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"WINDOWS OF SYNCHRONIZATION" AND "NON-CHAOTIC WINDOWS" *

Luís Mário Lopes[1], Sara Fernandes[2] and Clara Grácio[3]

Abstract. The dynamic of a chaotic dynamical system changes if we couple it together with another one. The changes depend on the dynamic of the systems and on the way the coupling is made. The strength of the coupling is controlled by the coupling strength constant. For some values of this constant synchronization may occur, defining a "window of synchronization". We investigate the existence of this "window" for different dynamics and different couplings. We verify that other "windows" may occur. They are related to the fact that some couplings seem to confine drastically the chaotic behavior of the dynamical systems. We investigate one of these "windows", the "fixed point non-chaotic window", for the Symmetric Linear Coupling.

Keywords. Couplings of chaotic dynamical systems, Complete synchronization, Delayed synchronization, Windows of synchronization, Non-chaotic windows

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Résumé. La dynamique d'un système dynamique chaotique change s'il est couplé avec un autre. Les variations dépendent de la dynamique des systèmes et sur la façon dont le couplage est effectué. La résistance de la liaison dépends de la constante de la force de couplage. Pour certaines valeurs de cette constante des synchronisations peuvent se produire définissant une "fenêtre de synchronisation". Nous étudions l'existence de cette "fenêtre" pour différentes dynamiques et différents couplages. Nous vérifions que d'autres "fenêtres" peuvent se produire. Elles sont liées au fait que certains couplage semblent limiter de façon drastique le comportement chaotique des systèmes dynamiques. Nous étudions une de ces "fenêtres", la "fenêtre non-chaotique du point fixe", pour le couplage linaire symétrique.

Mots clefs. Couplage de systèmes dynamiques chaotiques, and synchronization complète and synchronization avec du retard and fenêtre de synchronization, fenêtre de synchronization non-chaotic

1. Introduction

In the Nature, and also in our day life, almost everything is connected to everything. In other words, many systems are in some way coupled to others. The free behavior of a system may change drastically if it is coupled to another (or to others) even if it is an identical system. There are different models of couplings that try to reproduce the variety of interferences that two systems may cause to each other.

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[1] Department of Mathematics, ISEL, Rua Conselheiro Emídio Navarro, 1, 1959-007 Lisboa, Portugal, E-mail: llopes@adm.isel.pt

[2] Department of Mathematics, Escola Ciências e Tecnologia, Universidade de Évora, CIMA-UE, Rua Romão Ramalho, 59, 7000-585 Évora, Portugal, E-mail:saf@uevora.pt

[3] Department of Mathematics, Escola Ciências e Tecnologia, Universidade de Évora, CIMA-UE, Rua Romão Ramalho, 59, 7000-585 Évora, Portugal, E-mail:mgracio@uevora.pt

An important question when we couple together identical (or even non identical) systems is to know whether they synchronize and how they synchronize (since synchronization may occur in different ways) [2]. Synchronization may not occur at all but, when it does, it can only occur for some values of the coupling strength.

2. "Windows of synchronization" for the Symmetric Linear Coupling

We consider a discrete 1-dimensional chaotic dynamical system with the dynamic defined by the chaotic map f, i.e.

$$x(t+1) = f(x(t)), \qquad t \in \mathbb{N}_0 \tag{1}$$

and we want to investigate some features of coupling two of these systems together.

When we couple x_1 and x_2 together, an interaction term is introduced in (1) for both systems, originating the following coupled system:

$$\begin{cases} x_1(t+1) = f(x_1(t)) + c \cdot [g_{12}(x_2(t)) - g_{11}(x_1(t))] \\ x_2(t+1) = f(x_2(t)) + c \cdot [g_{21}(x_1(t)) - g_{22}(x_2(t))] \end{cases} \tag{2}$$

where $c \in \mathbb{R}$ is the global interaction strength and g_{ij} are real functions. Instead of a coupling of two systems, [1] considers a network of several interacting systems but just for $c \cdot g_{ij} = c_{ij} \cdot g$ (where $c_{ij} \in \mathbb{R}$ and g is a real function) and analyses the situation for $g = f$ and $c_{ij} = c$, that corresponds to symmetric interactions, linear in $f(x_i)$. The corresponding coupled system is

$$\begin{cases} x_1(t+1) = f(x_1(t)) + c\,(f(x_2(t)) - f(x_1(t))) \\ x_2(t+1) = f(x_2(t)) + c\,(f(x_1(t)) - f(x_2(t))) \end{cases} \tag{3}$$

We name it the Symmetric Linear Coupled System (SLCS)

Since

$$x_1(t) = x_2(t) = s(t) \tag{4}$$

with s satisfying $s(t+1) = f(s(t))$, is a solution of (3), we say that the coupled system admits a completely synchronized solution. This is true for any value of the coupling strength c but it doesn't mean that the coupled system completely synchronize for all values of c (see definition of complete synchronization hereafter, in Definition 1). In fact, for some values of c, (4) may be a non-attractive solution of (3), i.e. $(x_1(t), x_2(t))$ may not tend to $(s(t), s(t))$ even if we consider initial conditions such that $|x_1(0) - x_2(0)|$ is sufficiently small (but not zero, obviously).

Definition 1. *We say that a coupled system (2) completely synchronizes if the completely synchronized solution (4) is an exponentially stable solution of (2).*

Definition 2. *We name "window of complete synchronization" the set of values of the coupling constant c for which the coupled system completely synchronizes.*

For the SLCS, the "window of complete synchronization" isn't empty since the completely synchronized solution (4) is exponentially stable at least for $c = 0.5$. In fact, for this value of the coupling strength constant c we have

$$\begin{cases} x_1(t+1) = 0.5 \cdot f(x_1(t)) + 0.5 \cdot f(x_2(t)) \\ x_2(t+1) = 0.5 \cdot f(x_2(t)) + 0.5 \cdot f(x_1(t)) \end{cases} \Rightarrow x_1(t+1) = x_2(t+1)$$

This means that the coupled system completely synchronizes at first iterate, independently of the initial conditions. To check if it completely synchronizes for other values of c we use the following numerical approach: we calculate its iterates $(x_1(t), x_2(t))$ for t sufficiently large, namely between $t = 900$ and $t = 1000$, using random initial conditions; we use different random initial conditions for each value of c considered . We do that for four different chaotic maps. We choose maps with different behavior concerning piecewise linearity and number of extrema:

1. The tent map, which is a piecewise map with just one maximum:

$$f_1(x) = \begin{cases} 2x, x \in \left[0, \frac{1}{2}\right] \\ 2 - 2x, x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

2. The logistic map, which is the polynomial interpolation of the tent map using the vertices $x_0 = 0$, $x_1 = \frac{1}{2}$ and $x_2 = 1$ as nodes:

$$f_2(x) = 4x\,(1-x)\,, x \in [0,1]$$

3. The 3-piecewise linear map, which is a piecewise map with two extrema (one maximum and one minimum):

$$f_3(x) = \begin{cases} 2.4 \cdot x, x \in [0, x_1] \\ 1.7 - 2.4 \cdot x, x \in [x_1, x_2] \\ 2.4 \cdot x - 1.4, x \in [x_2, 1] \end{cases}, \text{ with } x_1 = \frac{17}{48} \text{ and } x_2 = \frac{31}{48}$$

4. The cubic-like map, which is the polynomial interpolation of the 3-piecewise linear map using the vertices $x_0 = 0$, x_1, x_2 and $x_3 = 1$ as nodes:

$$f_4(x) = \frac{x(x-x_2)(x-1)}{x_1(x_1-x_2)(x_1-1)} \cdot 0.85 + \frac{x(x-x_1)(x-1)}{x_2(x_2-x_1)(x_2-1)} \cdot 0.15 + \frac{x(x-x_1)(x-x_2)}{(1-x_1)(1-x_2)}$$

These maps are shown in figure 1.

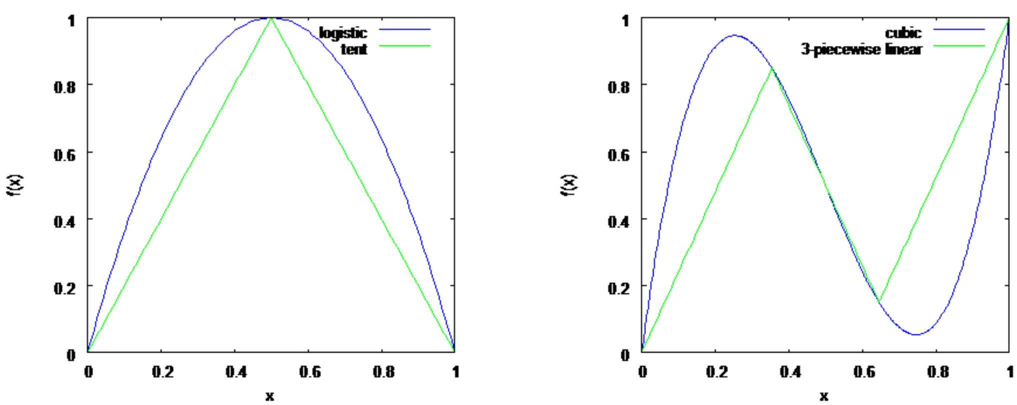

FIGURE 1. The tent and logistic maps (on the left) and the 3-piecewise linear and cubic-like maps (on the right)

Using the referred numerical approach for the SLCS and representing the difference of the iterates, $x_2(t) - x_1(t)$, as a function of c, for c in $[0,1]$ we obtain the graphs shown in figure 2. We observe that the coupled system completely synchronizes not only for $c = 0.5$ but also for values of c in a set that contains $c = 0.5$. In fact, for each of these graphs we observe there is an interval centered in $c = 0.5$ for which the only image is zero. This means that, for t sufficiently large, $x_2(t) - x_1(t) = 0 \Leftrightarrow x_2(t) = x_1(t)$, i.e. this means that for the values of c in those intervals the coupled system completely synchronizes.

We note that the "windows of complete synchronization" could contain values of c other than those that correspond to $x_2 - x_1 = 0$ in these graphs. In fact, since the initial conditions are chosen randomly, for a particular value of c corresponding to a coupled system that completely synchronizes it could happen that the initial conditions were chosen outside the basin of attraction of the completely synchronized solution. That

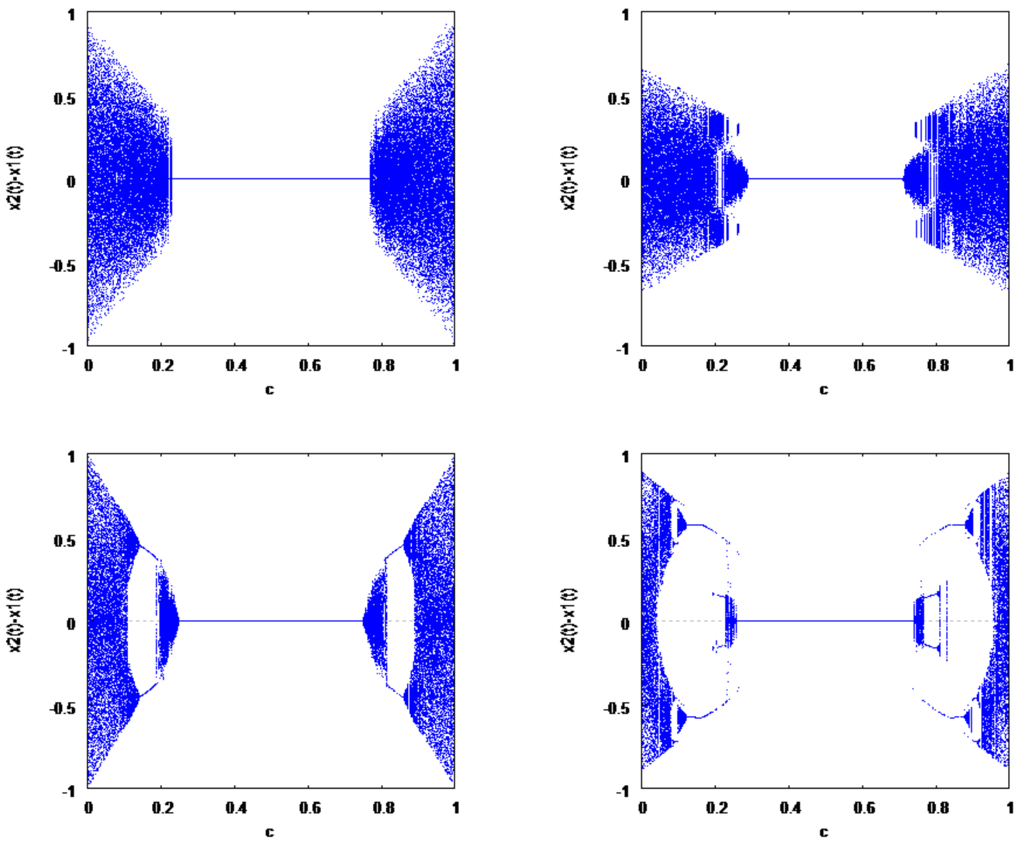

FIGURE 2. The $x_2(t) - x_1(t)$ as a function of c graphs for the tent map (top left), logistic map (bottom left), 3-piecewise linear map (top right) and cubic-like map (bottom right)

would mask that the coupled system completely synchronizes since for that value of c the graphs wouldn't show $x_2 - x_1 = 0$.

Neverthless, the obtained results are confirmed by the analytical calculations of the "window of complete synchronization". In fact, using

Proposition 3. *(A Simplified Version of a Result in [3]): Consider the dynamical network*

$$x_i(t+1) = f(x_i(t)) + c\sum_{j=1}^{N} a_{ij} f(x_j(t)), \qquad i = 1, 2, ..., N$$

Let $0 = \lambda_1 > \lambda_2 \geq ... \geq \lambda_N$ *be the eigenvalues of the coupling matrix* $A = (a_{ij})_{N\times N}$. *If* $\frac{1-e^{-h}}{|\lambda_2|} < c < \frac{1+e^{-h}}{|\lambda_N|}$, *where* h *is the Lyapunov exponent of each individual dynamical node, then the synchronized solution* $x_1(t) = x_2(t) = ... = x_N(t)$ *is exponentially stable.*

we conclude that

Proposition 4. *The "window of complete synchronization" for the SLCS is* $\left]\frac{1-e^{-h}}{2}, \frac{1+e^{-h}}{2}\right[$

Proof. Using Proposition 3, the result is obvious since the SLCS corresponds to a coupling matrix $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ whose eigenvalues are 0 and -2. ☐

Further, the definition of the Lyapunov exponent ($h = \lim_{N\to+\infty} \frac{\sum_{t=0}^{N} \ln\left|f'(x(t))\right|}{N}$) immediately provides the values of the Lyapunov exponent for the tent and 3-piecewise maps ($\ln 2$ and $\ln 2.4$, respectively) since $|f'(x)|$ is constant a.e. for these maps. Then, the corresponding "windows of complete synchronization" are $]0.25, 0.75[$ and $]0.291(6), 0.708(3)[$, respectively. The Lyapunov exponents for the logistic and tent map are identical (see [4], for instance) and so they are their "windows of complete synchronization". A simple numerical calculus provides 0.71623 as the value of the Lyapunov exponent for the cubic-like map. Then, its corresponding "window of complete synchronization" is $]0.25571, 0.74429[$.

These results confirm the ones obtained by the numerical approach.

This fact encourage us to use the same numerical approach with other couplings for which the analytical results in [3] are no longer valid. In the next section, we use it to estimate the "windows of complete synchronization" of three of those other couplings.

3. Three other couplings

Now, we consider three other ways of coupling two dynamical systems together.

In the first one - we name it the Commanded Linear Coupled System (CLCS) - we keep the linearity in $f(x_i)$ but we break the symmetry of the coupling creating a completely directed interaction: x_2 is affected by x_1, but x_1 is not affected by x_2 (x_1 is a free dynamical system):

$$\begin{cases} x_1(t+1) = f(x_1(t)) \\ x_2(t+1) = f(x_2(t)) + c\left(f(x_1(t)) - f(x_2(t))\right) \end{cases}$$

This coupling corresponds to choose $g_{11} = g_{12} = 0$ and $g_{21} = g_{22} = f$ in (2) and it admits a completely synchronized solution (4), with $s(t+1) = f(s(t))$, just like the SLCS does.

Further, it is easy to check that the "window of complete synchronization" isn't empty since the coupled system completely synchronizes for $c = 1$. In fact, for $c = 1$ we have

$$\begin{cases} x_1(t+1) = f(x_1(t)) \\ x_2(t+1) = f(x_1(t)) \end{cases} \Rightarrow x_1(t+1) = x_2(t+1)$$

This means that the coupled system completely synchronizes at first iterate. Figure 3 is the corresponding to figure 2 for the new coupling and it shows the new "windows of complete synchronization" that obviously include $c = 1$.

As second coupling we want to consider one that instead of admitting a completely synchronized solution would admit a delayed synchronized one. In order to do that we substitute the interaction term $f(x_1(t)) - f(x_2(t))$ of the previous coupling (that cancels out for the completely synchronized solution $x_1(t) = x_2(t) = s(t)$) by $f(x_1(t-1)) - f(x_2(t)) = x_1(t) - f(x_2(t))$ (that cancels out for the delayed synchronized solution $x_1(t) = x_2(t+1) = s(t)$), obtaining what we name the Commanded Coupled System with Delay (CCSD):

$$\begin{cases} x_1(t+1) = f(x_1(t)) \\ x_2(t+1) = f(x_2(t)) + c\left(x_1(t) - f(x_2(t))\right) \end{cases} \tag{5}$$

This coupled system is different from the ones considered in [1] but is an example of the ones that originate the generalized synchronization referred in [2].

The coupled system (5) only admits a completely synchronized solution if $(x_1(t), x_2(t))$ is a fixed point, i.e. $x_1(t) = x_2(t) = s(t)$ with $s(t+1) = f(s(t))$ is a solution of (5) only if $s(t+1) = s(t)$. Nevertheless, this isn't an attractive solution since f is a chaotic map. So, the coupled system doesn't completely synchronize for any

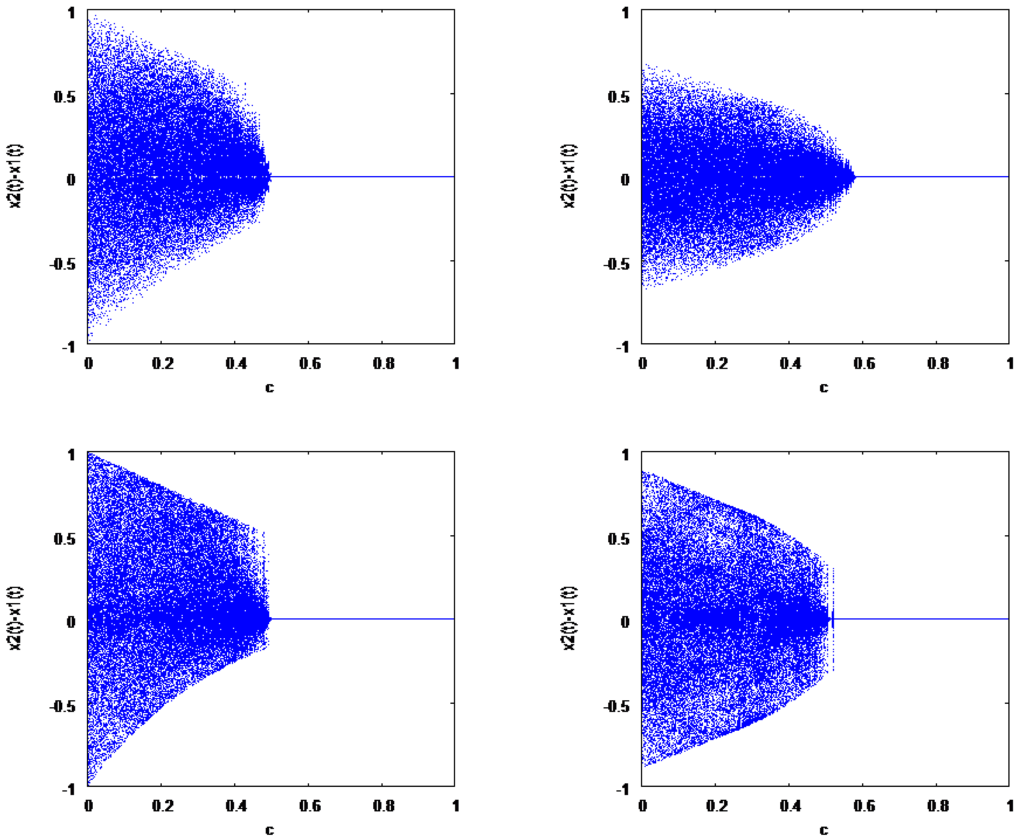

Figure 3. The $x_2(t) - x_1(t)$ as a function of c graphs for the CLCS and the tent map (top left), logistic map (bottom left), 3-piecewise linear map (top right) and cubic-like map (bottom right)

value of c and, according to that, the graphs in figure 4 (that are the corresponding of the ones in figure 3 for the CCSD) show empty "windows of complete synchronization".

On the other hand, the CCSD admits a delayed synchronized solution, i.e.

$$x_1(t) = x_2(t+1) = s(t) \tag{6}$$

with $s\,(t+1) = f\,(s\,(t))$, is a solution of (5) for any value of the coupling strength c. But, as previously for the completely synchronized solutions, this doesn't mean that the coupled system does synchronize with delay for all values of c. In fact, for some values of c, (6) may be a non-attractive solution of (5), i.e. $(x_1\,(t)\,, x_2\,(t))$ may not tend to $(s\,(t)\,, s\,(t-1))$ even if we consider initial conditions such that $|x_1\,(0) - f(x_2\,(0))|$ is sufficiently small (but not zero, obviously).

Definition 5. *We say that a coupled system (2) synchronizes with delay if the delayed synchronized solution (6) is an exponentially stable solution of (2).*

Definition 6. *We name "window of delayed synchronization" the set of values of the coupling constant c for which the coupled system synchronizes with delay.*

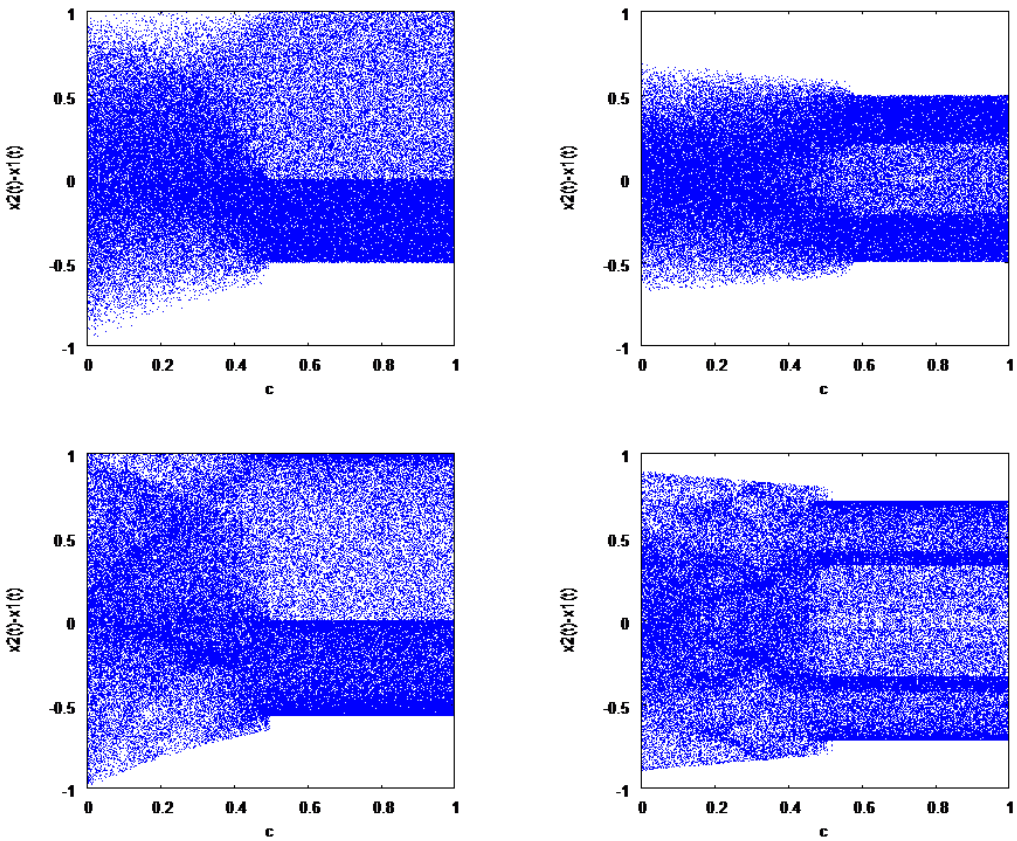

FIGURE 4. The $x_2(t) - x_1(t)$ as a function of c graphs for the CCSD and the tent map (top left), logistic map (bottom left), 3-piecewise linear map (top right) and cubic-like map (bottom right)

In order to numerically estimate the "windows of delayed synchronization", we use the same approach we used for the "windows of complete synchronization", calculating $x_2(t) - x_1(t-1)$ instead of $x_2(t) - x_1(t)$. Figure 5 is the corresponding to figure 4 for the delayed synchronization. Their graphs duplicate the ones obtained in figure 3, confirming that this coupling is a good corresponding to the CLCS for the delayed synchronization.

For the third coupling, we keep the delayed interaction term but we "close the coupling", breaking the commanded situation and returning in some way to the symmetry of (3), i.e. we consider the coupled system

$$\begin{cases} x_1(t+1) = f(x_1(t)) + c(x_2(t) - f(x_1(t))) \\ x_2(t+1) = f(x_2(t)) + c(x_1(t) - f(x_2(t))) \end{cases}$$

We name it the Symmetric Coupled System with Delay (SCSD). It admits a completely synchronized solution $x_1(t) = x_2(t) = s(t)$, with $s(t+1) = f(s(t)) + c(s(t) - f(s(t)))$, which doesn't need to be a fixed point (as it is needed in the corresponding commanded coupled system, i.e. in the CCSD). A synchronized solution with delay $x_1(t) = x_2(t+1) = s(t)$, with $s(t+1) = f(s(t)) + c(s(t-1) - f(s(t)))$, is also admissible but just if it is a period-2 solution (i.e. if $s(t+1) = s(t-1)$). Figure 6 show the graphs of $x_2(t) - x_1(t)$ as a function of c for this coupled system and figure 7 show the corresponding $x_2(t) - x_1(t-1)$ as a function of c ones.

We check that for all the maps considered there are non-empty "windows of delayed synchronization" but only the non-piecewise linear ones (i.e. the logistic and the cubic-like maps) have non-empty "windows of complete

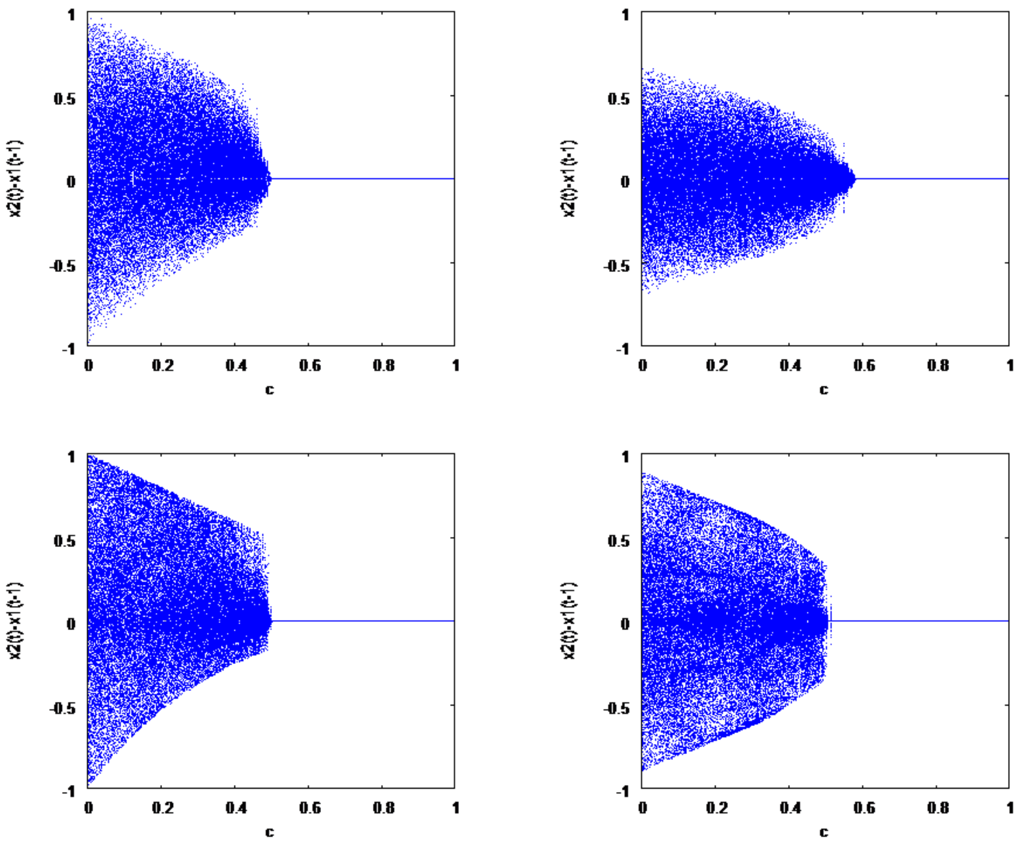

FIGURE 5. The $x_2(t) - x_1(t-1)$ as a function of c graphs for the CCSD and the tent map (top left), logistic map (bottom left), 3-piecewise linear map (top right) and cubic-like map (bottom right)

synchronization". In order to show these "windows of complete synchronization" more clearly we amplify the respective graphs around them in figure 8, using random initial conditions with $|x_1(0) - x_2(0)|$ small enough (but not zero, obviously) for they to be in the basin of attraction of the completely synchronized solution.

As we said, it was possible for the SCSD to completely synchronize having a chaotic behavior but we check that, instead of that, attractive period-2 orbits (or other non-chaotic orbits) appear in the "windows of complete synchronization". Figure 9 show that more clearly (for the logistic and the cubic-like maps we used initial conditions chosen in the same way we did for figure 8). These figures also show that for the considered maps with one extremum (the tent and the logistic maps) there is just one period-2 orbit in the "window of delayed synchronization" while for the ones with two extrema (the 3-piecewise linear map and the cubic-like map) there are two period-2 orbits (note that what seems to be a period-4 orbit is, in fact, the superposition of two period-2 orbits; the reason why both period-2 orbits appear in the graphs is related to the fact that we use different random initial conditions for each value of c).

4. "Fixed point non-chaotic windows" for the Symmetric Linear Coupling

In some of the obtained graphs $x_2(t) - x_1(t)$ as a function of c and $x_2(t) - x_1(t-1)$ as a function of c, in addition to the "windows of synchronization", other "windows" appear. We are going to characterize the ones

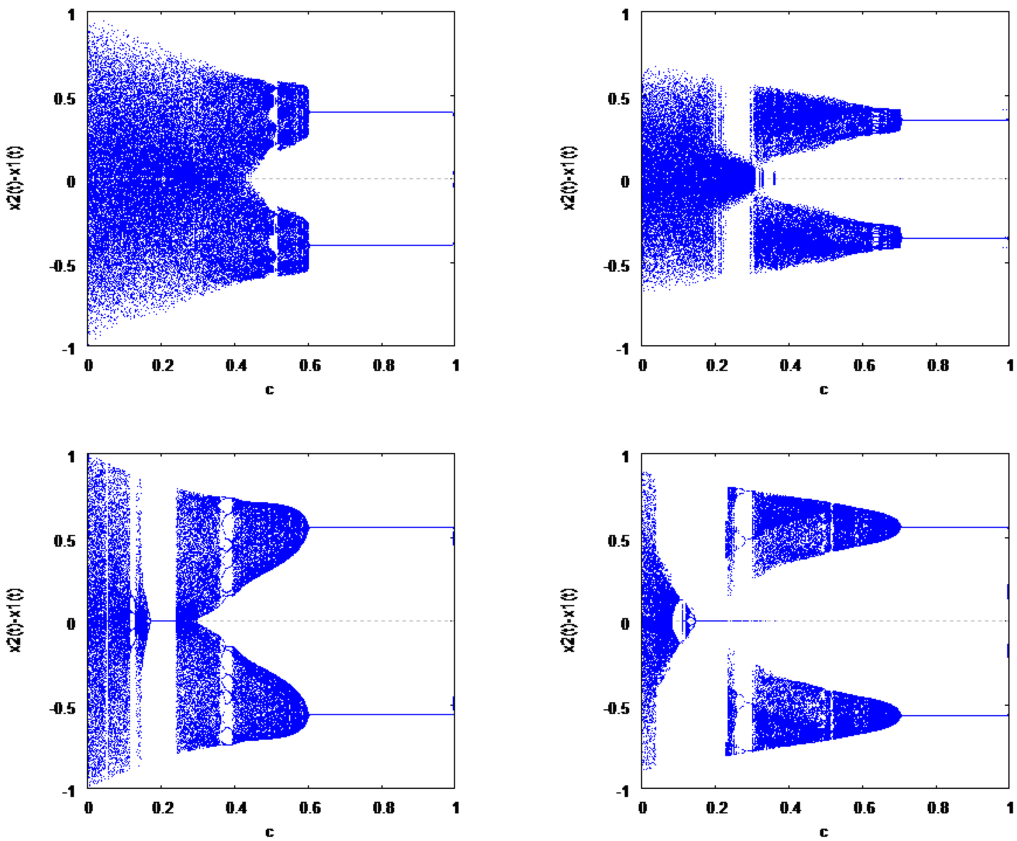

FIGURE 6. The $x_2(t) - x_1(t)$ as a function of c graphs for the SCSD and the tent map (top left), logistic map (bottom left), 3-piecewise linear map (top right) and cubic-like map (bottom right)

that appear in the Symmetric Linear Coupling and justify the existence or nonexistence of the one we call "fixed point non-chaotic window".

In figure 2 we verify that for the piecewise linear chaotic maps considered (the tent map and the 3-piecewise linear maps) only "windows of complete synchronization" appear. On the contrary, for the other two chaotic maps considered (the nonlinear ones, i.e. the logistic map and the cubic-like map), two other "windows" appear. We name "fixed point non-chaotic window" the one that appears for coupling strengths c larger than those of the "window of complete synchronization" (look at the "window" at the right side of the "window of complete synchronization" in the mentioned graphs of figure 2). We name "period-2 non-chaotic window" the one that appears for coupling strengths c smaller than those of the "window of complete synchronization" (look at the "window" at the left side of the "window of complete synchronization" in the mentioned graphs of figure 2).

The reason for these designations becomes clear if we look at (3) as a 2-dimensional dynamical system

$$\overline{x}(t+1) = F_c(\overline{x}(t))$$

with $\overline{x}(t) = (x_1(t), x_2(t))$ and

$$F_c(u,v) = ((1-c)\cdot f(u) + c\cdot f(v), (1-c)\cdot f(v) + c\cdot f(u))$$

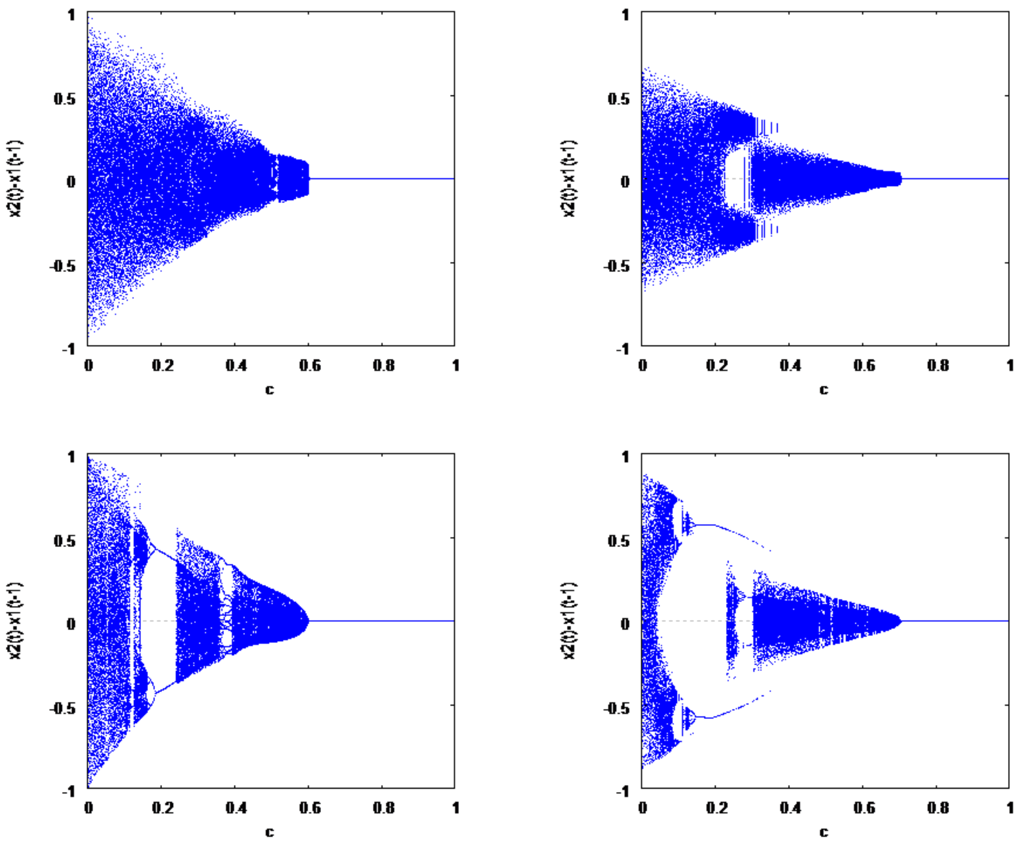

FIGURE 7. The $x_2(t)-x_1(t-1)$ as a function of c graphs for the SCSD and the tent map (top left), logistic map (bottom left), 3-piecewise linear map (top right) and cubic-like map (bottom right)

In fact, the "fixed point non-chaotic windows" corresponds to 2-dimensional dynamical systems with an attractive fixed point (x_{10}, x_{20}). This can be checked in figure 10 that show the iterates of $x_1(t)$ and $x_2(t)$ for t between 975 and 1000, for a value of c in the "fixed point non-chaotic window".

In the same way, the "period-2 non-chaotic windows" corresponds to 2-dimensional dynamical systems with an attractive period-2 orbit $\{(x_{1a}, x_{2a}), (x_{1b}, x_{2b})\}$. This can be checked in figure 11 that show the iterates of $x_1(t)$ and $x_2(t)$ for t between 975 and 1000, for a value of c in the "period-2 non-chaotic window".

In order to analytically obtain the "fixed point non-chaotic windows" and confront them with the estimates provided by figure 2, we use the definition of an attractive fixed point for a 2-dimensional dynamical system and the following proposition

Proposition 7. *Let $\overline{x_0} = (x_{10}, x_{20})$ be a fixed point of the 2-dimensional dynamical system $\overline{x}(t+1) = F(\overline{x}(t))$. Let $DF(\overline{x_0})$ be the Jacobian of F in $\overline{x_0}$. If the absolute values of both eigenvalues of $DF(\overline{x_0})$ are smaller than one then the fixed point $\overline{x_0}$ is an attractive one.*

Proof. Linearizing $\overline{x}(t+1) = F(\overline{x}(t))$ about $\overline{x_0}$, using $\overline{x}(t) = \overline{x_0} + \overline{\eta}(t)$, we have

$$\overline{x_0} + \overline{\eta}(t+1) = F(\overline{x_0}) + DF(\overline{x_0}).\overline{\eta}(t) \Leftrightarrow \overline{\eta}(t+1) = DF(\overline{x_0}).\overline{\eta}(t)$$

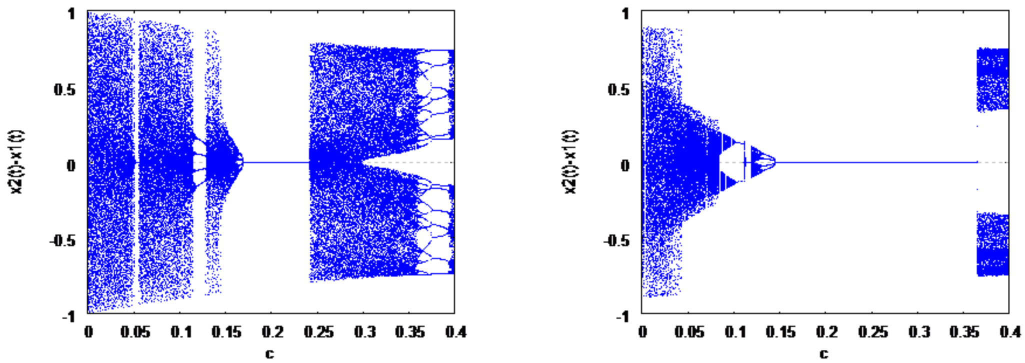

FIGURE 8. "Windows of complete synchronization" for the SCSD and the logistic map (on the left) and the cubic-like map (on the right)

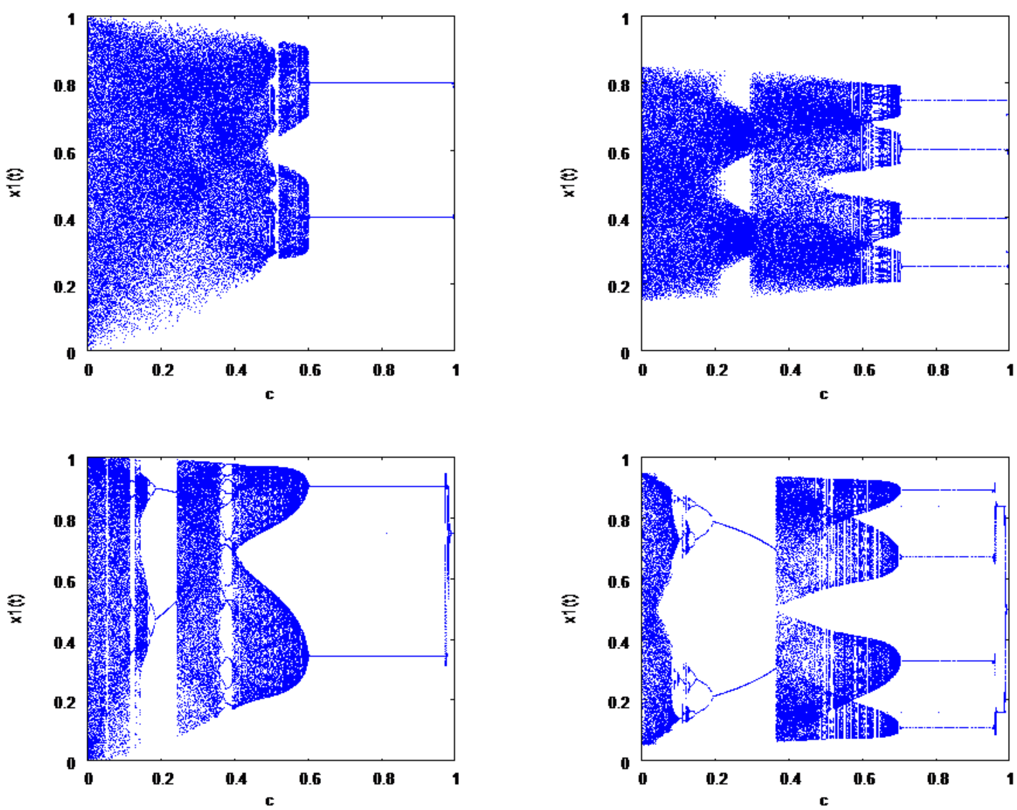

FIGURE 9. The $x_1(t)$ as a function of c graphs for the SCSD and the tent map (top left), logistic map (bottom left), 3-piecewise linear map (top right) and cubic-like map (bottom right)

Then, expanding $\overline{\eta}$ on basis $\left(\overline{\phi_1}, \overline{\phi_2}\right)$, where $\overline{\phi_1}$ and $\overline{\phi_2}$ are the generalized eigenvectors of $DF\left(\overline{x_0}\right)$, we have $\overline{\eta} = \Phi\overline{\upsilon}$, with $\Phi = \left[\overline{\phi_1}^T \overline{\phi_2}^T\right]$, and $\overline{\upsilon}$ satisfies $\overline{\upsilon}\left(t+1\right) = J\overline{\upsilon}\left(t\right)$, with $J = \Phi^{-1} \cdot DF\left(\overline{x_0}\right) \cdot \Phi$ being the Jordan canonical form of $DF\left(\overline{x_0}\right)$.

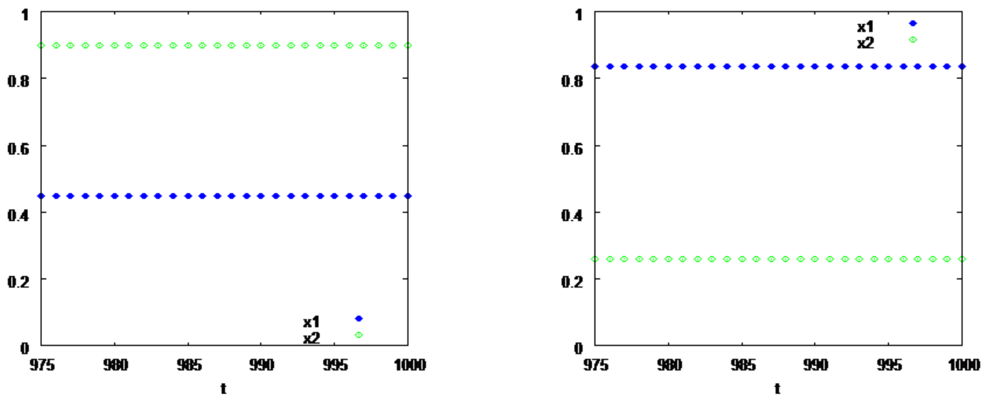

FIGURE 10. The $x_1(t)$ and $x_2(t)$ iterates for the logistic map (and $c = 0.86$, on the left) and for the cubic-like map (and $c = 0.86$, on the right)

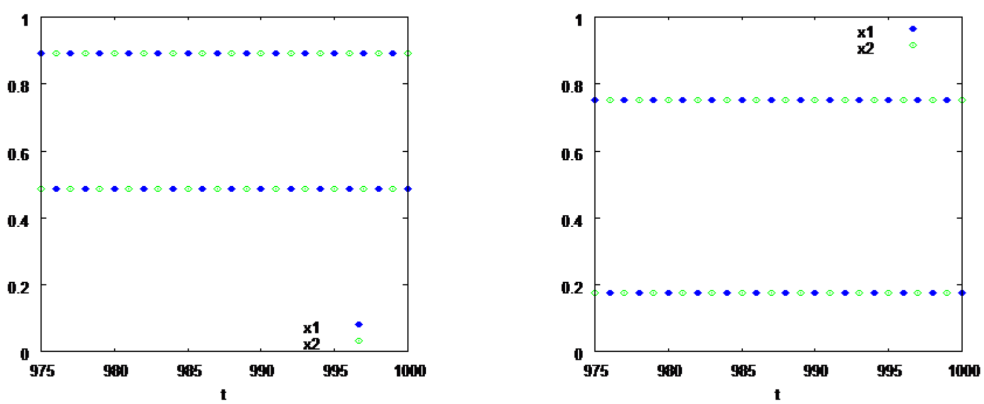

FIGURE 11. The $x_1(t)$ and $x_2(t)$ iterates for the logistic map (and $c = 0.17$, on the left) and for the cubic-like map (and $c = 0.15$, on the right)

If the geometric multiplicity of $DF(\overline{x_0})$ is two then $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $\overline{\upsilon}(t+n) = \begin{bmatrix} \upsilon_1(t+n) \\ \upsilon_2(t+n) \end{bmatrix} = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} \upsilon_1(t) \\ \upsilon_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1^n \upsilon_1(t) \\ \lambda_2^n \upsilon_2(t) \end{bmatrix}$; so, if $|\lambda_i| < 1$, for $i = 1, 2$, then $\lim\limits_{t\to+\infty} \left\|\overline{\upsilon(t)}\right\| = 0 \Rightarrow \lim\limits_{t\to+\infty} \left\|\overline{\eta(t)}\right\| = 0 \Rightarrow \lim\limits_{t\to+\infty} \overline{x}(t) = \overline{x_0}$, i.e. the fixed point $\overline{x_0}$ is an attractive one.

If the geometric multiplicity of $DF(\overline{x_0})$ is one then $J = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$ and $\overline{\upsilon}(t+n) = \begin{bmatrix} \upsilon_1(t+n) \\ \upsilon_2(t+n) \end{bmatrix} = \begin{bmatrix} \lambda_1^n & n\lambda_1^{n-1} \\ 0 & \lambda_1^n \end{bmatrix} \begin{bmatrix} \upsilon_1(t) \\ \upsilon_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1^n \upsilon_1(t) + n\lambda_1^{n-1} \upsilon_2(t) \\ \lambda_1^n \upsilon_2(t) \end{bmatrix}$; so, if $|\lambda_1| < 1$, then $\lim\limits_{t\to+\infty} \left\|\overline{\upsilon(t)}\right\| = 0 \Rightarrow \lim\limits_{t\to+\infty} \left\|\overline{\eta(t)}\right\| = 0 \Rightarrow \lim\limits_{t\to+\infty} \overline{x}(t) = \overline{x_0}$, i.e. the fixed point $\overline{x_0}$ is an attractive one. $\square$

For the SLCS each value of c corresponds to a different 2-dimensional dynamical system $\overline{x}(t+1) = F_c(\overline{x}(t))$. The fixed points $\overline{x_0} = (x_{10}, x_{20})$ of each of those systems are the solutions of $\overline{x_0} = F_c(\overline{x_0})$. Letting c assume values in $[0,1]$, this equation defines a curve in the (x_1,x_2) plan (if there's more than one fixed point for each c it may define more than one curve). In figure 12 we trace these curves for each of the four chaotic maps considered, marking in light the fixed points that are attractive. We used the previous proposition to verify if a fixed point is attractive or not. Namely, for each point in the curve we calculated the absolute values of the eigenvalues of $DF_c(\overline{x_0}) = \begin{bmatrix} (1-c)\,f'(x_{01}) & cf'(x_{02}) \\ cf'(x_{01}) & (1-c)\,f'(x_{02}) \end{bmatrix}$ and if both of them were smaller than one then we knew that that fixed point was attractive and we marked it in light. For the attractive fixed points we also registered the corresponding values of the coupling strength c.

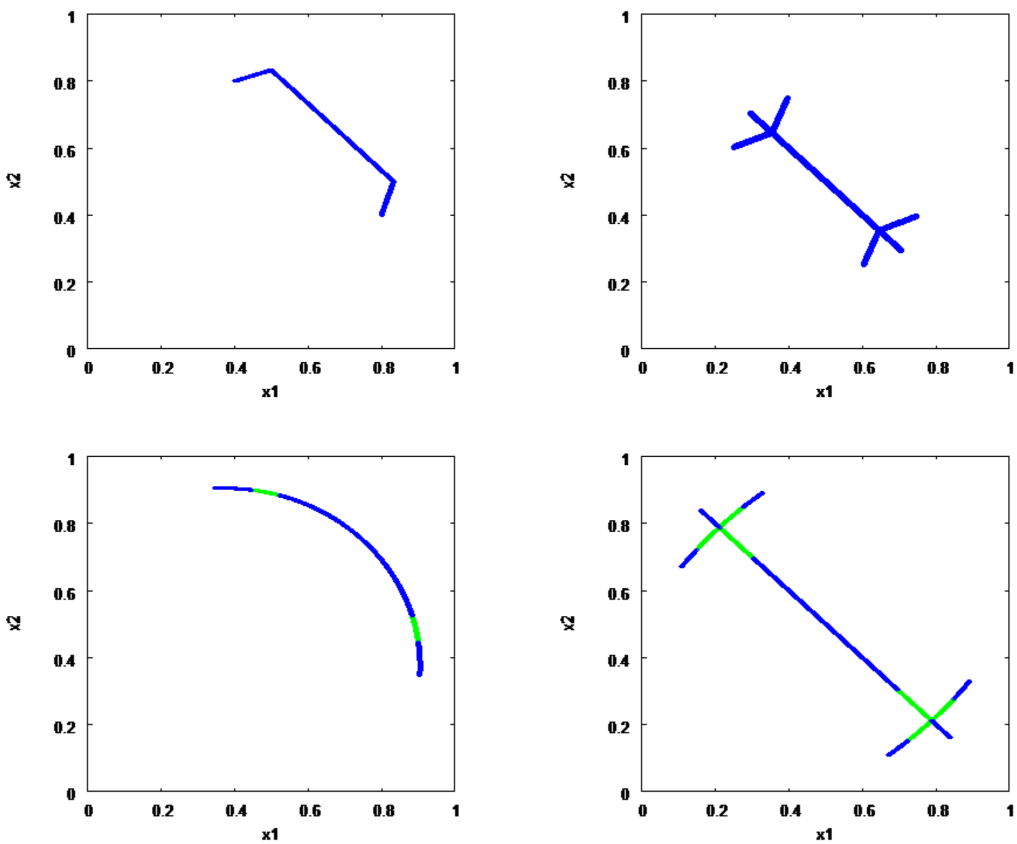

Figure 12. Fixed points for the tent map (top left), logistic map (bottom left), 3-piecewise linear map (top right) and cubic-like map (bottom right). The attractive fixed points are marked in light

We conclude that there aren't "fixed point non-chaotic windows" for the tent map nor for the 3-piecewise linear map and that for the logistic map and for the cubic-like map the "fixed point non-chaotic windows" are approximately $[0.806, 0.861]$ and $[0.734, 0.878]$, respectively. This confirms what figure 2 shows about the "fixed point non-chaotic window". To check more easily the accuracy of these intervals we amplify the "fixed point non-chaotic windows" of figure 2 and show them in figure 13. For these graphs, instead of using any random initial conditions as we did for the graphs of figure 2, we chose random initial conditions close enough to the fixed point in order to guarantee that if a fixed point was an attractive one the initial conditions would be in its basin of attraction. This way, in figure 13 the "fixed point non-chaotic windows" are "more perfect", i.e. they aren't invaded by trajectories corresponding to initial conditions outside the basin of attraction of the fixed point (trajectories that mask the existence of an attractive fixed point), as it happens in figure 2.

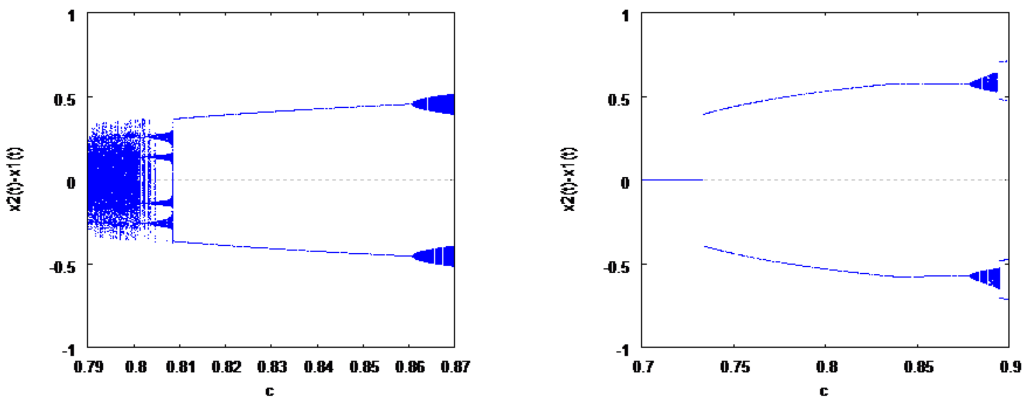

Figure 13. "Fixed point non-chaotic windows" for the logistic map (on the left) and for the cubic-like map (on the right)

5. Concluding remarks

We used a numerical approach to obtain the "windows of synchronization" of the couplings considered and to investigate if coupling together the chosen chaotic dynamical systems originates, for some values of the coupling strength, an attractive fixed-point or an attractive period-2 orbit.

For the Symmetric Linear Coupled System, that numerical approach estimated "windows of synchronization" that are confirmed by the analytical results previously obtained. We used the same numerical approach with three other couplings obtaining the corresponding "windows of synchronization". The Symmetric Linear Coupled System and the Commanded Linear Coupled System are able to completely synchronize but the Commanded Coupled System with Delay isn't. Nevertheless, the results show that the Commanded Coupled System with Delay is a good corresponding to Commanded Linear Coupled System when instead of a complete synchronization we want to consider a synchronization with delay. The Symmetric Coupled System with Delay is able to completely synchronize and also to synchronize with delay but just for the non-piecewise linear maps. For the piecewise linear maps only delayed synchronization appears. Anyway, for the Symmetric Coupled System with Delay, all the synchronizations (complete or in delay) correspond to non-chaotic behaviors.

Further, for the piecewise linear maps considered, the Symmetric Linear Coupling doesn't originate attractive fixed-point nor attractive period-2 orbits. But it does originate them for their polynomial interpolation using their vertices as nodes. For the Symmetric Linear Coupling, we analytically calculated the "fixed point non-chaotic window" and the results obtained also confirm the ones that the numerical approach provided.

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