**ESAIM: PROCEEDINGS AND SURVEYS, November 2014, Vol. 46, p. 196-212**
ECIT 2012 – Witold Jarczyk, Daniele Fournier-Prunaret, João Manuel Gonçalves Cabral

# CONVERGENCE OF ITERATES OF PRE-MEAN-TYPE MAPPINGS

Janusz Matkowski[1]

**Abstract.** Pre-mean in an interval $I$, being defined as a function $M : I^2 \to I$ such that $M(x,x) = x$ for $x \in I$, is an essential generalization of the mean. If $M$ and $N$ are pre-means, a map $(M,N) : I^2 \to I^2$ is called pre-mean-type mapping. The problem of convergence of iterates of pre-mean type mappings of the form $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$ with $s,t \in (0,1)$; $p,q \in \mathbb{R}$, $p \neq q$, where $B_{s,t}^{[p,q]} : (0,\infty)^2 \to (0,\infty)$,

$$B_{s,t}^{[p,q]} = \left(\frac{sx^p + (1-s)\,y^p}{tx^q + (1-t)\,y^q}\right)^{1/(p-q)}, \qquad x, y > 0,$$

is considered. It is proved, in particular, that for $p = 2r$, $q = r$ and $s \leq t < 2s$, the sequence of iterates at the point $(x,y)$ converges to $\left(\sqrt{xy}, \sqrt{xy}\right)$. For some $s$ and $t$ the iterates behave in "chaotic" way. An application in solving a functional equation is presented.

**Résumé.** Pré-moyenne en intervalle $I$, etant détérminé comme une fonction $M : I^2 \to I$ , telle che $M(x,x) = x$, $x \in I$, est une generalisation fondamentale de la moyenne. Si $M$ et $N$ sont les pré-moyennes, l'application $(M,N) : I^2 \to I^2$ s'appelle l'application qui pré-moyenne. Le problème de la convergence de l'itération des applications qui pré-moyennent qui ont la forme $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$ avec $s,t \in (0,1)$; $p,q \in \mathbb{R}$, $p \neq q$; $B_{s,t}^{[p,q]} : (0,\infty)^2 \to (0,\infty)$,

$$B_{s,t}^{[p,q]} = \left(\frac{sx^p + (1-s)\,y^p}{tx^q + (1-t)\,y^q}\right)^{1/(p-q)}, \qquad x, y > 0,$$

est ici considéré. On prouve, en particulier, que pour $p = 2r$, $q = r$ et $s \leq t < 2s$, une suite des itérations en un point $(x,y)$ est convergente á $\left(\sqrt{xy}, \sqrt{xy}\right)$. Pour certains $s$,$t$ les itérations se comportent d'une manière chaotique. Le résultat reçu est appliqué á résoudre une équation fonctionnelle.

[1] Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Szafrana 4a, PL-65-516 Zielona Góra, Poland, e-mail: : `J.Matkowski@wmie.uz.zgora.pl`

**Mathematics Subject Classification (2013).** Primary 26E60, 26A18 Secondary 39B12.

**Keywords.** mean, pre-mean, pre-mean-type mapping, invariant mean, invariant pre-mean, iterate, convergence, functional equation

**Les notions-clés et les phrases.** moyenne, pré-moyenne, pré-moyenne invariante, itération, convergence, équation fonctionnelle

## Introduction

It is known that, in general, the iterates of mean-type mappings display pleasant behavior. For instance [21], if $M,N : I^2 \to \mathbb{R}$ are continuous means in an interval $I \subset \mathbb{R}$ and

$$\max(M(x,y), N(x,y)) - \min(M(x,y), N(x,y)) < \max(x,y) - \min(x,y)$$

for all $x,y \in I$, $x \neq y$, then the sequence of iterates of the mean-type mapping $(M,N) : I^2 \to I^2$ converges to a mean-type mapping $(K,K)$, where $K : I^2 \to I$ is a continuous and $(M,N)$-invariant mean (cf. also [1], [4], [5], [7], [8], [10], [15]- [17], [19]- [27]). This result, important in applications, generalizes the Gauss theorem on the geometric and arithmetic means [6].

Every mean $M : I^2 \to \mathbb{R}$ in an interval $I$ is *reflexive*, that is $M(x,x) = x$ for all $x \in I$, and $M$ maps $I^2$ into $I$. A function $M$ mapping $I^2$ into $I$ and reflexive is called a *pre-mean* in $I$ (cf. [20]). It is obvious that if $M : I^2 \to I$ is a pre-mean and increasing with respect to both variables, then it is a mean ( [18], [20], [21]). However, the class of the pre-means is essentially bigger than the class of means. For instance, the function $M : (0,\infty)^2 \to (0,\infty)$, defined by

$$M(x,y) = \frac{\frac{1}{3}x^2 + \frac{2}{3}y^2}{\frac{2}{3}x + \frac{1}{3}y}, \qquad x,y > 0,$$

is a pre-mean and it is not a mean, as $M\left(\frac{1}{2},1\right) = \frac{9}{8} > \max\left(\frac{1}{2},1\right)$.

In this paper we examine the behavior of iterates of the special class of pre-mean type mappings of the form $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$ where $B_{t,s}^{[p,q]} : (0,\infty)^2 \to (0,\infty)$ is defined by

$$B_{s,t}^{[p,q]}(x,y) := \left(\frac{sx^p + (1-s)\,y^p}{tx^q + (1-t)\,y^q}\right)^{1/(p-q)}, \qquad x,y > 0,$$

for $p,q \in \mathbb{R}$, $p \neq q$, and $s,t \in (0,1)$. For $t = s$, the function $B_{s,t}^{[p,q]}$ is a well-known Gini mean (cf. for instance [3]).

In section 1 we recall some basic notions: mean, pre-mean, pre-mean-type mapping, invariant mean (pre-mean) and invariant curve. We show that the pre-mean $B_{s,t}^{[p,q]}$ is a mean if, and only if, $t = s$ or $pq \leq 0$ (Proposition 1.2). Thus, $B_{s,t}^{[p,q]}$ is not a mean if, and only if, $pq > 0$ and $t \neq s$. We also observe that for all $p,q \in \mathbb{R}$, $p \neq q$, and $s,t \in (0,1)$, the geometric mean is invariant with respect to the pre-mean-type mapping $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$ (Proposition 1.4). In general case, for each $a > 0$, the hyperbola $\mathcal{H}_a = \left\{\left(x, \frac{a^2}{x}\right) : x > 0\right\}$ is an invariant curve (Proposition 1.5). This fact is especially important and helpful in consideration of iterates in the case when $pq > 0$.

At the beginning of section 2 we conclude that in the case when $t = s$ or $pq \leq 0$, the sequence of iterates of $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$ is pointwise convergent in $(0,\infty)^2$ to the mean-type mapping $(\mathcal{G},\mathcal{G})$ (Theorem 2.1). Then we prove that, on each $\mathcal{H}_a$, the $n$-th iterate of the mapping $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$ can be expressed as the $n$-th iterate of a suitable single variable $f_{[s,t];a}^{[p,q]} : (0,\infty) \to (0,\infty)$ (Theorem 2.2). This result is the main tool in our investigations of convergence of iterates in the case $pq > 0$. However, it turns out that, in general case, it requires technically involved considerations. Therefore, in section 3, we confine an application of Theorem 2.2 to examine in detail the behavior of the sequence of iterates of this mapping in the case when $p = 2, q = 1$. Lemma 3.1, describing the iterative properties of the functions $f_{[s,t];a}^{[2,1]}$ plays here the crucial role. The most important result, Theorem 3.5, reads as follows. *If* $s,t \in (0,1)$ *are such that* $s \leq t < 2s$*, then the sequence of* *iterates of the pre-mean* $\left(B_{s,t}^{[2,1]}, B_{1-s,1-t}^{[-2,-1]}\right)$ *converges pointwise in* $(0,\infty)^2$*, and*

$$\lim_{n\to\infty}\left(B_{s,t}^{[2,1]}, B_{1-s,1-t}^{[-2,-1]}\right)^n = (\mathcal{G},\mathcal{G}),$$

*where* $G$ *is the geometric mean.* Similar convergence, in a neighborhood of the diagonal, occurs also in the case when $t = 2s-1$ (Theorem 3.7). Conversely than mean-type mappings, the iterates of the pre-mean-type mappings can behave in chaotic way. It follows from the fact that there are $s,t\in(0,1)$ such that the function $f_{[s,t];1}^{[2,1]}$ has a fixed point of the order 3 and from Theorem of Sharkovsky [29] (Remark 3.4).

If $M : I^2 \to I$ is a pre-mean in an interval $I$ and $\varphi : J \to I$ is homeomorphism, then $M^{[\varphi]} : J^2 \to J$ defined by

$$M^{[\varphi]}(u,v) = \varphi^{-1}\left(M\left(\varphi(u),\varphi(v)\right)\right), \qquad u,v\in J,$$

is a pre-mean in $J$. Since for all $r\in\mathbb{R}$, $r\neq 0$,

$$B_{s,t}^{[2r,r]} = \left(B_{s,t}^{[2,1]}\right)^{[\varphi]} \qquad \text{and} \qquad B_{1-s,1-t}^{[-2r,-r]} = \left(B_{1-s,1-t}^{[-2,-1]}\right)^{[\varphi]},$$

where $\varphi(u) = u^r$ $(u>0)$, in section 4, we show that the results of the previous section can be extended for the pre-mean-type mappings $\left(B_{s,t}^{[2r,r]}, B_{1-s,1-t}^{[-2r,-r]}\right)$.

In section 5 we apply Theorem 3.5 to determine all functions $\Phi : (0,\infty)^2 \to \mathbb{R}$, continuous at the points of diagonal, and satisfying the functional equation

$$\Phi\left(B_{s,t}^{[2r,r]}(x,y), B_{1-s,1-t}^{[-2r,-r]}(x,y)\right) = \Phi(x,y), \qquad x,y>0.$$

## 1. Some definitions, remarks and auxiliary results

Let $I\subset\mathbb{R}$ be an interval. A function $M : I^2\to\mathbb{R}$ is called a *mean in* $I$, if

$$\min(x,y) \le M(x,y) \le \max(x,y), \qquad x,y\in I;$$

$M$ is called a *strict mean*, if for all $x,y\in I$, $x\neq y$, these inequalities are sharp (cf. [2], [3]).

If $M$ is a mean in an interval $I$, then it is reflexive, i.e.

$$M(x,x) = x, \qquad x\in I,$$

and $M(J^2) = J$ for every subinterval $J\subset I$.

Note that, for a function $M : I^2\to\mathbb{R}$, the following conditions are equivalent:

*$M$ is a mean in $I$;*

*for all $x,y\in I$, $x<y$,*

$$M([x,y]\times[x,y]) = [x,y];$$

*for all $x,y\in I$, $x<y$,*

$$M([x,y]\times[x,y]) \subset [x,y].$$

Moreover, every reflexive and (strictly) increasing with respect to each variable function $M : I^2\to\mathbb{R}$ is a (strict) mean $I$ (cf. [20]; Remark 3 in [21]).

A function $M$ mapping $I^2$ into $I$ and reflexive is called a *pre-mean* in $I$.

If $M : I^2\to I$ and $N : I^2\to I$ are pre-means in $I$, then the map $(M,N) : I^2\to I^2$ such that

$$(M,N)(x,y) = (M(x,y), N(x,y)), \qquad (x,y)\in I^2,$$

is called a *pre-mean-type mapping*.

Let $M, N : I^2 \to I$ and $K : I^2 \to \mathbb{R}$ be some functions. We say that $K$ is *invariant* with respect to the map $(M,N)$ ,briefly $(M,N)$*-invariant*, if $K \circ (M,N) = K$, that is, if

$$K\left(M\left(x,y\right),N\left(x,y\right)\right) = K\left(x,y\right), \qquad x,y \in I.$$

If moreover $K$ is a pre-mean (or a mean) we say that $K$ is an $(M,N)$*-invariant pre-mean* (or $(M,N)$*-invariant mean*).

A special case of a more general result presented in [21] reads as follows.

**Remark 1.1.** *If $M, N : I^2 \to I$ are continuous means in $I$ and*

$$\max(M\left(x,y\right),N\left(x,y\right)) - \min(M\left(x,y\right),N\left(x,y\right)) < \max(x,y) - \min(x,y)$$

*for all $x, y \in I$, $x \neq y$, then the sequence of iterates $(M,N)^n$, $n \in \mathbb{N}$, of the mean-type mapping $(M,N) : I^2 \to I^2$ converges to the mean-type mapping $(K,K)$, where $K : I^2 \to I$ is a continuous and $(M,N)$-invariant mean.*

The continuity of the means $M$, $N$ and the $(M,N)$-invariance of $K$ imply its uniqueness. Moreover, if $M$ and $N$ are strict, then so is $K$ (cf. [16], [20]). Taking here $I = (0,\infty)$, $M = \mathcal{A}$ and $N = \mathcal{G}$, where $\mathcal{A}$ is the arithmetic mean and $\mathcal{G}$ is the geometric mean, we obtain a result obtained by Gauss [6].

Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to I$ is called a *pre-mean* $I$, if

$$M\left(x,x\right) = x, \qquad x \in I,$$

that is if $M$ is reflexive [20]. Of course every mean in $I$ is a pre-mean.

As we have noted in the Introduction, the class of the pre-means is essentially bigger than the class of means.

We shall consider a special class of pre-means. Given $p, q \in \mathbb{R}$, $p \neq q$, and $s, t \in (0,1)$, define $B_{t,s}^{[p,q]} : (0,\infty)^2 \to (0,\infty)$ by

$$B_{s,t}^{[p,q]}\left(x,y\right) := \left(\frac{sx^p + (1-s)\,y^p}{tx^q + (1-t)\,y^q}\right)^{1/(p-q)}, \qquad x,y > 0. \tag{1}$$

Let us note the following

**Proposition 1.2.** *Assume that $p, q \in \mathbb{R}$, $p \neq q$, and $s, t \in (0,1)$. Then*

- **(i):** $B_{s,t}^{[p,q]}$ *is a pre-mean;*
- **(ii):** $B_{s,t}^{[p,q]}$ *is a mean if, and only if, $s = t$ or $pq \leq 0$.*

*Proof.* Taking $x = y$ in (1) we get $B_{s,t}^{[p,q]}\left(x,x\right) = x$ for all $x > 0$, that is $B_{t,s}^{[p,q]}$ is reflexive, which proves part 1.

If $s = t$ then $B_{s,t}^{[p,q]}$ is a Gini mean; so in this case the "if" part in (ii) is true. If $pq \leq 0$ then, obviously, the function $B_{s,t}^{[p,q]}$ is increasing with respect to each of the variables. Since $B_{s,t}^{[p,q]}$ is reflexive, it is a strict mean (cf. [20]). To prove the "only if" result, assume that $pq > 0$ and $s \neq t$ and consider two possible cases.

**Case** $p > 0$ and $q > 0$.

If $B_{s,t}^{[p,q]}$ were a mean we would have

$$B_{s,t}^{[p,q]}\left(x,1\right) < 1 \quad \text{ and } \quad B_{s,t}^{[p,q]}\left(1,x\right) < 1 \quad \text{ for } \ x \in (0,1)\,,$$

that is

$$\left(\frac{sx^p + (1-s)}{tx^q + (1-t)}\right)^{1/(p-q)} < 1 \quad \text{and} \quad \left(\frac{s + (1-s)\,x^p}{t + (1-t)\,x^q}\right)^{1/(p-q)} < 1 \quad \text{for} \ \ x \in (0,1)\,.$$

Hence, if $p-q>0$ then

$$\frac{sx^p+(1-s)}{tx^q+(1-t)}<1 \quad \text{and} \quad \frac{s+(1-s)\,x^p}{t+(1-t)\,x^q}<1 \quad \text{for} \quad x\in(0,1)\,,$$

Letting $x\to 0$, we obtain

$$\frac{1-s}{1-t}\leq 1 \quad \text{and} \quad \frac{s}{t}\leq 1,$$

whence $s=t$, that is a contradiction. If $p<q$ then these inequalities are reversed and we again get $s=t$.

**Case** $p<0$ and $q<0$.

Assume, for the contrary, that $B_{s,t}^{[p,q]}$ is a mean. Thus, if for all positive $x,y$ such that $x<y$ we have

$$x\leq B_{s,t}^{[p,q]}\,(x,y)\leq y.$$

Since

$$B_{s,t}^{[-p,-q]}\,(x,y)=\frac{xy}{B_{s,t}^{[p,q]}\,(x,y)},$$

hence

$$x\leq B_{s,t}^{[-p,-q]}\,(x,y)\leq y,$$

and $B_{s,t}^{[-p,-q]}$ would be a mean, that contradicts the previous case. □

**Remark 1.3.** *Note that if $s\in(0,1)$ then $B_s^{[p,q]}:=B_{s,s}^{[p,q]}$ is the Gini weighted mean (cf. [22] where also another type of weighted extension of Gini means are presented).*

In the sequel $\mathcal{G}:(0,\infty)^2\to(0,\infty)$ denotes the geometric mean:

$$\mathcal{G}\,(x,y)=\sqrt{xy}, \qquad x,y>0.$$

**Proposition 1.4.** *Let $p,q\in\mathbb{R}$, $p\neq q$, and $s,t\in(0,1)$ be arbitrarily fixed. Then the geometric mean $\mathcal{G}$ is invariant with respect to the pre-mean-type mapping $\left(B_{s,t}^{[p,q]},B_{1-s,1-t}^{[-p,-q]}\right)$, that is*

$$\mathcal{G}\circ\left(B_{s,t}^{[p,q]},B_{1-s,1-t}^{[-p,-q]}\right)=\mathcal{G}.$$

.

*Proof.* Indeed, by the definition of $B_{s,t}^{[p,q]}$ we have, for all $x,y>0$,

$$\begin{aligned}
&\mathcal{G}\circ\left(B_{s,t}^{[p,q]},B_{1-s,1-t}^{[-p,-q]}\right)(x,y)\\
=\;&\sqrt{\left(\frac{sx^p+(1-s)\,y^p}{tx^q+(1-t)\,y^q}\right)^{1/(p-q)}\left(\frac{(1-s)\,x^{-p}+sy^{-p}}{(1-t)\,x^{-q}+ty^{-q}}\right)^{1/(-p+q)}}\\
=\;&\sqrt{\left(\frac{sx^p+(1-s)\,y^p}{tx^q+(1-t)\,y^q}\right)^{1/(p-q)}\left(\frac{(xy)^p}{(xy)^q}\,\frac{(1-t)\,y^q+tx^q}{(1-s)\,y^p+sx^p}\right)^{1/(p-q)}}\\
=\;&\sqrt{xy}=\mathcal{G}\,(x,y)\,.
\end{aligned}$$

□

Let $M, N : I^2 \to I$, $K : I^2 \to \mathbb{R}$, let $J \subset I$ be an interval and let $f : J \to I$ be a function. We say that the graph of the function $f$ is an *invariant curve with respect to the map* $(M, N)$, briefly, $(M, N)$*-invariant curve*, if

$$f(M(x, f(x))) = N(x, f(x)), \qquad x \in J.$$

A general problem of invariant curves, under some regularity conditions, was treated by Hadamard [9], Lattès [13], Montel [28], (cf. also Kuczma [14], p.275) and [24] where the invariant curves with respect to the mean-type mappings are considered.

**Proposition 1.5.** *Let $p, q \in \mathbb{R}$, $p \neq q$, and $s, t \in (0,1)$ be arbitrarily fixed. Then*

(1) *For every $a > 0$, the graph of the function $f : (0, \infty) \to (0, \infty)$,*

$$f(x) = \frac{a^2}{x}, \qquad x > 0,$$

*that is, the hyperbola*

$$\mathcal{H}_a := \left\{ \left( x, \frac{a^2}{x} \right) : x > 0 \right\},$$

*is $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$-invariant curve; in particular $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)(\mathcal{H}_a) \subset \mathcal{H}_a$;*

(2) *the point $(a, a)$ is a unique fixed point of the mapping $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$ restricted to the set $\mathcal{H}_a$;*

(3) *The family of sets $\{\mathcal{H}_a : a > 0\}$ forms a partition of $(0, \infty)^2$, that is $\mathcal{H}_a \cap \mathcal{H}_b = \emptyset$ for all $a, b > 0$, $a \neq b$, and $\bigcup_{a>0} \mathcal{H}_a = (0, \infty)^2$.*

*Proof.* Let us fix $a > 0$. We have

$$\mathcal{G}(x, f(x)) = \sqrt{x \frac{a^2}{x}} = a, \qquad x > 0.$$

Hence, by Proposition 1.4,

$$\mathcal{G}\left(B_{s,t}^{[p,q]}(x, f(x)), B_{1-s,1-t}^{[-p,-q]}(x, f(x))\right) = \mathcal{G}(x, f(x)) = a, \qquad x > 0,$$

that is

$$\sqrt{B_{s,t}^{[p,q]}(x, f(x))\, B_{1-s,1-t}^{[-p,-q]}(x, f(x))} = \sqrt{x f(x)} = a, \qquad x > 0.$$

Thus

$$\frac{a^2}{B_{s,t}^{[p,q]}(x, f(x))} = B_{1-s,1-t}^{[-p,-q]}(x, f(x)), \qquad x > 0,$$

or, equivalently,

$$f\left(B_{s,t}^{[p,q]}(x, f(x))\right) = B_{1-s,1-t}^{[-p,-q]}(x, f(x)), \qquad x > 0,$$

which completes the proof of the first result. The second result follows from the definition of $\mathcal{H}_a$ and part 2 of Proposition 1.4. Part 3 is obvious. □

## 2. Convergence of the sequences iterates of the pre-mean-type mappings $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$

From Proposition 1.2, Proposition 1.4 and Remark 1.1, we immediately obtain the following

**Theorem 2.1.** *Assume that $p, q \in \mathbb{R}$, $p \neq q$, and $s, t \in (0,1)$. If $s = t$ or $pq \leq 0$ then the sequence $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)^n$, $n \in \mathbb{N}$, is pointwise convergent in $(0,\infty)^2$ and*

$$\lim_{n\to\infty} \left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)^n (x,y) = \left(\sqrt{xy}, \sqrt{xy}\right), \qquad x, y > 0.$$

We shall see that in the case when $pq > 0$ and $s \neq t$ (not covered by Theorem 2.1) the convergence problem of the sequence of iterates of $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$ is much more difficult and complex.

Note that, in view of Proposition 1.5, this problem can be reduced to study the convergence of the sequence of the restrictions of this map to the invariant sets $\mathcal{H}_a$, for $a > 0$. The following results plays important role in the farther investigations.

**Theorem 2.2.** *Let $p, q \in \mathbb{R}$, $p \neq q$, and $s, t \in (0,1)$ be arbitrarily fixed. Then, for every $a > 0$ and $n \in \mathbb{N}$,*

$$\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)^n \left(x, \frac{a^2}{x}\right) = \left(\left(f_{[s,t];a}^{[p,q]}\right)^n (x), \frac{a^2}{\left(f_{[s,t];a}^{[p,q]}\right)^n (x)}\right), \qquad x > 0, \tag{2}$$

*where $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)^n$ denotes the $n$-th iterate of the mapping $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$ and $\left(f_{[s,t];a}^{[p,q]}\right)^n$ is the $n$-th iterate of the single variable function $f_{[s,t];a}^{[p,q]} : (0,\infty) \to (0,\infty)$ defined by*

$$f_{[s,t];a}^{[p,q]}(x) := B_{s,t}^{[p,q]}\left(x, \frac{a^2}{x}\right), \qquad x > 0,$$

*that is*

$$f_{[s,t];a}^{[p,q]}(x) = \left(\frac{sx^p + (1-s)\left(\frac{a^2}{x}\right)^p}{tx^q + (1-t)\left(\frac{a^2}{x}\right)^q}\right)^{1/(p-q)}, \qquad x > 0. \tag{3}$$

*Proof.* Put, for the simplicity of notations, $B = B_{s,t}^{[p,q]}$ and $f_a = f_{[s,t];a}^{[p,q]}$. Making use of the $\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)$-invariance of the geometric mean (Proposition 1.4), we have

$$\left(B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]}\right)(x,y) = \left(B(x,y), \frac{xy}{B(x,y)}\right), \qquad x, y > 0.$$

For $n=1$ formula (2) is obvious. Suppose that (2) holds true for some $n\in\mathbb{N}$. Then, for all $x>0$,

$$\begin{aligned}\left(B_{s,t}^{[p,q]},B_{1-s,1-t}^{[-p,-q]}\right)^{n+1}\left(x,\frac{a^2}{x}\right) &= \left(B_{s,t}^{[p,q]},B_{1-s,1-t}^{[-p,-q]}\right)^{n}\left(B\left(x,\frac{a^2}{x}\right),\frac{x\frac{a^2}{x}}{B\left(x,\frac{a^2}{x}\right)}\right)\\ &= \left(f_a^n\left(B\left(x,\frac{a^2}{x}\right)\right),\frac{a^2}{f_a^n\left(B\left(x,\frac{a^2}{x}\right)\right)}\right)\\ &= \left(f_a^n\left(f_a(x)\right),\frac{a^2}{f_a^n\left(f_a(x)\right)}\right)=\left(f_a^{n+1}(x),\frac{a^2}{f_a^{n+1}(x)}\right),\end{aligned}$$

and the induction completes the proof. □

From (3) we easily get the following

**Remark 2.3.** *Under the conditions of Theorem 2.2, for every $a>0$, we have*

$$f_{[s,t];a}^{[p,q]}(a)=a,\qquad \left(f_{[s,t];a}^{[p,q]}\right)'(a)=\frac{(2s-1)\,p-(2t-1)\,q}{p-q}.$$

Thus, $a$ is a fixed point of $\left(f_{[s,t];a}^{[p,q]}\right)$, for every $a>0$. Moreover $\left(f_{[s,t];a}^{[p,q]}\right)'(a)$, the "multiplier" of the fixed point $a$ of the function $f_{[s,t];a}^{[p,q]}$, does not depend on $a$. It follows that the following result holds true.

**Corollary 2.4.** *Let $p,q\in\mathbb{R}$, $p\neq q$, and $s,t\in(0,1)$ be arbitrarily fixed. If*

$$\left|\frac{(2s-1)\,p-(2t-1)\,q}{p-q}\right|<1,$$

*then there is an open set $U\subset(0,\infty)^2$ containing the diagonal $\Delta:=\{(x,x):x>0\}$ such that*

$$\lim_{n\to\infty}\left(B_{s,t}^{[p,q]},B_{1-s,1-t}^{[-p,-q]}\right)^{n}(x,y)=\left(\sqrt{xy},\sqrt{xy}\right),\qquad (x,y)\in U.$$

In the sequel we are going to consider the problem of convergence of the iterates of $\left(B_{s,t}^{[p,q]},B_{1-s,1-t}^{[-p,-q]}\right)$ in some special cases.

## 3. Convergence in the case $p=2,\ q=1$

In this section we describe the convergence of the sequence iterates of the pre-mean-type mapping $\left(B_{s,t}^{[2,1]},B_{1-s,1-t}^{[-2,-1]}\right)$. Note that

$$\begin{aligned}B_{s,t}^{[2,1]}(x,y) &= \frac{sx^2+(1-s)\,y^2}{tx+(1-t)\,y},\\ B_{1-s,1-t}^{[-2,-1]}(x,y) &= \left(\frac{(1-s)\,x^{-2}+sy^{-2}}{(1-t)\,x^{-1}+ty^{-1}}\right)^{-1}=\frac{xy}{B_{s,t}^{[2,1]}(x,y)},\end{aligned}$$

and, assuming the notation

$$f_{[s,t];a}=f_{[s,t];a}^{[2,1]},$$

we have

$$f_{[s,t];a}(x) = \frac{sx^4 + (1-s)a^4}{x\left[tx^2 + (1-t)a^2\right]}, \qquad x > 0. \tag{4}$$

**Lemma 3.1.** *Let $s, t \in (0,1)$, and $a > 0$ be arbitrarily fixed. Then*

**(i):** *the function $f_{[s,t];a} : (0,\infty) \to (0,\infty)$ is of the class $C^\infty$,*

$$\lim_{x\to 0+} f_{[s,t];a}(x) = \lim_{x\to\infty} f_{[s,t];a}(x) = \infty, \qquad \lim_{x\to\infty} \frac{f_{[s,t];a}(x)}{x} = \frac{s}{t},$$

$$f_{[s,t];a}(a) = a, \qquad f'_{[s,t];a}(a) = 4s - 2t - 1; \tag{5}$$

**(ii):** *there exists a unique $z_a = z_a(s,t) > 0$ such that $f_{[s,t];a}$ is strictly decreasing in $(0, z_a]$ and strictly increasing in $[z_a, \infty)$; moreover, the function $(a,s,t) \longmapsto z_a(s,t)$ is continuous in $(0,\infty)\times(0,1)^2$;*

**(iii):** *the point $a$ is the only fixed point of $f_{[s,t];a}$ if, and only if,*
*$t = 2s - 1$ or $s < t$;*
*moreover,*
*if $t = 2s-1$, then $t < s$,*

$$f'_{[s,t];a}(a) = 1,$$

*there exists a unique for $\xi_a = \xi_a(s) \in (0,a)$ such that $f_{[s,2s-1];a}(\xi_a) = a$,*

$$f_{[s,2s-1];a}(x) < a \Leftrightarrow x \in (\xi_a, a); \qquad f_{[s,2s-1];a}(x) > \max(a,x) \quad \text{for } x \in (0,\xi_a)\cup(a,\infty);$$

*and the function $(a,s) \longmapsto \xi_a(s)$ is continuous;*
*if $s < t < 2s$ then $-1 < f'_{[s,t];a}(a) < 2s-1$, in particular, $\left|f'_{[s,t];a}(a)\right| < 1$;*
*if $t = 2s$ then $f'_{[s,t];a}(a) = -1$;*
*if $2s < t$ then $f'_{[s,t];a}(a) < -1$;*

**(iv):** *if $t < s$ and $t \neq 2s-1$, then*

$$a_1 := \sqrt{\frac{1-s}{s-t}}a$$

*is the only fixed point of $f_{[s,t];a}$ different from $a$; moreover,*
*if $t < 2s-1$ then $s > \frac{1}{2}$, $a_1 < a$, $f'_{[s,t];a}(a) > 2s$, in particular, $f'_{[s,t];a}(a) > 1$;*

$$f_{[s,t];a}(x) < x \quad \text{for } x \in (a_1, a); \qquad f_{[s,t];a}(x) > a \quad \text{for } x < a_1; \qquad f_{[s,t];a}(x) > x \quad \text{for } x > a;$$

*and*

$$\left|f'_{[s,t];a}(a_1)\right| < 1 \quad \text{iff} \quad s(2t-1) - t^2 < 0;$$

*if $t > 2s-1$ then $a_1 > a$,*

$$-1 < 2s - 1 < f'_{[s,t];a}(a) < 1$$

*in particular, $\left|f'_{[s,t];a}(a)\right| < 1$, and*

$$f_{[s,t];a}(x) > a \quad \text{for } x \in (0,a); \qquad f_{[s,t];a}(x) < x \quad \text{for } x \in (a, a_1); \qquad f_{[s,t];a}(x) > x \quad \text{for } x > a;$$

*Proof.* Part (i) follows from (4) and Remark 2.3. To prove (ii) note that

$$f'_{[s,t];a}(x) = \frac{stx^6 + 3a^2s(1-t)x^4 - 3a^4(1-s)tx^2 - a^6(1-s)(1-t)}{x^2\left[tx^2 + (1-t)a^2\right]^2} = 0$$

for an $x > 0$ iff

$$\varphi(z) = stz^3 + 3a^2 s(1-t) z^2 - 3a^4(1-s) tz - a^6 (1-s)(1-t) = 0$$

for $z = x^2$. Since the derivative

$$\varphi'(z) = 3\left[stz^2 + 2a^2 s(1-t) z - 3a^4 (1-s) t\right]$$

has only one positive root, the function $f_{[s,t];a}$ has a unique local minimum at a point $z_a(s,t) > 0$. Calculating $z = z_a(s,t)$ (or applying the implicit function theorem), we can easily check that the function $(a,s,t) \longmapsto z_a(s,t)$ is continuous in $(0,\infty) \times (0,1)^2$.

To prove (iii) note that, by (4), equation $f_{[s,t];a}(x) = x$, equivalent to

$$(s-t)x^4 - (1-t)a^2x^2 + (1-s)a^4 = 0,$$

can be written in the form

$$(x-a)(x+a)\left[x^2 - \frac{1-s}{s-t}a^2\right] = 0. \tag{6}$$

If $s < t$ then a positive $x$ satisfies this equation if, and only if, $x = a$.

If $s > t$ than $\frac{1-s}{s-t}$ is positive and

$$x = \sqrt{\frac{1-s}{s-t}}a$$

satisfies equation (6). This number coincides with $a$ if, and only if, $t = 2s-1$. Since $f_{[s,2s-1];a}(a) = a$, $f'_{[s,2s-1];a}(a) = 1$ (what is easy to verify) and $\lim_{x\to 0} f_{[s,2s-1];a}(x) = \infty$, there exists $\xi_a \in (0,a)$ such that $f_{[s,2s-1];a}(\xi_a) = a$. The uniqueness of $\xi_a$ follows from the fact that the equation $f_{[s,2s-1];a}(x) = a$, written explicitly,

$$\frac{sx^4 + (1-s)a^4}{x\left[(2s-1)x^2 + 2(1-s)a^2\right]} = a$$

has a unique solution in the interval $(0,a)$. Since this equation can be equivalently written in the form

$$(x-a)\left[sx^3 + (1-s)ax^2 + (1-s)a^2x - (1-s)a^3\right] = 0,$$

it is enough to show that there is a unique $x = \xi_a$ in the interval $(0,a)$ satisfying the equation

$$h(x) := sx^3 + (1-s)ax^2 + (1-s)a^2x - (1-s)a^3 = 0.$$

As $h(0) = -(1-s)a^3 < 0$ and $h(a) = a^3 > 0$, the existence of $\xi_a \in (0,a)$ follows from the Darboux property. Since $s \in (0,1)$, it easy to check that

$$h'(x) = 3sx^2 + 2(1-s)ax + (1-s)a^2 > 0, \qquad x \in \mathbb{R},$$

whence $h$ is strictly increasing. It follows that $\xi_a$ is unique.

Hence, taking into account that

$$\lim_{x\to\infty} \frac{f_{[s,2s-1];a}(x)}{x} = \frac{s}{t} > 1,$$

we conclude that $f_{[s,2s-1];a}(x) > x$ for $x > 0$, $x \neq \xi_a$.

The remaining properties of $\xi_a$ as well as the continuity of the function $(a,s) \longmapsto \xi_a(s)$ is obvious.

If $s < t < 2s$ then $-2s < -2t < -4s$, and from (5), $-1 < f'_{[s,t];a}(a) < 2s-1$. The estimations of this derivative in the remaining cases are similar.

This completes the proof of (iii).

If $t < 2s - 1$ then $-2t > -4s + 2$ and from (5) we get

$$f'_{[s,t];a}(a) = 4s - 2t - 1 > 4s + (-4s + 2) - 1 = 1.$$

Since

$$f_{[s,t];a}(x) - x = \frac{(s-t)\left(x^2\right)^2 - (1-t)\,a^2x^2 + (1-s)\,a^4}{x\left(tx^2 + (1-t)\,a^2\right)},$$

$f_{[s,t];a}(a_1) - a_1 = 0$, $f_{[s,t];a}(a) - a = 0$ and $s - t > 0$, the numerator takes negative values for $x \in (a_1, a)$, and, consequently,

$$f_{[s,t];a}(x) < x, \qquad x \in (a_1, a).$$

The inequality $f_{[s,t];a}(x) > x$ for $x > a$ follows from the inequality $f'_{[s,t];a}(a) > 1$. We have

$$f'_{[s,t];a}(a_1) = \frac{4s^2 - s(4t+3) + t(t+2)}{s(2t-1) - t^2}.$$

Hence the inequality

$$\left|f'_{[s,t];a}(a_1)\right| < 1$$

is satisfied iff

$$f'_{[s,t];a}(a_1) - 1 = \frac{2(s-t)(2s-t-1)}{s(2t-1) - t^2} < 0$$

and

$$f'_{[s,t];a}(a_1) + 1 = \frac{2(s-1)(2s-t)}{s(2t-1) - t^2} > 0.$$

Since $s - 1 < 0$, $s > t$ and $t < 2s - 1$, these inequalities are satisfied iff

$$s(2t-1) - t^2 < 0.$$

If $t > 2s - 1$ then, in this case, $-2s < -2t < -4s + 2$, whence, by (5)

$$2s - 1 < f'_{[s,t];a}(a) = 4s - 2t - 1 < 1.$$

The argument in the remaining part is analogous. □

**Lemma 3.2.** *Let $s, t \in (0,1)$ such that $s \leq t < 2s$ be arbitrarily fixed. Then, for all $a, x > 0$,*

$$\lim_{n\to\infty}\left(f^{[2,1]}_{[s,t];a}\right)^n(x) = a.$$

*Proof.* Take arbitrary $a > 0$.In view of Lemma 3.1, $a$ is the only fixed point of the function $f^{[2,1]}_{[s,t];a}$. We shall show that

$$-1 < \frac{f^{[2,1]}_{[s,t];a}(x) - a}{x - a} < 1, \qquad x > a. \tag{7}$$

From (4), after some calculations, we get

$$\frac{f^{[2,1]}_{[s,t];a}(x) - a}{x - a} - 1 = \frac{(x+a)\left[(s-t)\,x^2 + (s-1)\,a^2\right]}{x\left[tx^2 + (1-t)\,a^2\right]}, \qquad x > 0,\ x \neq a.$$

Since, by assumption, $s - t \leq 0$ and $s - 1 < 0$, it follows that

$$\frac{f^{[2,1]}_{[s,t];a}(x) - a}{x - a} - 1 < 0, \qquad x > a,$$

which proves the second of inequalities (7). Note that

$$\frac{f_{[s,t];a}^{[2,1]}(x)-a}{x-a}+1=\frac{h(x)}{x\left[tx^2+(1-t)a^2\right]},\qquad x>0, x\neq a,$$

where

$$h(x):=(s+t)x^3+(s-t)ax^2+(s-2t+1)a^2x+(s-1)a^3.$$

Since

$$h'(x)=3(s+t)x^2+2(s-t)ax+(s-2t+1)a^2,$$

we have, for $x>a$,

$$\begin{aligned} h'(x) &> 3(s+t)ax+2(s-t)ax+(s-2t+1)a^2 \\ &= (5s+t)ax+(s-2t+1)a^2>(5s+t)a^2+(s-2t+1)a^2 \\ &= (6s-t+1)a^2>0, \end{aligned}$$

and, consequently, the function $h$ is strictly increasing in $[a,\infty)$. Since $h(a)=2(2s-t)a^4\geq 0$ and, by the assumption, $2s-t\geq 0$, we have $h(a)\geq 0$. It follows that $h$ is positive in $(0,\infty)$, which completes the proof of (7).

Inequality (7) implies that for all $x>0$,

$$\lim_{n\to\infty}\left(f_{[s,t];a}^{[2,1]}\right)^n(x)=a.$$

Since, obviously, $f((0,a))\subset(a,\infty)$, this equality remains true for all $x\in(0,\infty)$. □

**Lemma 3.3.** *Let $s\in(0,1)$. If $t=2s-1$ then $s\in\left(\frac{1}{2},1\right)$, $0<t<s$ and for every $a>0$ there exists $\xi_a\in(0,a)$ such that*

$$\lim_{n\to\infty}\left(f_{[s,t];a}^{[2,1]}\right)^n(x)=\begin{cases} a & \text{for} \quad x\in(\xi_a,a] \\ \infty & \text{for} \quad x\in[(0,\infty)\setminus(\xi_a,a]] \end{cases}.$$

*Proof.* Assume that $t=2s-1$ and fix $a>0$. Note first that

$$f_{[s,2s-1];a}^{[2,1]}(x)>x,\qquad x>0,\ x\neq a.$$

Indeed, by (4), this inequality takes the form

$$\frac{sx^4+(1-s)a^4}{x\left[(2s-1)x^2+2(1-s)a^2\right]}>x,\qquad x>0,\ x\neq a,$$

that is equivalent to the obvious inequality

$$(1-s)\left(x^2-a^2\right)^2>0,\qquad x>0,\ x\neq a.$$

By Lemma 3.1 there exists $\xi_a\in(0,a)$ such that

$$f_{[s,2s-1];a}(x)<a,\quad x\in(\xi_a,a);\quad f_{[s,2s-1];a}(\xi_a)=a;\quad f_{[s,2s-1];a}(x)>a,\quad x\in(0,\xi_a).$$

In particular we have

$$0<\frac{f_{[s,2s-1];a}^{[2,1]}(x)-a}{x-a}<1,\qquad x\in(\xi_a,a),$$

which implies that

$$\lim_{n\to\infty}\left(f_{[s,t];a}^{[2,1]}\right)^n(x)=a, \quad x\in(\xi_a,a].$$

□

**Remark 3.4.** *For $s\in(0,1)$ consider the family of functions $f_{[s,t];a}$ with $a=1$ and $t=1-s^3$ that is one-paremeter family of functions $\left\{f_{[s,1-s^3];1}:s\in(0,1)\right\}$. Obviously the function $\varphi:(0,1)\to\mathbb{R}$ defined by the formula*

$$\varphi(s)=\left(f_{[s,1-s^3];1}\right)^3(2)-2, \qquad s\in(0,1),$$

*where $\left(f_{[s,1-s^3];1}\right)^3$ denotes the third iterate of $f_{[s,1-s^3];1}$, is continuous. Making simple calculations we obtain that $\varphi\left(\frac{1}{10}\right)>1$ and $\varphi\left(\frac{2}{10}\right)<-\frac{1}{2}$. By the Darboux property of continuous functions, there is $s_0\in\left(\frac{1}{10},\frac{2}{10}\right)$ such that $\varphi(s_0)=0$ whence*

$$\left(f_{[s_0,1-s_0^3];1}\right)^3(2)=2.$$

*Since*

$$\left(f_{[s,1-s^3];1}\right)(2)-2=\frac{3\left(s^3+5s-5\right)}{2\left(4-3s^3\right)}, \qquad s\in(0,1),$$

*it is easy to verify that 2 is a fixed point of only one function of this family that is indexed by $s=s_1$, where*

$$s_1=\sqrt[3]{\frac{5\sqrt{6}}{18}+\frac{5}{8}}-\sqrt[3]{\frac{5\sqrt{6}}{18}-\frac{5}{8}}\in\left(\frac{7}{10},\frac{8}{10}\right).$$

*It follows that 2 is a fixed point of the order 3 for the function $f:=f_{[s_0,1-s_0^3];1}$. By the Sharkovsky theorem [29], the function $f$ has fixed points of all orders, which implies that the behavior of iterates of this function is extremely chaotic.*

*Setting $s:=s_0$ we have $s\in\left(\frac{1}{10},\frac{2}{10}\right)$ and $t:=1-s^3\in\left(\frac{96}{100},\frac{99}{100}\right)$ and, obviously, the inequality $s<t<2s$ is not satisfied. This shows that if the conditions of the second statement of Lemma 3.1 (iii) are not satisfied, the functions of the family $\left\{f_{[s,t];a}:s<t\right\}$ can exhibit strongly unpleasant iterative behavior.*

From Theorem 2.2 and Lemma 3.2 we immediately obtain the following

**Theorem 3.5.** *Let $s,t\in(0,1)$ such that $s\le t<2s$ be arbitrarily fixed. Then the sequence of iterates of the pre-mean $\left(B_{s,t}^{[2,1]},B_{1-s,1-t}^{[-2,-1]}\right)$ converges pointwise in $(0,\infty)^2$, and*

$$\lim_{n\to\infty}\left(B_{s,t}^{[2,1]},B_{1-s,1-t}^{[-2,-1]}\right)^n=(\mathcal{G},\mathcal{G}).$$

**Remark 3.6.** *As $B=B_{s,t}^{[p,q]}$ for $t=s$ is a mean, this result in the case $t=s$ follows from the main theorems in [15], [20], [21].*

From Theorem 2.2 and Lemma 3.3 with $a=\sqrt{xy}$, and setting $\eta(a):=\xi_a$ for $a>0$, we obtain

**Theorem 3.7.** *Let $s\in(0,1)$. If $t=2s-1$ then there exists a function $\eta:(0,\infty)\to(0,\infty)$, $\eta(z)<z$ for all $z>0$, such that*

$$\lim_{n\to\infty}\left(B_{s,t}^{[2,1]},B_{1-s,1-t}^{[-2,-1]}\right)^n(x,y)=\left(\sqrt{xy},\sqrt{xy}\right), \qquad \eta(xy)<x\le xy,\ y>0.$$

## 4. A result for pre-mean-type mappings $\left(B_{s,t}^{[2r,r]}, B_{1-s,1-t}^{[-2r,-r]}\right)$

The following is easy to verify.

**Remark 4.1.** *Let $I, J \subset \mathbb{R}$ be an intervals. For $M : I^2 \to I$ and a homeomorphism $\varphi : J \to I$ define $M^{[\varphi]} : J^2 \to J$ by*

$$M^{[\varphi]}(u,v) := \varphi^{-1}\left(M\left(\varphi(u), \varphi(v)\right)\right), \qquad u, v \in J.$$

*The $M$ is a pre-mean (mean) in $I$ if, and only if, $M^{[\varphi]} : J^2 \to J$ is a pre-mean (mean) in $J$.*

**Remark 4.2.** *Let $I \subset \mathbb{R}$ be an interval and $M, N : I^2 \to I$ be arbitrary functions. Assume that the sequence$\left((M,N)^n\right)_{n\in\mathbb{N}}$ of the iterates of the mapping $(M,N) : I^2 \to I^2$ converges and*

$$\lim_{n\to\infty}\left((M,N)^n\right) = (K, L).$$

*If $\varphi : J \to I$ is a homeomorphism, then the sequence of iterates $\left(\left(M^{[\varphi]}, N^{[\varphi]}\right)^n\right)_{n\in\mathbb{N}}$ converges and*

$$\lim_{n\to\infty}\left(\left(M^{[\varphi]}, N^{[\varphi]}\right)^n\right) = \left(K^{[\varphi]}, L^{[\varphi]}\right).$$

*Proof.* Put

$$(M_n, N_n) := (M,N)^n, \quad (M_{\varphi,n}, N_{\varphi,n}) := \left(M^{[\varphi]}, N^{[\varphi]}\right)^n, \qquad n \in \mathbb{N}.$$

We shall show that, for all $n \in \mathbb{N}$, and $u, v \in J$,

$$M_{\varphi,n}(u,v) = \varphi^{-1} \circ M_n\left(\varphi(u), \varphi(v)\right), \quad N_{\varphi,n}(u,v) = \varphi^{-1} \circ M_n\left(\varphi(u), \varphi(v)\right). \tag{8}$$

By the definition of $M^{[\varphi]}$ and $N^{[\varphi]}$ it obviously true for $n = 1$.

Assume that these equalities hold true for some $n \in \mathbb{N}$. Since

$$M_{n+1} = M_n \circ (M,N), \qquad N_{n+1} = N_n \circ (M,N), \qquad n \in \mathbb{N},$$

$$M_{n+1,\varphi} = M_{n,\varphi} \circ \left(M^{[\varphi]}, N^{[\varphi]}\right), \qquad N_{n+1,\varphi} = N_{n,\varphi} \circ \left(M^{[\varphi]}, N^{[\varphi]}\right), \qquad n \in \mathbb{N}.$$

we hence get, for all $u, v \in J$,

$$\begin{aligned} M_{\varphi,n+1}(u,v) &= M_{n,\varphi}\left(M^{[\varphi]}(u,v), N^{[\varphi]}(u,v)\right) \\ &= \varphi^{-1} \circ M_n\left(\varphi\left(M^{[\varphi]}(u,v)\right), \varphi\left(N^{[\varphi]}(u,v)\right)\right) \\ &= \varphi^{-1} \circ M_n\left(\varphi\left(\varphi^{-1}\left(M\left(\varphi(u), \varphi(v)\right)\right)\right), \varphi\left(\varphi^{-1}\left(N\left(\varphi(u), \varphi(v)\right)\right)\right)\right) \\ &= \varphi^{-1} \circ M_n\left(M\left(\varphi(u), \varphi(v)\right), N\left(\varphi(u), \varphi(v)\right)\right) \\ &= \varphi^{-1} \circ M_{n+1}\left(\varphi(u), \varphi(v)\right). \end{aligned}$$

Since the same argument shows that

$$M_{\varphi,n+1}(u,v) = \varphi^{-1} \circ M_{n+1}\left(\varphi(u), \varphi(v)\right), \qquad u, v \in J,$$

the induction completes the proof of (8).

Now our remark immediately follows from (8). □

**Remark 4.3.** *Take arbitrary $r \in \mathbb{R}$, $r \neq 0$, and put $\varphi(u) := u^r$ for $u > 0$. It is easy to see that*

$$B_{s,t}^{[2r,r]}(u,v) = \left( \frac{s(u^r)^2 + (1-s)(v^r)^2}{tv^r + (1-t)v^r} \right)^{1/r} = \left( B_{s,t}^{[2,1]} \right)^{[\varphi]}(u,v), \qquad u,v > 0,$$

*and, similarly,*

$$B_{1-s,1-t}^{[-2r,-r]}(u,v) = \left( B_{1-s,1-t}^{[-2,-1]} \right)^{[\varphi]}(u,v), \qquad u,v > 0.$$

Applying Remark 4.2, Remark 4.3 and Theorem 3.5 we obtain the following

**Theorem 4.4.** *Let $s,t \in (0,1)$ such that $s \leq t < 2s$ be arbitrarily fixed. Then the sequence of iterates of the pre-mean $\left( B_{s,t}^{[2r,r]}, B_{1-s,1-t}^{[-2r,-r]} \right)$ converges pointwise in $(0,\infty)^2$, and*

$$\lim_{n\to\infty} \left( B_{s,t}^{[2r,r]}, B_{1-s,1-t}^{[-2r,-r]} \right)^n = (\mathcal{G}, \mathcal{G}).$$

**Remark 4.5.** *To answer the question if the analogous result is true for all pre-mean type mappings $\left( B_{s,t}^{[p,q]}, B_{1-s,1-t}^{[-p,-q]} \right)$ where $p,q \in \mathbb{R}$, $p \neq q$, it is enough to decide it holds true for the pre-mean type mapping of the form*

$$\left( B_{s,t}^{[k+1,k]}, B_{1-s,1-t}^{[-k-1,-k]} \right),$$

*where $k \in \mathbb{R}$. Indeed, taking $k = \frac{q}{p-q}$ and $\varphi(u) = u^r$ with $r = p - q$ we have*

$$B_{s,t}^{[p,q]} = \left( B_{s,t}^{[k+1,k]} \right)^{[\varphi]} \quad \text{and} \quad B_{1-s,1-t}^{[-p,-q]} = \left( B_{1-s,1-t}^{[-k-1,-k]} \right)^{[\varphi]}$$

*which allows to apply Remark 4.2.*

## 5. An application

**Theorem 5.1.** *Let $s,t \in (0,1)$ such that $s \leq t < 2s$ and $r \in \mathbb{R}$ be fixed. Suppose that a function $\Phi : (0,\infty)^2 \to \mathbb{R}$ is continuous on the diagonal $\Delta := \{(x,x) : x > 0\}$. Then $\Phi$ satisfies the functional equation*

$$\Phi\left( B_{s,t}^{[2r,r]}(x,y), B_{1-s,1-t}^{[-2r,-r]}(x,y) \right) = \Phi(x,y), \qquad x,y > 0, \tag{9}$$

*if, and only if, there exists a continuous function of a single variable $\varphi : (0,\infty) \to \mathbb{R}$ such that*

$$\Phi(x,y) = \varphi(xy), \qquad x,y > 0$$

*(that is, if, and only if, there exists a continuous function of a single variable $\varphi : (0,\infty) \to \mathbb{R}$ such that*

$$\Phi = \varphi \circ \mathcal{G}.)$$

*Proof.* Assume that a function $\Phi : (0,\infty)^2 \to \mathbb{R}$ is continuous on the diagonal $\Delta$ and satisfies equation (9) that is

$$\Phi(x,y) = \Phi\left( B_{s,t}^{[2r,r]}, B_{1-s,1-t}^{[-2r,-r]} \right)(x,y), \qquad x,y > 0.$$

Hence, by induction,

$$\Phi = \Phi \circ \left( B_{s,t}^{[2r,r]}, B_{1-s,1-t}^{[-2r,-r]} \right)^n, \qquad n \in \mathbb{N},$$

where $\left(B_{s,t}^{[2r,r]}, B_{1-s,1-t}^{[-2r,-r]}\right)^n$ denotes the $n$-th iterate of the mean-type mapping $\left(B_{s,t}^{[2r,r]}, B_{1-s,1-t}^{[-2r,-r]}\right)$. In view of Theorem 4.4, for all $x, y > 0$,

$$\lim_{n\to\infty} \left(B_{s,t}^{[2r,r]}, B_{1-s,1-t}^{[-2r,-r]}\right)^n (x,y) = \left(\sqrt{xy}\sqrt{xy}\right), \qquad x, y > 0.$$

Hence, taking into account the continuity of $\Phi$ on the diagonal $\Delta(I^k)$, and setting

$$\varphi(u) := \Phi\left(\sqrt{u}, \sqrt{u}\right), \qquad \text{for} \quad u > 0,$$

we obtain, for all $x, y > 0$,

$$\begin{aligned} \Phi(x,y) &= \lim_{n\to\infty} \Phi \circ \left(B_{s,t}^{[2r,r]}, B_{1-s,1-t}^{[-2r,-r]}\right)^n (x,y) = \Phi\left(\sqrt{xy}\sqrt{xy}\right) \\ &= \varphi(xy), \end{aligned}$$

which completes the "only if" part of the theorem. The "if" part is obvious. □

The convergence of the iterates of the mean-type mappings have appeared to be useful in solving some functional equations ( [20], [21], [24]- [27] (cf. also [11] where logarithmic mean in the context some functional equations is treated). In [12] it shown that the set of homogeneous means is equipped with a metric.

**Acknowledgement.** *The author would to thank the referee for helping in improving the contents of this paper.*

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