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A NOTE ON PERIODS OF POWERS *

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Abstract. Let $f : X \to X$ be a continuous map defined from a topological space X into itself. We discuss the problem of analyzing and computing explicitly the set $\mathrm{Per}(f^p)$ of periods of the p-th iterate f^p from the knowledge of the set $\mathrm{Per}(f)$ of periods of f. In the case of interval or circle maps, that is, $X = [0,1]$ or $X = \mathbb{S}^1$, this question was solved in [11]. Now, we present some remarks and advances concerning the set $\mathrm{Per}(f^p)$ for a continuous interval map, and on the other hand we study and solve the problem when we consider σ-permutation maps, namely, when $X = [0,1]^k$ for some integer $k \geq 2$ and the map has the form $F(x_1, x_2, \dots, x_k) = (f_{\sigma(1)}(x_{\sigma(1)}), f_{\sigma(2)}(x_{\sigma(2)}), \dots, f_{\sigma(k)}(x_{\sigma(k)}))$, being each f_j a continuous interval map and σ a cyclic permutation of $\{1, 2, \dots, k\}$. This paper can be seen as the continuation of [11].

Résumé. Soit $f : X \to X$ une application continue définie sur une espace topologique X. Nous discutons ici le problème d'analyser et de calculer explicitement l'ensemble $\mathrm{Per}(f^p)$ des périodes de la composée f^p, une fois que nous connaissons l'ensemble $\mathrm{Per}(f)$ des périodes de f. Quand $X = [0,1]$ ou $X = \mathbb{S}^1$, la question fut résolue à [11]. Maintenant, d'un côté nous présentons quelques remarques et quelques progrès pour le case d'applications de l'intervalle $I = [0,1]$, et d'un autre côté nous étudions et résoudrons le problème quand nous considérons produits directs σ-permutés, c'est-à-dire, applications continues $F : I^k \to I^k$, pour quelque entier $k \geq 2$, qui ont la suivante forme $F(x_1, x_2, \dots, x_k) = (f_{\sigma(1)}(x_{\sigma(1)}), f_{\sigma(2)}(x_{\sigma(2)}), \dots, f_{\sigma(k)}(x_{\sigma(k)}))$, où chaque f_j est continue et σ est une permutation cyclique de $\{1, 2, \dots, k\}$. Donc, on peut voir cette article comme la continuation naturelle de [11].

Keywords: Discrete dynamical systems; set of periods; iterates; Sharkovsky's Theorem; cyclic permutation; direct product; σ-permutation map; Sharkovsky type; functional equation.
Mots clefs: Système dynamique discret; ensemble des périodes; composée; Théorème de Sharkovsky; permutation cyclique; produit direct; produit direct σ-permuté; type de Sharkovsky; équation fonctionnelle.
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1. Introduction

In the present paper we are interested in the study of the following question:

Question. *Given a topological space X and a continuous map $f : X \to X$, assume that* $\mathrm{Per}(f)$ *is the set of periods of f. Consider a positive integer p. What is then the set of periods* $\mathrm{Per}(f^p)$ *of the "power" f^p?*

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Let us introduce some concepts and the notation needed to develop our analysis. We say that (X, f) is a discrete dynamical system, that is, X is a topological space and $f : X \to X$ is a continuous map, in short $f \in C(X)$. By f^0 we denote the identity map Id_X, and the n-iterate of f or the n-"power" of f is given by $f^n = f^{n-1} \circ f$, $n > 1$, $f^1 = f$. Recall that $x \in X$ is a periodic point if $f^n(x) = x$ for some positive integer n. The smallest positive value holding that $f^n(x) = x$ is the period or order of x under f, which will be denoted by $\mathrm{ord}_f(x)$. The set of periods of f is denoted by $\mathrm{Per}(f)$ keeping the notation $\mathrm{P}(f)$ for the set of periodic points of f. Notice that $\mathrm{P}(f) = \mathrm{P}(f^k)$ for any $k > 0$. Similar formulas are true for another well–known sets like uniformly recurrent, recurrent or ω–limit sets (see e.g. [10]), but however, for topological entropy it is also well–known the power formula $h(f^k) = k \cdot h(f)$ (see e.g. [1] or [10]). Even more, for the non–wandering set $\Omega(f)$, it is held that ... $\subseteq \Omega(f^4) \subseteq \Omega(f^2) \subseteq \Omega(f)$ and there are examples showing that the inclusions may be strict, while $\Omega(f) = \Omega(f^{2n+1})$ for any positive integer n (see e.g. [10] for definition and the above mentioned properties). So, computing the set of periods of a power of continuous map f from its set of periods it is not trivial.

Nevertheless, in some cases the answer to our Question is immediate. For instance, if $\mathrm{Per}(f) = \{1, 2, 2^2\}$, then $\mathrm{Per}(f^2) = \{1, 2\}$, $\mathrm{Per}(f^{2^m}) = \{1\}$ for $m \geq 2$, and $\mathrm{Per}(f^{2m+1}) = \{1, 2, 2^2\}$ for $m \geq 0$.

The reader is referred to [1, Lemma 2.1] to see the proof of the following combinatorial result.

Lemma 1. *Let $f : X \to X$ be a map from a topological space X into itself. Assume that $x \in X$ is periodic with $\mathrm{ord}_f(x) = n$ and for some $p \in \mathbb{Z}^+$, assume that $ord_{f^p}(x) = m$. Then*

$$n = m \cdot \gcd(n, p),$$

where $\gcd(n, p)$ denotes the greatest common divisor of n and p.

Taking into account the above result, we are in a position for doing the answer to our problem in terms of the following set of positive integers, which gets us the general solution to the problem:

$$\mathrm{Per}(f^p) = \left\{ \frac{m}{\gcd(m, p)} : m \in \mathrm{Per}(f) \right\}. \tag{1}$$

If we consider the interval case, when $X = I := [0, 1]$, or the circle case $X = \mathbb{S}^1$, we could try to express the set of periods of periods $\mathrm{Per}(f^p)$ via initial segments of the Sharkovsky's ordering or under the sets $M(\rho_\ell, \rho_u) := \{m \in \mathbb{N} : \rho_\ell < l/m < \rho_u$ for some integer $l\}$, being $[\rho_\ell, \rho_u]$ the rotation interval of the circle map (for more details about these sets, consult [1]). This was done in [11] and provides us a more pleasant presentation of the set of periods of a power than the combinatorial solution given by Eq. (1).

Now, as a continuation of [11], in the real case, we give some remarks to the main result of [11] and obtain the set of periods corresponding to powers of σ-permutation maps, a particular class of p-dimensional maps strongly related with one–dimensional maps. In Section 2 we give some preliminaries and state the main result of [11], jointly with some appropriate remarks. In Section 3 we get the set of periods of F^p, where F is a σ-permutation map.

2. Some remarks to the interval case

The well-known Sharkovsky's ordering of the natural numbers is given by

$$\begin{aligned} &3 >_s 5 >_s 7 >_s ... >_s 2 \cdot 3 >_s 2 \cdot 5 >_s 2 \cdot 7 >_s ... \\ &2^n \cdot 3 >_s 2^n \cdot 5 >_s 2^n \cdot 7 >_s ... >_s 2^{n+1} >_s 2^n >_s ... >_s 2 >_s 1. \end{aligned}$$

For $n \in \mathbb{N} := \{1, 2, \ldots\}$ define the segment

$$\mathcal{S}(n) = \{m \in \mathbb{N} : n >_s m\} \cup \{n\},$$

and put

$$\mathcal{S}(2^\infty) = \{2^n : n \in \mathbb{N} \cup \{0\}\}.$$

Then, Sharkovsky's theorem ([22]) establishes that for $f \in C(I)$ there exists $n \in \mathbb{N} \cup \{2^\infty\}$ such that $\mathrm{Per}(f) = \mathcal{S}(n)$.

Let p and q be positive integers and write

$$\begin{aligned} t(p,q) &= \min\{m : m \text{ is odd}, m \geq 3, \text{ and } q \leq m \cdot p\}, \text{ if } q > 1, \\ t(p,1) &= 1. \end{aligned}$$

For instance, $t(1,5) = 5$, $t(3,3) = 3$, $t(7,1) = 1$ and $t(5,39) = 9$. Now we are able to state the main result on the set of periods of the iterates of an interval map. It is interesting to emphasize that its proof is merely combinatorial, so it remains true for triangular maps, that is, two–dimensional continuous maps with the form $T(x,y) = (f(x), g(x,y))$, $(x,y) \in [0,1]^2$ (see [16]). The reader is referred to [11] to check all the details of the proof as well as the analogous result for circle maps.

Theorem 2. *([11, Theorem 2.1]) Let $f \in C(I)$ be such that $\mathrm{Per}(f) = \mathcal{S}(q \cdot 2^r)$, with $q \geq 1$ odd and $r \in \mathbb{N} \cup \{0, \infty\}$. Let $p \in \mathbb{N}$, $p \neq 1$.*

- (a) *If $p \geq 3$ is odd, then*
  - (a1) *If $q = 1$, $\mathrm{Per}(f^p) = \mathcal{S}(2^r)$.*
  - (a2) *If $q > 1$, $\mathrm{Per}(f^p) = \mathcal{S}(t(p,q) \cdot 2^r)$.*
- (b) *If $p = l \cdot 2^k$, $l \geq 1$ odd, $k \geq 1$, then*
  - (b1) *If $q = 1$,*

$$\mathrm{Per}(f^p) = \begin{cases} \mathcal{S}(1) & \text{if } k \geq r, \\ \mathcal{S}(2^{r-k}) & \text{if } k < r. \end{cases}$$

(here if $2^k = 2^\infty$, we consider that $2^{k-r} = 2^\infty$).

  - (b2) *If $q > 1$,*

$$\mathrm{Per}(f^p) = \begin{cases} \mathcal{S}(3) & \text{if } k > r, \\ \mathcal{S}(t(l,q)) & \text{if } k = r, \\ \mathcal{S}(t(l,q) \cdot 2^{r-k}) & \text{if } k < r. \end{cases}$$

In [11], the number $t(\ell, q)$ was defined to be an odd number greater than one. If we define

$$\widetilde{t}(\ell, 1) = 1,$$

and for ℓ and q are odd numbers

$$\widetilde{t}(\ell \cdot 2^k, q \cdot 2^r) = \begin{cases} t(\ell, q) & \text{if } k \leq r \text{ or } k > r, q = 1, \\ 3 & \text{otherwise} \end{cases}$$

we can obtain a **compact formula** for the set of periods of a power.

Theorem 3. *Let $f \in C(I)$. If $\mathrm{Per}(f) = \mathcal{S}(q \cdot 2^r)$, with $q \geq 1$ odd and $r \in (\mathbb{N} \cup \{0, \infty\})$, and $p = \ell \cdot 2^k$, $\ell \geq 1$ odd, $k \geq 0$, then*

$$\mathrm{Per}(f^p) = \mathcal{S}(\widetilde{t}(\ell, q) \cdot 2^{\max\{r-k, 0\}}).$$

Theorem 2 reports us that the level of complexity of two maps f, g and its iterates f^p, g^p with respect to their types are the same. Remember that we say that a map $f \in C(I)$ has Sharkovsky type equals to $m \in \mathbb{N} \cup \{2^\infty\}$ whenever $\mathrm{Per}(f) = \mathcal{S}(m)$. We denote by $T(f)$ the type of f.

Corollary 4. *Let $f, g \in C(I)$, $p \in \mathbb{N}$. If $T(f) \geqslant_{\mathrm{s}} T(g)$, then $T(f^p) \geq_{\mathrm{s}} T(g^p)$.*

Notice that the equality can be obtained although f and g have different types.

We finish this section by connecting our main result Theorem 2 with certain functional equations related with iterates of fractional order. To be more precise, consider the functional equation $f^p = g$, where f, g are continuous maps defined from an interval into itself, g is given and f is a unknown map so-called a p-th root of g, $f = g^{\frac{1}{p}}$. This problem was already treated by Babbage at the beginning of the XIX century ([3], [4]). Some other interesting contributions to the problem are, among others, [15], [20], [13], [17], [19]. As a general monograph on functional equations the reader is referred to [18], and to the survey [9] to see the state-of-art of the problem of the iterative roots.

Keeping in mind the functional equation $f^p = g$, if the type of the given map g is $T(g) = 1$, Theorem 2 implies that the type $T(f)$ of f must be equal to 2^r, with $0 \leq r \leq p$. So, a natural question arises: If we previously prove that $f^p = g$ has solutions, is it possible to find p-th roots f_ℓ such that $T(f_\ell) = 2^\ell, \ell = 0, 1 \ldots, p$? In general the answer is negative, for instance, consider $f^p(x) = x$, with $p > 2$: it is well known that the only continuous solutions are $f_0(x) = x$ and any convolution f_1, so $T(f_1) = 2$. What conditions must be added to g in order to obtain a positive answer, if possible? The same question can be posed for the case in which the type of g is any element of $\mathbb{N} \cup \{2^\infty\}$.

3. The case of σ-permutation maps

Let us present an interesting class of p-dimensional maps defined from I^p into itself, where I is a real subinterval. We say that the continuous map $F : I^p \to I^p$ is a $\sigma-$permutation map if there exist a cyclic permutation $\sigma : \{1, \ldots, p\} \to \{1, \ldots, p\}$ and continuous maps $f_j : I \to I$, $j = 1, \ldots, p$ such that

$$F(x_1, x_2, \ldots, x_p) = (f_{\sigma(1)}(x_{\sigma(1)}), f_{\sigma(2)}(x_{\sigma(2)}), \ldots, f_{\sigma(p)}(x_{\sigma(p)})).$$

This class of maps is strongly related with the compositions $\varphi_j := f_{\sigma(j)} \circ f_{\sigma^2(j)} \circ \ldots \circ f_{\sigma^p(j)}$, $j = 1, \ldots, p$ and, consequently, their dynamics are essentially one dimensional ([6], [7], [8], [14],....).

When $p = 2$ and $I = [0, 1]$, this type of maps appears associated with certain economical model so called Cournot duopoly ([12], [21],....).

In [11] the authors ask for the possibility of checking whether similar results to the described for the case of periods of powers of interval maps can be carried to spaces whose structure of periods are known, for instance tree or graph maps (see [1]) or some two dimensional maps (see, again [1] or [6, 7]). In this note, we give the answer for the case of σ-permutation maps defined in the p-dimensional cube I^p. In order to do it, we need to recall the main result of [6].

When $X = I$ the periodic structure of σ-permutation maps have been obtained in [6]. Next, we explain it. For $p \in \mathbb{N}$ and $m \in (\mathbb{N} \cup \{2^\infty\})$, we introduce

$$S_p(m) = \left\{t \in \mathbb{N} : t \nmid p \text{ and } \frac{t}{\gcd(t,p)} \in \mathcal{S}\,(m)\right\} \cup \{1\} \; = \left\{kt : k \mid p,\; t \in (\mathcal{S}(m) \setminus \{1\})\,,\; \gcd(\frac{p}{k}, t) = 1\right\} \cup \{1\}$$

where $t \nmid p$ means that t does not divide p.

Theorem 5. *([6]) (Periodic structure of σ-permutation maps on I^p)*

(1) *Let F be a σ-permutation map on I^p, $p \geq 2$. Then there exists $m \in (\mathbb{N} \cup \{2^\infty\})$ such that*

$$\mathrm{Per}(F) = S_p(m) \text{ or}$$
$$\mathrm{Per}(F) = S_p(m) \cup \{t : t|p\}.$$

We have $\mathrm{Per}(f_i^{(p)}) = \mathcal{S}(m)$ for every $i \in \{1, 2, ..., p\}$. Moreover, $\mathrm{Per}(F) \neq S_p(m)$ if and only if $f_i^{(p)}$ possesses at least two different fixed points.

(2) *Suppose that* $\mathcal{P} = S_p(m)$ *or* $\mathcal{P} = S_p(m) \cup \{t : t|p\}$ *for some* $m \in \mathbb{N} \cup \{2^\infty\}$. *Then there exists a* σ*-permutation map* $F : [0,1]^p \to [0,1]^p$ *with* $\mathrm{Per}(F) = \mathcal{P}$.

Remark 6. Another form of presenting the periodic structure of σ-permutation maps can be performed by using the notation and ideas of [2] as follows. Define the blocks of periods $\mathcal{A}_{p,q} = \{n : \mathrm{lcm}(n,p) = pq\}$ and order them according to Sharkovsky's Theorem, that is, $\mathcal{A}_{p,r_1} \geqslant_s \mathcal{A}_{p,r_2}$ if and only if $r_1 \geqslant_s r_2$. The following diagram explains this:

$$\begin{array}{rcl}
\mathcal{A}_{p,3} & \prec & \mathcal{A}_{p,5} \prec \mathcal{A}_{p,7} \prec \cdots \\
\mathcal{A}_{p,2\cdot 3} & \prec & \mathcal{A}_{p,2\cdot 5} \prec \mathcal{A}_{p,2\cdot 7} \prec \cdots \\
 & \vdots & \\
\mathcal{A}_{p,2^n\cdot 3} & \prec & \mathcal{A}_{p,2^n\cdot 5} \prec \mathcal{A}_{p,2^n\cdot 7} \prec \cdots \\
 & \vdots & \\
\cdots & \prec & \mathcal{A}_{p,2^m} \prec \cdots \prec \mathcal{A}_{p,2^2} \prec \mathcal{A}_{p,2} \prec \mathcal{A}_{p,1}.
\end{array}$$

Realize that $\mathcal{A}_{p,q}$ can be described as

$$\mathcal{A}_{p,q} = \left\{ r \in \mathbb{N} : q = \frac{r}{\gcd(p,r)} \right\}.$$

and that $\mathcal{A}_{p,1}$ is precisely the set of divisors of p. At the same time, notice that if $\mathcal{A}_{p,q} \cap \mathrm{Per}(F) \neq \emptyset$, for some q non divisor of p, then automatically we have $\mathcal{A}_{p,q} \subset \mathrm{Per}(F)$. Indeed, if $r \in \mathcal{A}_{p,q} \cap \mathrm{Per}(F), q \nmid p$, then Theorem 5 and the definition of the block $\mathcal{A}_{p,q}$ get $q = \frac{r}{\gcd(p,r)} \in \mathcal{S}(m)$ for some $m \in \mathbb{N} \cup \{2^\infty\}$. If $u \in \mathcal{A}_{p,q}$, so $u \nmid p$, we also have $q = \frac{u}{\gcd(p,u)}$, therefore $u \in \mathrm{Per}(F)$. We conclude that there exists $m \in \mathbb{N} \cup \{2^\infty\}$ for which either

$$\mathrm{Per}(F) = \left(\bigcup_{m \geq_s q, q \neq 1} \mathcal{A}_{p,q} \right) \cup \{1\}$$

or

$$\mathrm{Per}(F) = \bigcup_{m \geq_s q} \mathcal{A}_{p,q}.$$

For instance, for the case of 2-dimensional permutation maps, we have the following frame of forcing on $\mathbb{N} \setminus \{2\}$ (see [5]):

$$\begin{array}{c}
\mathcal{A}_{2,3} = \{3 \Leftrightarrow 2\cdot 3\} \Rightarrow \mathcal{A}_{2,5} = \{5 \Leftrightarrow 2\cdot 5\} \Rightarrow ... \Rightarrow \mathcal{A}_{2,2n+1} = \{2n+1 \Leftrightarrow 2\cdot(2n+1)\} \Rightarrow ... \\
\mathcal{A}_{2,2\cdot 3} = \{2^2\cdot 3\} \Rightarrow \mathcal{A}_{2,2\cdot 5} = \{2^2\cdot 5\} \Rightarrow ... \Rightarrow \mathcal{A}_{2,2\cdot(2n+1)} = \{2^2\cdot(2n+1)\} \Rightarrow ... \\
\mathcal{A}_{2,2^2\cdot 3} = \{2^3\cdot 3\} \Rightarrow \mathcal{A}_{2,2^2\cdot 5} = \{2^3\cdot 5\} \Rightarrow ... \Rightarrow \mathcal{A}_{2,2^2\cdot(2n+1)} = \{2^3\cdot(2n+1)\} \Rightarrow ... \\
... \\
\mathcal{A}_{2,2^k\cdot 3} = \{2^{k+1}\cdot 3\} \Rightarrow \mathcal{A}_{2,2^k\cdot 5} = \{2^{k+1}\cdot 5\} \Rightarrow ... \Rightarrow \mathcal{A}_{2,2^k\cdot(2n+1)} = \{2^{k+1}\cdot(2n+1)\} \Rightarrow ... \\
... \\
... \Rightarrow \mathcal{A}_{2,2^k} = \{2^{k+1}\} \Rightarrow ... \Rightarrow \mathcal{A}_{2,2^2} = \{2^3\} \Rightarrow \mathcal{A}_{2,2} = \{2^2\} \Rightarrow 1.
\end{array}$$

Before giving our main result on periods of powers of $\sigma-$permutation maps, it is necessary to know the periodic structure of a direct product of interval maps.

Lemma 7. *Let $H : I^p \to I^p$ be a direct product*

$$H(x_1, x_2, \ldots, x_p) = (h_1(x_1), h_2(x_2), \ldots, h_p(x_p)),$$

where each $h_j : I \to I$ is a continuous map, $j = 1, \ldots, p$. Suppose that $\mathrm{Per}(h_j) = \mathcal{S}(m_j)$ for some $m_j \in \mathbb{N} \cup \{2^\infty\}$, $j = 1, \ldots, p$. Then

$$\mathrm{Per}(H) = \{t : t = \mathrm{lcm}(t_1, \ldots, t_p) \text{ for some } t_j \in \mathcal{S}(m_j), j = 1, \ldots, p\} = \mathcal{S}(M),$$

where M is the biggest element of $\{m_1, m_2, \ldots, m_p\}$ in the Sharkovsky's ordering.

Proof. The equality $\mathrm{Per}(H) = \{t : t = \mathrm{lcm}(t_1, \ldots, t_p)$ for some $t_j \in \mathcal{S}(m_j), j = 1, \ldots, p\}$ is immediate. Now we prove that $\mathrm{Per}(H) = \mathcal{S}(M)$, where $M = m_j$ for some j and $M \geq_{\mathrm{s}} m_i$ for all $i = 1, \ldots, p$.

Taking into account that $1 \in \mathrm{Per}(h_i), i = 1, \ldots, p$, it is straightforward to check that $\mathcal{S}(m_i) \subseteq \mathrm{Per}(H)$ for any i and in particular $\mathcal{S}(M) \subseteq \mathrm{Per}(H)$. To finish, we prove that $\mathrm{Per}(H) \subseteq \mathcal{S}(M)$. To this end, consider an element $t \in \mathrm{Per}(H)$. Then t can be described as $t = \mathrm{lcm}(t_1, \ldots, t_p)$, with $t_j = 2^{s_j} q_j \in \mathcal{S}(m_j)$, $s_j \geq 0$, $q_j \geq 1$ odd, $j = 1, \ldots, p$. Hence, $t = 2^{\max\{s_j : j=1,\ldots,p\}} \cdot \mathrm{lcm}(q_1, \ldots, q_p)$. Suppose that $M = 2^\alpha \cdot q$, where $\alpha \in (\mathbb{N} \cup \{0, \infty\})$ and $q \geq 1$ is an odd natural number. We claim that $M \geq_{\mathrm{s}} t$.

We distinguish several cases. If $\alpha = 0, q = 1$, that is, $M = 1$ the claim is true since $m_j = 1$ for any j and $t = 1$. If $\alpha > 0$ and $q = 1$ again we obviously obtain $M \geq_{\mathrm{s}} t$ since all the periods are powers of two. If $\alpha = 0$ and $q \geq 3$, so $M = q$, if in addition we have $\max\{s_j\} > 0$ then M is odd and t even and consequently $M \geq_{\mathrm{s}} t$; if $\max\{s_j\} = 0$ now either $t = \mathrm{lcm}(q_j : j = 1, \ldots, p)$ is equal to 1 or t is an odd number greater than M: in both cases, Sharkovsky's ordering gets $M \geq_{\mathrm{s}} t$. Finally, if $\alpha > 0$ and $q \geq 3$ in a similar way is a simple matter to see that $M \geq_{\mathrm{s}} t$, by analyzing the subcases $\max\{s_j\} < \alpha$ and $\max\{s_j\} \geq \alpha$ and considering the Sharkovsky's ordering. □

Theorem 8. *Let F be a $\sigma-$permutation map. Assume that $\mathrm{Per}(f_{\sigma(1)} \circ f_{\sigma^2(1)} \circ \ldots \circ f_{\sigma^p(1)}) = \mathcal{S}(q \cdot 2^r)$, with $q \geq 1$ odd and $r \in (\mathbb{N} \cup \{0, \infty\})$. Given a positive integer $k = \ell \cdot 2^s$, $\ell \geq 1$ odd, $s \geq 0$, we have:*

(1) *If $p|k$, that is, $k = p \cdot u$ for some positive integer u, then*

$$\mathrm{Per}(F^k) = \mathrm{Per}((f_{\sigma(1)} \circ f_{\sigma^2(1)} \circ \ldots \circ f_{\sigma^p(1)})^u) = \mathcal{S}(m_u),$$

where $u = \frac{k}{p} := 2^v \cdot w$, and $m_u = \widetilde{t}(w \cdot 2^v, q \cdot 2^r) \cdot 2^{\max\{r-v,0\}}$.

(2) *If $p \nmid k$, then either*

$$\mathrm{Per}(F^k) = S_p(m_k) \;\; \textit{or} \;\; \mathrm{Per}(F^k) = S_p(m_k) \cup \{t : t|p\},$$

where $\mathcal{S}(m_k)$ is the set of periods of $(f_{\sigma(1)} \circ f_{\sigma^2(1)} \circ \ldots \circ f_{\sigma^p(1)})^k$, hence $m_k = \widetilde{t}(\ell \cdot 2^s, q \cdot 2^r) \cdot 2^{\max\{r-s,0\}}$.

Proof. (1) It is straightforward to check that F^k is a direct product, that is,

$$F^k = F^{p \cdot u} = \left((f_{\sigma(1)} \circ f_{\sigma^2(1)} \circ \ldots \circ f_{\sigma^p(1)})^u \times \ldots \times (f_{\sigma(p)} \circ f_{\sigma^2(p)} \circ \ldots \circ f_{\sigma^p(p)})^u\right).$$

Then, having in mind that $\mathrm{Per}(f_{\sigma(1)} \circ f_{\sigma^2(1)} \circ \ldots \circ f_{\sigma^p(1)}) = \mathrm{Per}(f_{\sigma(j)} \circ f_{\sigma^2(j)} \circ \ldots \circ f_{\sigma^p(j)})$ for any $j \in \{1, \ldots, p\}$ (see [6]), it suffices to apply Lemma 7 and Theorem 2.

(2) If $p \nmid k$ the map F^k is again a $\sigma-$permutation map. If we denote it by

$$F^k(x_1, \ldots, x_p) = (g_{\rho(1)}(x_{\rho(1)}), \ldots, g_{\rho(p)}(x_{\rho(p)})),$$

where ρ is a suitable cyclic permutation, it is a simple task to see that $(g_{\rho(j)} \circ g_{\rho^2(j)} \circ \ldots \circ g_{\rho^p(j)}) = (f_{\sigma(j)} \circ f_{\sigma^2(j)} \circ \ldots \circ f_{\sigma^p(j)})^k$. Then the fact that $\mathrm{Per}(f_{\sigma(1)} \circ f_{\sigma^2(1)} \circ \ldots \circ f_{\sigma^p(1)})^k = \mathrm{Per}(f_{\sigma(j)} \circ f_{\sigma^2(j)} \circ \ldots \circ f_{\sigma^p(j)})^k$ for $j = 1, \ldots, p$, jointly with Theorem 5 and Theorem 2 give us the desired result. □

Final remark

Let us mention that the study of set of periods, and forcing among periods, is a theory that goes further continuous interval and circle maps. Namely, for one–dimensional spaces like trees and graphs the analysis of such forcing relationship has been developed in recent years (see e.g. [1]). For continuous maps in such spaces the relationship between the set of periods of the maps f^p, $p \in \mathbb{N}$, and f is an open question that makes sense.

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