersections of preimages of the straight line $x = \mu + 1$ with the preimages of the critical line $x = \mu$ in double and triple points, which lie in the set Π_1.*

Proposition 2.9. *Let F_μ be the map (1), $\mu \in (0, 1]$, satisfying conditions*
(2.9.1) *$f_{\mu,n}(x_0; y_0) = \mu + 1$ for some point $(x_0; y_0) \in \mathbf{R^2}$, $n \geq 1$;*
(2.9.2) *partial derivatives $\frac{\partial f_{\mu,n}(x;y)}{\partial x}$ and $\frac{\partial f_{\mu,n}(x;y)}{\partial y}$ are continuous in the point $(x_0; y_0)$, and*

$$(\frac{\partial f_{\mu,n}(x_0;\, y_0)}{\partial x})^2 + (\frac{\partial f_{\mu,n}(x_0;\, y_0)}{\partial y})^2 \neq 0;$$

(2.9.3) *in addition, there is a closed neighborhood*

$$\overline{U}_\varepsilon(x_0;\, y_0) = [x_0 - \varepsilon;\, x_0 + \varepsilon] \times [y_0 - \varepsilon;\, y_0 + \varepsilon]$$

(ε is a positive number) such that the partial derivative $\frac{\partial f_{\mu,n}(x;y)}{\partial x}$ $(\frac{\partial f_{\mu,n}(x;y)}{\partial y})$ equals zero in not greater than a countable set of the points of $\overline{U}_\varepsilon(x_0; y_0)$.
Then there exists the C^1-smooth function $y = u_{n,\,loc}(x)$ $(x = v_{n,\,loc}(y))$, which is the solution of the equation $f_{\mu,n}(x; y) = \mu+1$ on the interval $(x_0-\varepsilon'; x_0+\varepsilon')$ $((y_0-\varepsilon''; y_0+\varepsilon''))$ with values in the interval $(y_0-\varepsilon''; y_0+\varepsilon'')$ $((x_0 - \varepsilon'; x_0 + \varepsilon'))$ for some positive ε', $\varepsilon'' \leq \varepsilon$.

In the proof of the claim (2.1.2) we set $x_0 = \mu + 1$, $y_0 = 1$. Then using Propositions 2.8, 2.9 and Baire-Hausdorff theorem, for $\mu \in (0, 1]$ we construct extensions of the functions $y = u_{n,\,loc}(x)$ $(x = v_{n,\,loc}(y))$ and obtain strictly decreasing functions $y = \eta_n(x)$ $(x = \widetilde{\eta}_n(y))$ defined on the interval $(\mu, +\infty)$ $((0, +\infty))$ with values on the interval $(0, +\infty)$ $((\mu, +\infty))$. Using arguments analogous to those, which are used in Lemmas 2.6

and 2.7, we prove the existence of F_μ-invariant curve $\Gamma_\mu^{(2)}$ defined on $(\mu, +\infty)$ with the values on $(0, +\infty)$. The claim (2.1.2) of Theorem 2.1 is proved. □

The invariant curve $\Gamma_\mu^{(2)}$, $\mu \in (0, 1]$, separates domains of the first quadrant with different asymptotic behavior of F_μ-trajectories. Trajectories of the points above $\Gamma_\mu^{(2)}$ satisfy equalities

$$\lim_{n\to+\infty} f_{\mu,n}(x, y) = +\infty \qquad \lim_{n\to+\infty} g_{\mu,n}(x, y) = +\infty.$$

Trajectories of the points below $\Gamma_\mu^{(2)}$ tend to the fixed point $A_{1,\mu}(0; \mu^2)$ for $\mu \in (0, 1]$. The curve $\Gamma_\mu^{(2)}$ makes it possible to construct the partition of the first quadrant and prove that F_μ is a singular Morse - Smale endomorphism for every $\mu \in (0, 1]$ [14].

3. The birth of the closed invariant curve (CIC)

Let us formulate Theorem 3.1 for the birth of the closed invariant curve (CIC) from the elliptic fixed point $A_{3,3/2}(1/2; 1)$.
Denote by $r(\mu) = \sqrt{2\mu - 2}$ and $\phi(\mu) = arctg\sqrt{8\mu - 9}$ the radius and the argument of the complex number $\lambda = \frac{1+i\sqrt{8\mu-9}}{2}$ respectively, where λ is one of the eigenvalues of the fixed point $A_{3,\mu}(\mu - 1; 1)$.
For $\mu = 3/2$ the resonance of the 3-d order takes place, and the eigenvalues of the point $A_{3,\mu}(\mu - 1; 1)$ belong to Siegel's domain.

Theorem 3.1. *The map F_μ for $\mu = 3/2$ possesses the following properties:*
(3.1.1) $r(\frac{3}{2}) \neq 0$;
(3.1.2) $e^{ik\phi(\frac{3}{2})} \neq 1$ *for* $k = 1, 2, 3, 4$;
(3.1.3) $a(\frac{3}{2}) = -6$, *where*

$$a(\mu) = -Re\left(\frac{(1 - 2e^{i\phi(\mu)})e^{-2i\phi(\mu)}g_{20}(\mu)g_{11}(\mu)}{2(1 - e^{i\phi(\mu)})}\right) - \frac{1}{2}\left|g_{11}(\mu)\right|^2 - \frac{1}{4}\left|g_{02}(\mu)\right|^2,$$

the coefficients $g_{ij}(\mu)$ of the normal form are defined by the equalities

$$g_{20}(\mu) = \frac{(i - \sqrt{8\mu - 9})(2\mu - 6)}{\sqrt{8\mu - 9}};$$

$$g_{11}(\mu) = \frac{\sqrt{8\mu - 9}(2\mu - 2) + i(2\mu - 6)}{\sqrt{8\mu - 9}};$$

$$g_{02}(\mu) = \frac{-\sqrt{8\mu - 9}(2\mu + 1) - 3i(2\mu - 1)}{\sqrt{8\mu - 9}};$$

and there exists a neighborhood of the elliptic fixed point $A_{3,3/2}(\frac{1}{2}; 1)$, in which CIC is born from the fixed point $A_{3,3/2}(\frac{1}{2}; 1)$, when μ crosses through the value $\frac{3}{2}$.
If $\mu \in [\frac{3}{2}, \frac{3}{2} + \varepsilon)$ and $\varepsilon > 0$ is sufficiently small, then the born invariant curve is the circle of the radius $\sqrt{\mu - \frac{3}{2}}$.
The following asymptotic formula for the map F_μ in the variables (r, ϕ) is correct

$$F_\mu(r, \phi) = r \cdot e^{i(\phi(\mu)+\theta)} + O(|r|^3). \tag{2}$$

Theorem 3.1 shows that the family (1) gives the new example of quadratic maps, which admit the bifurcation of the birth of the CIC from the elliptic fixed point (Neimark-Saker bifurcation). As it follows from formula (2) the restriction $F_{\mu|CIC}$ is a diffeomorphism close to the rotation on the CIC for $\mu \in [\frac{3}{2}, \frac{3}{2} + \varepsilon)$, where $\varepsilon > 0$ is sufficiently small.

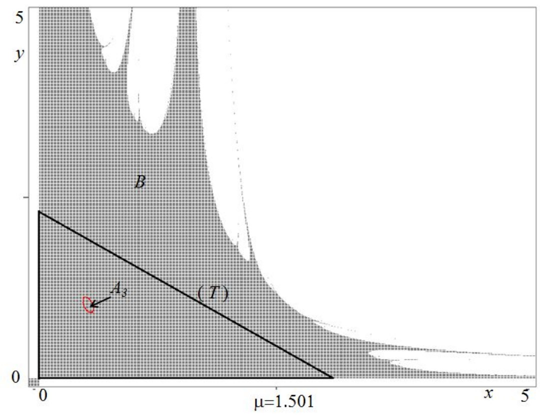

FIG. 2. *Basin B (in grey) of the attractive invariant closed curve (in red) around the repelling fixed point A_3 for $\mu = 1.501$.*

As it follows from Theorem 3.1 (see the claim (3.1.3)), the CIC is attracting for $\mu = \frac{3}{2}$. In Figure 2 is plotted the CIC in red and its basin of attraction B in grey. For each initial condition (x_0, y_0) taken in the basin B, the iterated sequence issued from (x_0, y_0) converges towards the CIC (when the rotation number of the restriction $F_{\mu|CIC}$ is irrational) or an attracting periodic orbit, which can exist on the CIC, when the rotation number of the restriction $F_{\mu|CIC}$ is rational. Note also that intransitive case can not be realized since F_μ is an analytic map [30].

In the proof of Theorem 3.1 the standard technique of the normal forms is used. Note that we apply the change of variables and of the parameter

$$\begin{cases} \widetilde{x} = x - \frac{(\beta+1)^2}{2}, \\ \widetilde{y} = y - 1 \\ \mu = \frac{(\beta+1)^2}{2} + 1, \end{cases}$$

which moves the one-parameter family F_μ into the one-parameter family

$$\widetilde{F}_\beta(\widetilde{x}, \widetilde{y}) = J(\widetilde{F}_\beta(\widetilde{x}, \widetilde{y}))|_{A'_3(0;0)} \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} + \begin{pmatrix} \widetilde{x}\widetilde{y} \\ \widetilde{x}^2 \end{pmatrix}, \tag{3}$$

where $\beta \in [0, \sqrt{2} - 1]$, $J(\widetilde{F}_\beta(\widetilde{x}, \widetilde{y}))|_{A'_3(0;0)}$ is the Jacobian matrix of the map $\widetilde{F}_\beta$ in the fixed point $A'_3(0;0)$ satisfying

$$J(\widetilde{F}_\beta(\widetilde{x}, \widetilde{y}))|_{A'_3(0;0)} = \begin{pmatrix} 1 & \frac{(\beta+1)^2}{2} \\ -2 & 0 \end{pmatrix}.$$

The following equalities are valid for the multiplicators $\lambda_{1,2}(\beta)$ of the fixed point $A'_3(0;0)$

$$\lambda_{1,2}(\beta) = \frac{1 \pm \sqrt{4(\beta+1)^2 - 1}}{2} = r(\beta) \cdot e^{\pm i\phi(\beta)},$$

where $r(\beta) = (1 + \beta)$, $\phi(\beta) = arctg\sqrt{4(\beta+1)^2 - 1}$.

The following theorem contains the main technical result for the proof of Theorem 3.1.

Theorem 3.2. *Let $\widetilde{F}_\beta$ be the quadratic map (3), $v(\widetilde{x},\widetilde{y})$ be a vector in the plane $\mathbb{R}^2$, $\beta \in [0, \sqrt{2}-1]$. Then after the change of variables*

$$\widetilde{x} = -(z \cdot (1 + i\sqrt{4(\beta+1)^2 - 1}) + \overline{z} \cdot (1 - i\sqrt{4(\beta+1)^2-1})), \quad \widetilde{y} = 4 \cdot (z + \overline{z})$$

the map (3) takes the form

$$\widetilde{F}_\beta(z,\overline{z}) = \lambda_1(\beta) \cdot z + g(z,\overline{z},\beta),$$

where $z \in \mathbb{C}$ and $\overline{z} \in \mathbb{C}$ (here $\overline{(\cdot)}$ is Hermitian conjugation) are the coordinates of the decomposition of the vector v in the basis of the eigenvectors of the matrix $J(\widetilde{F}_\beta(\widetilde{x},\widetilde{y}))|_{A_3'(0;0)}$; $g(z,\overline{z},\beta)$ is the complex-valued function of the variables z, $\overline{z}$ and the parameter β satisfying

$$g(z,\overline{z},\beta) = \frac{g_{20}(\beta)}{2} \cdot z^2 + g_{11}(\beta) \cdot z \cdot \overline{z} + \frac{g_{02}}{2} \cdot \overline{z}^2,$$

where the coefficients $g_{ij}(\beta)$, $i,j = 0,1,2$ are defined by the equalities

$$g_{20}(\beta) = \frac{((\beta+1)^2 - 4) \cdot (i - \sqrt{4(\beta+1)^2-1})}{\sqrt{4(\beta+1)^2-1}};$$

$$g_{11}(\beta) = \frac{(\beta+1)^2\sqrt{4(\beta+1)^2-1} + i \cdot ((\beta+1)^2 - 4)}{\sqrt{4(\beta+1)^2-1}};$$

$$g_{02}(\beta) = \frac{-(3 + (\beta+1)^2)\sqrt{4(\beta+1)^2-1} - 3i \cdot ((\beta+1)^2+1)}{\sqrt{4(\beta+1)^2-1}}.$$

4. Evolution of the closed invariant curve (numerical experiment)

Different scenarios of destructions of CIC were considered, for example, in [2] – [6], [29], [20] – [21], [27]. In this paragraph, we study the evolution of the closed invariant curve (CIC) born in the family (1) when $\mu = \frac{3}{2}$, from $\mu = \frac{3}{2}$ to $\mu = 2$ by considering numerical simulations.

- The CIC appears at $\mu = \frac{3}{2}$ (see Theorem 3.1).
- $\mu = 1.71049$ (cf. Figures 3), there is a contact between the CIC and the critical curve C_μ at a point A^*. The image of A^* by F_μ is a point $F_\mu(A^*)$ located on C_0'. Then $F_\mu^2(A^*)$ is itself located on the line C_0^+, $F_\mu^3(A^*)$) is the fixed point A_1. This rebuilding of CIC is connected with two features of F: firstly, A^* is a critical homoclinic point of $F_{|CIC}$ (isolated for $\mu = 1.71049$) and, secondly, the fixed point $A_1(0,\, \mu^2)$ is the flat singularity of F_μ for every $\mu \in [0,\, 2]$ such that $F_\mu(C_0) = (0;\, \mu^2)$.
  Enlargements close to the contact bifurcation value of μ and close to the fixed point A_1 are given on Figures 4-5. Before the contact bifurcation, two different horns (compare with [2]) with vertexes in the points P_1 and P_2 exist on the CIC. In the points P_1 and P_2 CIC is not smooth. At the contact bifurcation the points P_1 and P_2 will merge together with the fixed point A_1, and we observe the appearance of the first small loop containing the point A_1 on CIC. From this value of μ CIC became the ramified continuum, i.e. a compact connected metric space with ramification points (points of an order, greater then 2).
- $\mu = 1.72$ (Figures 6-7). We observe the appearance of the second new loop on the CIC containing the point A_1. This loop is the image by F^3 of the part of the CIC located on the right side of the line C_μ.
  Loops increase in number and create what is called a WCR (for Weakly Chaotic Ring) [28].
  All evolution of the CIC is connected with the properties of the map on the CIC. The appearance of countable number of new loops on CIC is related to homoclinic Ω-blow up, i.e. blow up of the nonwandering set of the map on the CIC. In this moment we observe transition from a single rotation

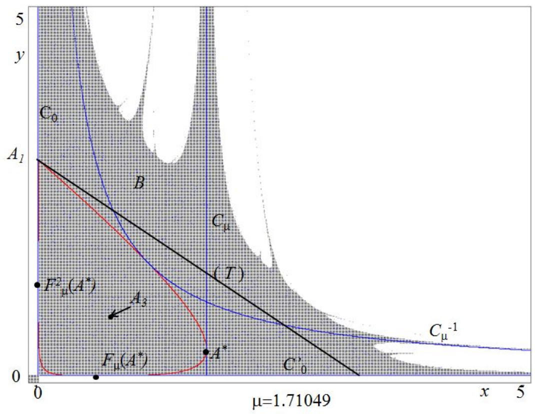

Fig. 3. *Contact between the CIC (in red) and the critical lines* C_μ *at the point* A^* *for* $\mu = 1.71049$.

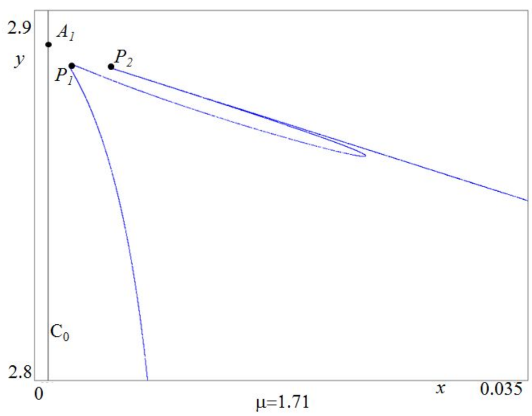

Fig. 4. *Before the first contact between the CIC and the critical lines, two different points* P_1 *and* P_2 *exist on the CIC. At the contact bifurcation, they will merge together with the fixed point* A_1.

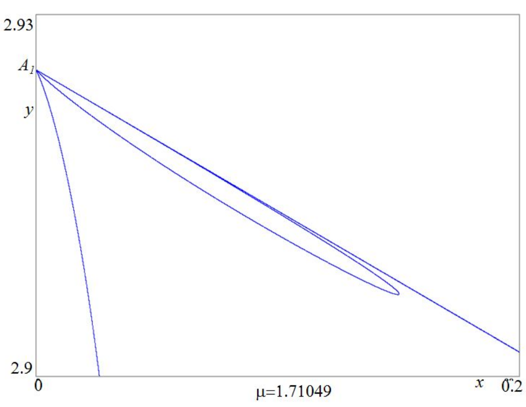

Fig. 5. *Enlargment of the Figure 3, the fixed point* A_1 *belongs to the CIC and a new loop is setting up.*

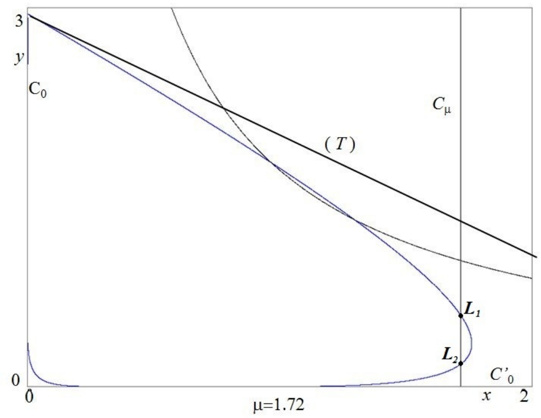

FIG. 6. *After the first contact between the CIC and the critical lines, a part of the CIC is located in the domain $x > \mu$. This part of the CIC has preimages on the CIC which form a second small loop (see enlargment). The first loop was created by the merging of points P_1 and P_2 with A_1 (cf. Figures 4-5).*

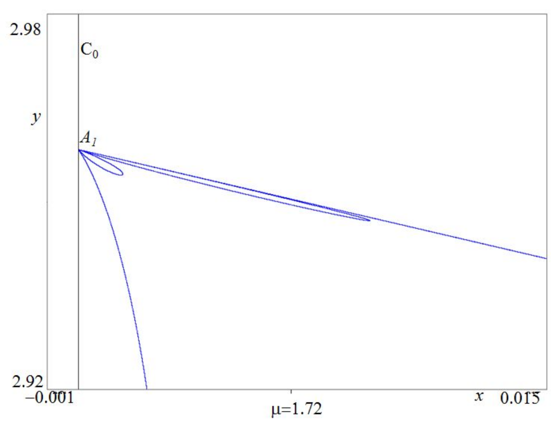

FIG. 7. *Enlargment of the Figure 6, the two loops can be seen.*

number of F on CIC to nondegenerate rotation interval (see, e.g., [18])[5]. As a result we observe the appearance of the WCR.

- $\mu = 1.8$ (cf. Figure 8), the CIC has become an attractive WCR, that means an attractor, one Lyapunov exponent of which is slightly positive. The WCR progressively turns into an annular chaotic attractor (ACA), as it has been obtained in [29].
- $\mu = 1.89$ (cf. Figure 9), there is a value of μ belonging to [1.89, 1.9], for which $A_3 \in$ WCR. After this μ-value, the annular chaotic attractor becomes a one piece chaotic attractor (CA). This situation has already been obtained in [29], [19].

In all above cases the basins of attraction of CIC, WCR, ACA or CA have interior points, and CIC, WCR, ACA and CA are connected sets. Numerical experiment permits to assume that the topological limit of the attraction basins when $\mu \to 2$ exists and doesn't contain interior points.

[5]Let us consider a multifunction defined on the set of maps on CIC such that the value of this multifunction for every map on CIC equals the nonwandering set of this map on CIC. Ω-blow up phenomenon for some μ^* means that this multifunction is not upper semicontinuous for $\mu = \mu^*$ (see, e.g., [31]).

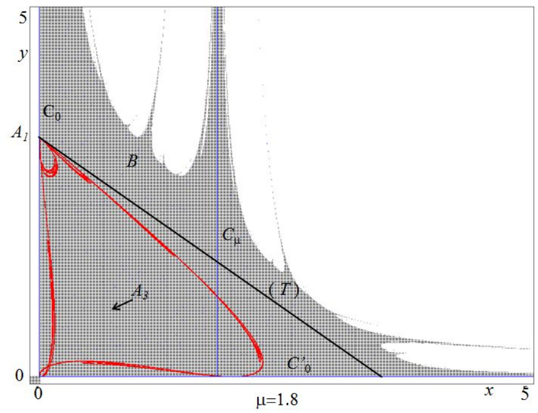

FIG. 8. *The CIC (in red) has become a WCR for $\mu = 1.8$. We can see that many loops have appeared on the CIC.*

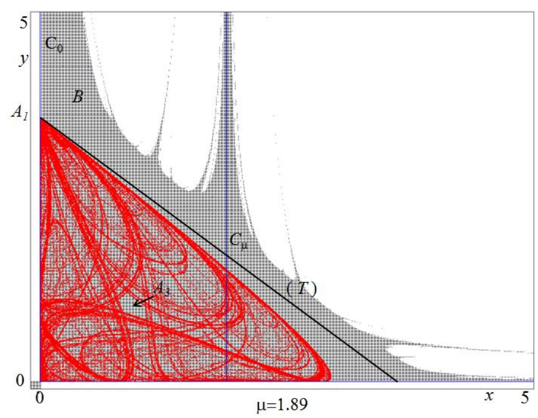

FIG. 9. *The WCR is going to become a connected chaotic attractor due to the contact bifurcation between the repelling point A_3 and itself (figure for $\mu = 1.89$ before the contact bifurcation).*

- $\mu = 2$. The chaotic attractor disappears [29], [19]. As it follows from [12], the invariant triangle Δ_2 is not an attractor of F_2. The triangle is a repeller in Sharkovskii's sense, i.e. it is the ω-limit set, which is not contained in a bigger ω-limit set, and it is not an attractor [31].

The comparison of numerical results of this paper with results of the numerical experiment from [21] shows that these results have analogous features: among them the lost of differentiability of CIC and appearance of the ramification points. But in contrast to [21] complication of topological structure of CIC in this paper is connected with appearance of loops. The more simple analytic formulas for the considered in this paper one-parameter family, than for Lorenz's one-parameter family of quadratic maps in the plane introduced in [26] and considered in [21], makes it possible not only to give more detailed and transparent description of the rebuildings of CIC, but also to give the rigorous proofs of some observable phenomena.

Summarizing results of the numerical investigation we give Figure 11, where both Lyapunov exponents are presented. The larger one becomes strictly positive when WCR begins to exist and increases with the size of the chaotic attractor. One can see the frequency lockings that appear on the graph of the

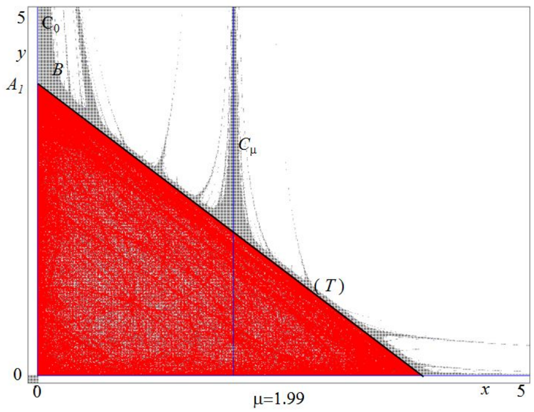

FIG. 10. *The chaotic attractor is going to undergo a contact bifurcation with its basin boundary, which will make it disappear (figure for $\mu = 1.99$ before the contact bifurcation) and which occurs at $\mu = 2$.*

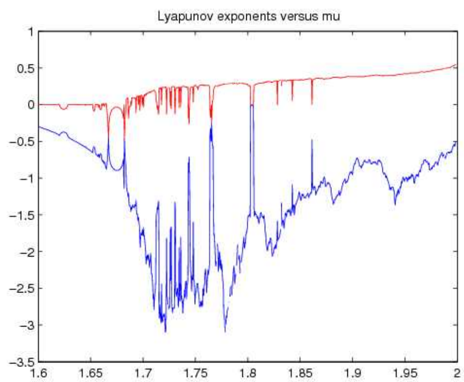

FIG. 11. *In red the lager Lyapunov exponent, when it becomes slightly positive, the CIC becomes a WCR, then when it becomes larger, a chaotic attractor; in blue a smaller one, always negative, indicates the stability of the system (see, e.g., [22]).*

larger Lyapunov exponent from time to time, when there are peaks on the graph of the smaller Lyapunov exponent with negative values.

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References

[1] R.H. Abraham, L. Gardini, C. Mira, *"Chaos in Discrete Dynamical Systems"*, Springer (TELOS), New York, 1997.

[2] D. G. Aronson, M. A. Chory, G. R. Hall, R. P. McGehee, *"Bifurcations from an invariant circle for two-parameter families of maps of the plane: A computer- assisted study"*, Commun. Math. Phys., vol.83, 1982, p.p.303-354.

[3] V.S. Afraimovich, L.P. Shil.nikov, *"Annulus principle and problem of interaction of two self-oscillating systems"*, Applied Math. and Mechanics, vol.42(4), 1977, p.p. 618-627.

[4] V.S. Afraimovich, L.P. Shil.nikov, *"Invariant two-dimensional tori, their destruction and stochasticity"*, Gorkii University, Gorkii, Russia, 1983, p.p. 3-26.

[5] V.S. Anishchenko, M.A. Safonova, U. Feudel, J. Kurths, *"Bifurcations and transition to chaos through three-dimensional tori"*, Int. J. Bif. and Chaos, vol. 4(3), 1994, pp. 595-607.

[6] V.I. Arnold, V.S. Afraimovich, Yu.S. Iliashenko, L.P. Shil.nikov, *"Dynam- ical Systems"*, vol. 5, Springer, Berlin, 1991.

[7] Y. Avishai, D. Berend, *"Transmission through a Thue-Morse chain"*, Physical Review B, vol.45, 1992, p.p. 2717-2724.

[8] Y. Avishai, D. Berend, V. Tkachenko, *"Trace Maps"*, Int. Journal of Modern Physics B, vol.11(30), 1997, p.p. 3525-3542.

[9] F. Balibrea, J. G. Guirao, M. Lampart, J. Llibre, *Dynamics of a Lotka-Volterra map*, Fund. Math, vol. 191, 2006, p.p. 265-279.

[10] S. Bel'mesova, L. Efremova, *On the one-parameter family of quadratic maps of the plane*, Proceed. of 50-th scientific conference of MIPT "Contemporary problems of fundamental and applyed sciences" [in Russian], vol. 1 (part VII), 2007, p.p. 8-11.

[11] S. Bel'mesova, L. Efremova, *On invariant sets of some quadratic maps of the plane*, Vestnik NNGU, named after N.I. Lobachevskii [in Russian], No 2(1), 2012, p.p. 152-158.

[12] S. Bel'mesova, L. Efremova, *"On unbounded trajectories of a certain quadratic mapping of the plane"*, Journ. Math. Sci. (N.Y.), vol. 157(3), 2009, p.p. 433-441.

[13] S. Bel'mesova, L. Efremova, *On Quadratic Maps from Some One-Parameter Family Closed to Unperturbed Map*, Proceed. of MIPT [in Russian], vol. 2, No 2, p.p. 46-57.

[14] S. S. Bel'mesova, L. S. Efremova, *A one-parameter family of quadratic maps of a plane including Morse – Smale endomorphisms*, Russian Math., No 8, 2013, p.p. 70-74.

[15] D. Damanik, A. Gorodetski, *Hyperbolicity of the Trace Map for the Weakly Coupled Fibonacci Hamiltonian*, Nonlinearity, vol. 22, 2009, p.p.123-143.

[16] D. Damanik, A. Gorodetski, *The Spectrum of the Weakly Coupled Fibbonacci Hamiltonian*, Elect. Research Announcements. in Math. Sci, vol. 16, 2009, p.p. 23-29.

[17] W. de Melo, S. van Strien, *One-Dimensional Dynamics*, Cambridge Univ. Press. Cambridge, 1992.

[18] L. S. Efremova, *A quotient of periods other than a power of two leads to chaos on the circle*, Russ. Math. Surveys, vol. 40, 1985, p.p. 217-218.

[19] D. Fournier-Prunaret, *Dynamical behaviour in DPCM system with an order 2 predictor. Case of a constant input*, Circuits, System and Signal Processing, special issue on "Applications of Chaos in Communications, vol. 24(5), 2005, p.p. 615-638.

[20] C.E. Frouzakis, L. Gardini, I.G. Kevrekidis, G. Millereoux, C. Mira, *On some properties of invariant sets of two-dimensional noninvertible maps*, Intern. Journ. Bifurcations and Chaos, vol.7, No 6, 1997, p.p. 1167-1194.

[21] C.E. Frouzakis, I.G. Kevrekidis, B.B. Peckham, *A route to computational chaos revisited: noninvertibility and the breakup of an invariant circle*, Physica D, vol. 177, 2003, p.p. 101-121.

[22] K. Geist, U. Parlitz, W. Lauterborn, *Comparison of different methods for computing Lyapunov exponents*, Prog. Theor. Phys., vol. 83, 1990, p.p. 875.

[23] M. Henon, *Numerical study of quadratic area preserving mapping*, J. Appl. Math., vol. 27, 1969, p.p. 291-312.

[24] A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia Math. Appl., vol. 54, Cambridge Univ. Press. Cambridge. 1995.

[25] R. Lopez-Ruiz, D. Fournier-Prunaret, *Complex behavior in a discrete coupled logistic model for the symbolic interaction of two species*, Math. Biosci. Eng., vol. 1, 2004, p.p. 307-324.

[26] E.N. Lorenz,*Computational chaosa prelude to computational instability*, Physica D, vol. 35, 1989, p.p. 299-317.

[27] V. Maistrenko, Yu. Maistrenko, E. Mosekilde, *"Torus breakdown in noninvertible maps"*, Physical Review E, vol. 67, 2003 046215.

[28] C. Mira, *"Chaotic dynamics from the One-Dimensional Endomophism to the Two-Dimensional Diffeomorphism"*, World Scientific, Singapore, 1987.

[29] C. Mira, L. Gardini, A. Barugola, J. C. Cathala, *Chaotic Dynamics in Two-Dimensional Noninvertible Maps*, series A, vol. 20, World Scientific Publishing, Singapore. 1996.

[30] Z. Nitecki, *Differentiable Dynamics. An Introduction to the Orbit Structure of Diffeomorphisms*, The MIT Press. 1971.

[31] A. N. Sharkovskii, Y. L. Maistrenko, E. Y. Romanenko, *Difference Equations and Their Applications* [in Russian], Naukova Dumka. Kiev. 1986; English transl., Kluwer, Academic Publishers. Dordrecht. 1993.

[32] S. Sternberg, *Lectures on Differential Geometry*, Prentice Hall, Inc. Englewood Cliffs, N.J. 1964.