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LOCAL WELL-POSEDNESS OF THE GENERALIZED CUCKER-SMALE MODEL WITH SINGULAR KERNELS

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Abstract. In this paper, we study the local well-posedness of two types of generalized kinetic Cucker-Smale (in short C-S) equations. We consider two different communication weights in space with nonlinear coupling of the velocities, $v|v|^{\beta-2}$ for $\beta > \frac{3-d}{2}$, where singularities are present either in space or in velocity. For the singular communication weight in space, $\psi^1(x) = 1/|x|^\alpha$ with $\alpha \in (0, d-1)$, $d \geq 1$, we consider smooth velocity coupling, $\beta \geq 2$. For the regular one, we assume $\psi^2(x) \in (L^\infty_{loc} \cap \mathrm{Lip}_{loc})(\mathbb{R}^d)$ but with a singular velocity coupling $\beta \in (\frac{3-d}{2}, 2)$. We also present the various dynamics of the generalized C-S particle system with the communication weights $\psi^i, i = 1, 2$ when $\beta \in (0, 3)$. We provide sufficient conditions of the initial data depending on the exponent β leading to finite-time alignment or to no collisions between particles in finite time.

1. Introduction

In the last years, collective behavior patterns, as a dynamic feature of autonomous agents, have received a great deal of attention from many different disciplines such as statistical physics, mathematics, biology, control theory..., due to its engineering, physical, and biological applications [2, 3, 9, 13, 21, 24, 25, 27, 28]. In this work, we focus on two particular types of generalized flocking models, among the large number of mathematical descriptions, based on Individual Based Models, dealing with interactions between individuals.

More precisely, let $f = f(x, v, t)$ be the one-particle distribution function at a spatial domain $x, v \in \mathbb{R}^d$ at time t in dimension $d > 1$. The probability density function f in phase space is determined by a Vlasov-like equation of the form

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \big[F(f)f\big] = 0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \tag{1.1}$$

subject to initial data

$$f(x, v, 0) = f^0(x, v), \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{1.2}$$

where F denotes the alignment force between particles:

$$F(f)(x, v, t) = -\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x - y)\nabla_v \phi(v - w) f(y, w, t)\, dy dw. \tag{1.3}$$

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Here, the potential function $\phi(v)$ for the velocity coupling is given by

$$\phi(v) = \frac{1}{\beta}|v|^{\beta}, \quad \text{where} \quad \beta > 0.$$

Depending on the communication weight ψ in space, we will work with two different cases:

- **(HA)** Singular communication weight and super-linear velocity coupling:

$$\psi(x) = \psi^1(x) := \frac{1}{|x|^{\alpha}} \quad \text{with} \quad \alpha \in (0, d-1) \quad \text{and} \quad \beta \geq 2.$$

- **(HB)** Regular communication weight and sub-linear velocity coupling:

$$\psi(x) = \psi^2(x) \geq 0 \quad \text{symmetric, with} \quad \psi^2 \in (L^{\infty}_{loc} \cap \mathrm{Lip}_{loc})(\mathbb{R}^d) \quad \text{and} \quad \beta \in \left(\frac{3-d}{2}, 2\right).$$

The notation for the space $\psi^2 \in \mathrm{Lip}_{loc}(\mathbb{R}^d)$ means that for any compact set $K \subset \mathbb{R}^d$, there is some constant $L_K > 0$ such that

$$|\psi^2(x) - \psi^2(y)| \leq L_K|x-y|, \quad x, y \in K.$$

Note that the alignment force of the original Cucker-Smale(in short C-S) model $F_{cs}(f)$ is given by $F_{cs}(f) = F(f)$ with $\psi = \widetilde{\psi}$ and $\beta = 2$, where

$$\widetilde{\psi}(x) = \frac{1}{(1+|x|^2)^{\gamma}}, \quad \gamma > 0.$$

For the original C-S model, rigorous asymptotic flocking estimates depending on the decay rate of the regular communication weight are discussed in [11,12]. Later, these estimates are improved and refined in the literature [7,16,18].

As particle approximations of the kinetic equations (1.1), we consider the following ODE system:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= v_i(t), \qquad (x_i(t), v_i(t)) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \\ \frac{dv_i(t)}{dt} &= \frac{1}{N}\sum_{j=1}^{N} \psi(x_j(t) - x_i(t))\frac{v_j - v_i}{|v_j - v_i|^{2-\beta}}, \quad i = 1, \cdots, N, \end{aligned} \tag{1.4}$$

with the initial data

$$(x_i(0), v_i(0)) =: (x_i^0, v_i^0).$$

Note that the kinetic equations (1.1) can be formally derived from the particle system (1.4) as the number of particles N goes to infinity.

Despite the interest of the C-S model and its variants for flocking dynamics, there are several drawbacks for real applications. Among them, our study is dealing with two issues; collision avoidance between individuals and general coupling of their velocities. For the real applications of C-S model, e.g., unmanned aerial vehicles [25], it is important to avoid any collision between the individuals. However the original C-S model [11,12] does not take into account this, and as a consequence, there are some studies to prevent the collisions by adding new forcing terms to control the distance between individuals [10, 23] or considering a singular communication weight [1]. Singular communication rates with linear velocity coupling have been treated in [1,26] dealing mainly with the existence theory and large-time behavior for the particle system (1.4). In [1], the authors identified the initial configurations preventing the pairwise collisions in a finite-time when the singularity of the communication weight is strong enough, $\alpha \geq 1$. The one-dimensional C-S particle system (1.4) was treated in [26]. The author showed existence of piecewise weak solutions when the singularity of the communication weight is sufficiently weak, $\alpha < 1$. Concerning the velocity coupling, the original C-S model has a linear coupling for velocities which can be obtained as a rigorous singular limit of a damped chain of oscillators in one dimension [17]. In this paper, we consider the nonlinear velocity coupling which is averaged over the strength of the relative speed with the exponent β.

The main results of this work are two-fold. We first establish the local well-posedness of weak solutions to the kinetic C-S equations (1.1)-(1.2), where the communication weight and velocity coupling are given by either assumptions **(HA)** or **(HB)**. In both cases, the general framework for the well-posedness of solutions can not be applied due to the singularity either in position or velocity, and it has not been addressed in the literature to the best of our knowledge. We adapted the strategy of our recent work [6] to overcome these difficulties. In case of enough regularity of the communication weight and coupling for velocity in the forcing term, the well-posedness for these kinetic equations describing the collective behavior is discussed in [4]. We also refer to [19, 20] for the issue of the derivation of mean-field Vlasov-type equations with a singular kernel.

On the other hand, we present several qualitative properties of the dynamics for the C-S particle system (1.4). In [5, 15], the formation of asymptotic flocking for the particle system (1.4) is discussed with the regular communication weight $\widetilde{\psi}$. According to the different strengths of the nonlinearity $\beta \in (0,3)$ in the velocity coupling, we provide the sufficient conditions on the initial data leading to finite-time alignment or to no collisions between particles in finite time.

The rest of this paper is organized as follows. In Section 2, we briefly provide definition and properties of Wasserstein distances, and state our main results on well-posedness and large-time behavior. Section 3 is devoted to give the details of the proof of a unique weak solution to the system (1.1) for each case. Our strategy is first to construct approximate solutions, and obtain the uniform bounds of approximate solutions with respect to the regularization. Then we finally show that the approximate solutions are Cauchy sequences, and let the parameter of regularization tend to zero to have the existence of the weak solutions. Finally, in the last section we investigate the dynamics of the generalized C-S particle system (1.4) with different strengths of the nonlinearity of the velocity coupling.

Notations: $|\cdot|$ denotes the Euclidean distance, and $\mathcal{P}_p(\mathbb{R}^d)$ stands for the set of probability measures with bounded moments of order $p \in [1,\infty)$. For notational simplicity, we also use the following notations throughout the paper: For $1 \le p \le \infty$,

$$\|g\|_{L^p} := \|g\|_{L^p(U)} \quad \text{where } U \text{ can be either } \mathbb{R}^d \text{ or } \mathbb{R}^d \times \mathbb{R}^d,$$

$$\|g\|_{L^1\cap L^p} := \|g\|_{L^1} + \|g\|_{L^p}, \quad \text{and} \quad \|g\| := \|g\|_{L^\infty(0,T;L^1\cap L^p)}.$$

2. Preliminaries and main results

2.1. Mathematical tools

In this part, we present several definition and properties of Wasserstein distances that will be mainly used in our arguments for the well-posedness.

Definition 2.1. *(Wasserstein p-distance) Let ρ_1, ρ_2 be two Borel probability measures on $\mathbb{R}^d$. Then the Euclidean Wasserstein distance of order $1 \le p < \infty$ between ρ_1 and ρ_2 is defined as*

$$d_p(\rho_1,\rho_2) := \inf_\gamma \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \, d\gamma(x,y) \right)^{1/p},$$

and, for $p = \infty$ (this is the limiting case, as $p \to \infty$),

$$d_\infty(\rho_1,\rho_2) := \inf_\gamma \left(\sup_{(x,y)\in supp(\gamma)} |x-y| \right),$$

where the infimum runs over all transference plans, i.e., all probability measures γ on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals ρ_1 and ρ_2 respectively,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x) d\gamma(x,y) = \int_{\mathbb{R}^d} \phi(x)\rho_1(x)dx,$$

and

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(y) d\gamma(x,y) = \int_{\mathbb{R}^d} \phi(y)\rho_2(y)dy,$$

for all $\phi \in \mathcal{C}_b(\mathbb{R}^d)$.

Note that $\mathcal{P}_p(\mathbb{R}^d), 1 \le p < \infty$ is a complete metric space endowed with the p-Wassertein distance d_p, see [29]. We refer to [14, 22] for more details in the case of the d_∞ distance.

In particular for $p = 1$, Wasserstein-1 distance d_1 is equivalent to the bounded Lipschitz distance which is also called *Monge-Kantorovich-Rubinstein distance*:

$$d_1(\rho_1,\rho_2) = \sup\left\{ \int_{\mathbb{R}^d} \varphi(\xi)(\rho_1(\xi)-\rho_2(\xi))d\xi \middle| \varphi \in \mathrm{Lip}(\mathbb{R}^d),\ \mathrm{Lip}(\varphi) \le 1 \right\},$$

where $\mathrm{Lip}(\mathbb{R}^d)$ and $\mathrm{Lip}(\varphi)$ denote the set of Lipschitz functions on $\mathbb{R}^d$ and the Lipschitz constant of a function φ, respectively. We also remind the definition of the push-forward of a measure by a mapping in order to give the relation between Wasserstein distances and optimal transportation.

Definition 2.2. *Let ρ_1 be a Borel measure on $\mathbb{R}^d$ and $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable mapping. Then the push-forward of ρ_1 by $\mathcal{T}$ is the measure ρ_2 defined by*

$$\rho_2(B) = \rho_1(\mathcal{T}^{-1}(B)) \quad \text{for} \quad B \subset \mathbb{R}^d,$$

and denoted as $\rho_2 = \mathcal{T}\#\rho_1$.

We recall in the next proposition some classical properties, which proofs may be found in [29].

Proposition 2.1. *(i) The definition of $\rho_2 = \mathcal{T}\#\rho_1$ is equivalent to*

$$\int_{\mathbb{R}^d} \phi(x)\, d\rho_2(x) = \int_{\mathbb{R}^d} \phi(\mathcal{T}(x))\, d\rho_1(x)$$

for all $\phi \in \mathcal{C}_b(\mathbb{R}^d)$. Given a probability measure with bounded p-th moment ρ_0, consider two measurable mappings $X_1, X_2 : \mathbb{R}^d \to \mathbb{R}^d$, then the following inequality holds:

$$d_p^p(X_1\#\rho_0, X_2\#\rho_0) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\gamma(x,y) = \int_{\mathbb{R}^d} |X_1(x)-X_2(x)|^p d\rho_0(x).$$

Here, we used as transference plan $\gamma = (X_1 \times X_2)\#\rho_0$ in Definition 2.1.

(ii) Given $\{\rho_k\}_{k=1}^N$ and ρ in $\mathcal{P}_1(\mathbb{R}^d)$, the following statements are equivalent:

- *$d_1(\rho_k,\rho) \to 0$ as $k \to +\infty$.*
- *ρ_k converges to ρ weakly-* as measures and*

$$\int_{\mathbb{R}^d} |\xi|\rho_k(\xi)d\xi \to \int_{\mathbb{R}^d} |\xi|\rho(\xi)d\xi, \quad \text{as} \quad n \to +\infty.$$

Finally, we recall *a priori* energy estimates of kinetic Cucker-Smale model.

Lemma 2.1. *Let f be any smooth solutions to the system* (1.1). *Then we have*

$$(i)\quad \frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f\,dxdv = 0, \qquad \frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} vf\,dxdv = 0,$$
$$(ii)\quad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f\,dxdv = -\frac{1}{2}\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \psi(x-y)|v-w|^\beta f(x,v,t)f(y,w,t)dxdydvdw.$$

2.2. Main results

In this part, we introduce the notion of weak solution in our frameworks, and state our main results on the well-posedness to the kinetic equations (1.1) and large-time behavior of solutions to the particle system (1.4).

We first present two cases depending on the singularity of communication weight and the strength of nonlinearity for velocity coupling.

- *Case A (Singular communication weight and super-linear velocity coupling)*: For the alignment force defined by the communication weight and velocity coupling satisfying **(HA)**, we assume that the initial data f^0 has compact support in velocity, and it belongs to L^p for some $1 < p \le \infty$, and the exponent α satisfies
$$(\alpha+1)p' < d,$$
where p' is the conjugate of p, i.e., $p' := p/(p-1)$.
- *Case B (Regular communication weight and sub-linear velocity coupling)*: For the alignment force term defined by the communication weight and velocity coupling satisfying **(HB)**, we assume that the initial data f^0 has compact support in position and velocity, it belongs to L^p for some $1 < p \le \infty$, and the exponent β of nonlinear velocity coupling satisfies
$$(3-2\beta)p' < d \ \text{ for } \ \beta \in \left(\frac{3-d}{2}, 1\right) \quad \text{and} \quad (2-\beta)p' < d \ \text{ for } \ \beta \in (1,2).$$

Definition 2.3. *For a given $T \in (0,\infty)$, f is a weak solution of* (1.1) *on the time-interval $[0,T)$ if and only if the following condition are satisfied:*

(1) $f \in L^\infty(0,T;(L^1_+ \cap L^p)(\mathbb{R}^d\times\mathbb{R}^d)) \cap \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$,
(2) *For all $\Psi \in \mathcal{C}_c^\infty(\mathbb{R}^d\times\mathbb{R}^d\times[0,T])$,*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,v,T)\Psi(x,v,T)dxdv - \int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} f\big(\partial_t\Psi + \nabla_x\Psi\cdot v + \nabla_v\Psi\cdot F(f)\big)dxdvdt$$
$$= \int_{\mathbb{R}^d\times\mathbb{R}^d} f^0(x,v)\Psi^0(x,v)dxdv,$$

where $\Psi^0(x,v) := \Psi(x,v,0)$.

We now state our first result on the local existence of a unique weak solution.

Theorem 2.1. *Suppose that either the assumptions of Case A or those of Case B hold, and the initial data f^0 satisfies*
$$f^0 \in (L^1_+ \cap L^p)(\mathbb{R}^d\times\mathbb{R}^d) \cap \mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d).$$
Then there exist $T > 0$ and unique weak solution f to the system (1.1) *in the sense of Definition* 2.3 *on the time interval $[0,T]$. Furthermore, if $f_i, i = 1,2$ are two such solutions to* (1.1), *then we have the following d_1-stability estimate.*
$$\frac{d}{dt}d_1(f_1(t),f_2(t)) \le Cd_1(f_1(t),f_2(t)), \quad \textit{for} \quad t \in [0,T],$$
where C is a positive constant.

Our second result is on the dynamics of the generalized Cucker-Smale particle system (1.4). For this, we assume further that either $\psi = \psi^1(s)$ with $\alpha \geq 1$, or $\psi = \psi^2(s)$ satisfying that

$$0 < \psi^2(s_1) \leq \psi^2(s_2) \quad \text{for} \quad 0 \leq s_2 \leq s_1 < \infty, \quad \text{and} \quad \psi^2(s) \to 0 \quad \text{as} \quad s \to \infty\,.$$

Note that the original alignment force of Cucker-Smale model satisfies the conditions of ψ^2. Without loss of generality, we may assume that

$$\sum_{k=1}^N v_i(0) = 0.$$

We also set

$$\|x(t)\|_\infty := \max_{1\leq i\leq N} |x_i(t)|, \quad \|v(t)\|_\infty := \max_{1\leq i\leq N} |v_i(t)|, \quad \text{and} \quad \eta_{m,X}(t) := \min_{1\leq i,j\leq N} |x_i(t) - x_j(t)|.$$

Then our second result is stated as follows.

Theorem 2.2. *Let (x, v) be any smooth solutions to the system (1.4) with initial data (x_0, v_0) satisfying*

$$\|x_0\|_\infty > 0, \quad \|v_0\|_\infty^{3-\beta} < \frac{(3-\beta)C_0}{2} \min\left\{\int_0^{2\|x_0\|_\infty} \psi(s)ds, \int_{2\|x_0\|_\infty}^{\infty} \psi(s)ds\right\}, \tag{2.1}$$

where C_0 is a positive constant independent of t, $\|x_0\|_\infty := \|x(0)\|_\infty$, and $\|v_0\|_\infty := \|v(0)\|_\infty$. Then the followings hold:

- *If $\beta = 2$, we have exponential alignment,*

$$\|v(t)\|_\infty \leq \|v_0\|_\infty \exp\left\{-\frac{C_0\psi(2x_M)t}{2}\right\},$$

where x_M is a positive constant defined by

$$\|v_0\|_\infty^{3-\beta} = \frac{(3-\beta)C_0}{2} \int_{2\|x_0\|_\infty}^{2x_M} \psi(s)ds.$$

Furthermore, if $\eta^0_{m,X} := \eta_{m,X}(0) > \frac{\|v_0\|_\infty}{C_0\psi(2x_M)}$, then we have no finite-time collision between particles and

$$\|v(t)\|_\infty \geq \|v_0\|_\infty \exp\left\{-\psi(\eta^*_{m,X})t\right\},$$

where $\eta^*_{m,X} := \eta^0_{m,X} - \frac{\|v_0\|_\infty}{C_0\psi(2x_M)} > 0$.

- *If $\beta \in (0, 2)$, we have finite-time alignment,*

$$\|v(t)\|_\infty \leq \left(\|v_0\|_\infty^{2-\beta} - \frac{(2-\beta)C_0\psi(2x_M)t}{2}\right)^{\frac{1}{2-\beta}}.$$

Furthermore if $\eta^0_{m,X} > T^*\|v_0\|_\infty$, then we have no collision between particles, where

$$T^* := \frac{4\|v_0\|^{2-\beta}}{(2-\beta)C_0\psi(2x_M)}.$$

- *If $\beta \in (2,3)$, we have polynomial alignment,*

$$\|v(t)\|_\infty \le \left(\|v_0\|_\infty^{2-\beta} + \frac{(\beta-2)C_0\psi(2x_M)t}{2} \right)^{-\frac{1}{\beta-2}}.$$

Remark 2.1. *1. Note that if $\alpha \in [1, d-1)$ for $d > 2$, then $\int_0^{2\|x_0\|_\infty} \psi^1(s)ds = \infty$ and this yields that we only need the following condition for v_0 in (2.1):*

$$\|v_0\|_\infty^{3-\beta} < \frac{(3-\beta)C_0}{2} \int_{2\|x_0\|_\infty}^\infty \psi^1(s)ds.$$

2. In the case of $\beta = 2$, if we choose the initial data for position x_0 such that $\eta^0_{m,X} > \frac{\|v_0\|_\infty}{C_0\psi(2x_M)}$, then there is no collision between particles and alignment for velocities in a finite time. Similarly, if we select the initial data x_0 satisfying $\eta^0_{m,X} > T^*\|v_0\|_\infty$ when $\beta \in (0,2)$, then the particles do not collide with each other until T^*.

3. Local well-posedness of the generalized Cucker-Smale models

In this section, we provide a detailed proof of Theorem 2.1 under the assumptions of Case A. Since the arguments for the Case B are similar to this, we will give a sketch of proof for it in the last part of this section. We also notice that it is enough to show Theorem 2.1 in the Case A when $\beta = 2$ due to the estimate on the support of f in velocity (see Lemma 3.1 below). We will denote by $F_1(f)$ the alignment force (1.3) in case $\psi = \psi^1(s)$.

3.1. **A regularized model**

In this part, we will consider a regularized model. For this, we first introduce a standard mollifier θ:

$$\theta(x) = \theta(-x) \ge 0, \quad \theta \in \mathcal{C}_0^\infty(\mathbb{R}^d), \quad \text{supp}\,\theta \subset B(0,1), \quad \int_{\mathbb{R}^d} \theta(x)dx = 1,$$

and we set a sequence of smooth mollifiers:

$$\theta_\varepsilon(x) := \frac{1}{\varepsilon^d}\theta\left(\frac{x}{\varepsilon}\right).$$

Here $B(0,1) := \{x \in \mathbb{R}^d : |x| \le 1\}$. Then we define the mollified communication weight ψ^1_ε as $\psi^1_\varepsilon := \psi^1 * \theta_\varepsilon$ for each $\varepsilon > 0$. Since $\psi^1_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$, we deduce from well-posedness theories in [4,16,18] that there exists a unique global solution f_ε which has compact support in velocity to the following equations.

$$\begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \nabla_v \cdot \left[F_1^\varepsilon(f_\varepsilon)f_\varepsilon\right] = 0, & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \\ F_1^\varepsilon(f_\varepsilon)(x,v,t) := \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} \psi^1_\varepsilon(x-y)(w-v)f_\varepsilon(y,w,t)dydw, & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \\ f_\varepsilon(x,v,0) =: f^0(x,v), & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{3.1}$$

For the solution f_ε to the system (3.1), we will show the uniform L^p-bound of f_ε in ε. For this, we first need to estimate the growth of the kinetic velocity. Consider the forward bi-characteristics $Z_\varepsilon(s) := (X_\varepsilon(s;0,x,v), V_\varepsilon(s;0,x,v))$ satisfying the following ODE system:

$$\begin{aligned} \frac{dX_\varepsilon(s)}{ds} &= V_\varepsilon(s), \\ \frac{dV_\varepsilon(s)}{ds} &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \psi^1_\varepsilon\left(X_\varepsilon(s)-y\right)\left(w - V_\varepsilon(s)\right)f_\varepsilon(y,w,s)dydw, \end{aligned} \tag{3.2}$$

where f_ε is the unique global solution to the system (3.1). Set $\Omega_\varepsilon(t)$ and $R^v_\varepsilon(t)$ the v-projection of compact supp$f_\varepsilon(\cdot,t)$ and maximum value of v in $\Omega_\varepsilon(t)$, respectively:

$$\Omega_\varepsilon(t) := \overline{\{v \in \mathbb{R}^d : \exists (x,v) \in \mathbb{R}^d \times \mathbb{R}^d \text{ such that } f_\varepsilon(x,v,t) \neq 0\}}, \quad R^v_\varepsilon(t) := \max_{v \in \Omega_\varepsilon(t)} |v|.$$

Then we have the following growth estimate for support of f_ε in velocity.

Lemma 3.1. *Let $Z_\varepsilon(t)$ be the solution to the particle trajectory* (3.2) *emanating from an initial point in the support of f^0. Then we have*

$$R^v_\varepsilon(t) \le R^v_\varepsilon(0) = R^v_0 := \max_{v \in \Omega(0)} |v|,$$

i.e., the support of $f(x,v,t)$ in velocity is uniformly bounded by the one of $f^0(x,v)$.

Proof. For the proof, we employ the same idea in [7, Section 4]. We choose $V_\varepsilon(t)$ that make the value of $R^v_\varepsilon(t)$ such that $\frac{dR^v_\varepsilon(t)}{dt}$ is well-defined to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(R^v_\varepsilon(t)\right)^2 &= \frac{1}{2}\frac{d}{dt}|V_\varepsilon(t)|^2 = V_\varepsilon(t) \cdot \frac{d}{dt}V_\varepsilon(t)\\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi^1_\varepsilon\left(X_\varepsilon(t) - y\right)\left(w - V_\varepsilon(t)\right) \cdot V_\varepsilon(t) f_\varepsilon(y,w,t)\,dydw\\ &\le 0.\end{aligned}$$

Here we used the fact that for any $w \in \Omega_\varepsilon(t)$, $(w - V_\varepsilon(t)) \cdot V_\varepsilon(t) \le 0$. This completes the proof. □

Remark 3.1. *Set $\tilde{\Omega}_0 := B(0, R^v_0)$. Then it follows from Lemma 3.1 that $\Omega_\varepsilon(t) \subset \tilde{\Omega}_0$ for $t \ge 0$.*

We now show the L^p-estimate of f_ε with the help of the estimate in Lemma 3.1.

Proposition 3.1. *Let f_ε be the solution to system* (3.1)*. Then there exists a $T > 0$ such that the uniform $L^1 \cap L^p$-estimate of f_ε*

$$\sup_{t \in [0,T]} \|f_\varepsilon\|_{L^1 \cap L^p} \le C,$$

holds, where C is a positive constant independent of ε.

Proof. First, we easily find

$$\frac{d}{dt}\int_{\mathbb{R}^d \times \mathbb{R}^d} f_\varepsilon\,dxdv = 0,$$

and this yields $\|f_\varepsilon\|_{L^1} = \|f^0\|_{L^1}$. We next turn to L^p-estimate of f_ε. It is a straightforward to get

$$\frac{d}{dt}\int_{\mathbb{R}^d \times \mathbb{R}^d} f^p_\varepsilon\,dxdv = -(p-1)\int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\nabla_v \cdot (F^\varepsilon_1(f_\varepsilon))\right) f^p_\varepsilon\,dxdv.$$

For the estimate of $\|\nabla_v \cdot (F^\varepsilon_1(f_\varepsilon))\|_{L^\infty}$, we use a cut-off function $\chi_1 \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ defined by

$$\chi_1(x) := \begin{cases} 1 & |x| \le 1,\\ 0 & |x| > 1.\end{cases}$$

Note that ψ^1_ε can be decomposed into two terms:

$$\psi^1_\varepsilon(x) = \psi^1 * \theta_\varepsilon = (\psi^1(\chi_1 + (1-\chi_1))) * \theta_\varepsilon = (\psi^1\chi_1) * \theta_\varepsilon + (\psi^1(1-\chi_1)) * \theta_\varepsilon,$$

and

$$\|(\psi^1\chi_1) * \theta_\varepsilon\|_{L^{p'}} \le \|\psi^1\chi_1\|_{L^{p'}} \le C, \quad \|(\psi^1(1-\chi_1)) * \theta_\varepsilon\|_{L^\infty} \le \|\psi^1(1-\chi_1)\|_{L^\infty} \le 1,$$

due to $\alpha p' < d$. Here C is a positive constant depending only on d, α, and p. Thus we obtain

$$
\begin{aligned}
|\nabla_v \cdot (F_1^\varepsilon(f_\varepsilon))| &\le d \int_{\mathbb{R}^d \times \mathbb{R}^d} |(\psi^1 \chi_1) * \theta_\varepsilon||f_\varepsilon| dy dw + d \int_{\mathbb{R}^d \times \mathbb{R}^d} |(\psi^1(1-\chi_1)) * \theta_\varepsilon||f_\varepsilon| dy dw \\
&\le C(R_0^v)^{\frac{1}{p'}} \|\psi^1 \chi_1\|_{L^{p'}} \|f_\varepsilon\|_{L^p} + \|\psi^1(1-\chi_1)\|_{L^\infty} \|f_\varepsilon\|_{L^1} \\
&\le C \|f_\varepsilon\|_{L^1 \cap L^p},
\end{aligned}
$$

where C is a positive constant independent of ε. Hence we have

$$
\frac{d}{dt} \|f_\varepsilon\|_{L^1 \cap L^p} \le Cd \left(1 - \frac{1}{p}\right) \|f_\varepsilon\|_{L^1 \cap L^p}^2,
$$

and this yields that there exists a $T > 0$,

$$
\sup_{t \in [0,T]} \|f_\varepsilon\|_{L^1 \cap L^p} \le C,
$$

where C is a positive constant depending on the p, d, T, α, R_0^v, and $\|f^0\|_{L^1 \cap L^p}$, but not ε.

□

Remark 3.2. *1. It is easy to find the estimate of first moments of f_ε. In fact, it directly follows from* (3.1) *that*

$$
\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v| f_\varepsilon dx dv &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{v}{|v|} \cdot F_1^\varepsilon(f_\varepsilon) f_\varepsilon dx dv \\
&\le \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \psi_\varepsilon^1(x-y) |w| f_\varepsilon(x,v) f_\varepsilon(y,w) dx dv dy dw \\
&\quad - \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \psi_\varepsilon^1(x-y) |v| f_\varepsilon(x,v) f_\varepsilon(y,w) dx dv dy dw \\
&= 0,
\end{aligned}
$$

where we used $\psi_\varepsilon^1(x) = \psi_\varepsilon^1(-x)$, and the change of variables $(x,v) \leftrightarrow (y,w)$. This yields

$$
\|v f_\varepsilon\|_{L^\infty(0,T;L^1)} \le \|v f^0\|_{L^1}. \tag{3.3}
$$

Since

$$
\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x| f_\varepsilon dx dv \le \int_{\mathbb{R}^d \times \mathbb{R}^d} |v| f_\varepsilon dx dv,
$$

we deduce from (3.3) *that*

$$
\|x f_\varepsilon\|_{L^\infty(0,T;L^1)} \le \|x f^0\|_{L^1} + T \|v f^0\|_{L^1}.
$$

2. It follows from the definition of ψ_ε^1 that

$$
\begin{aligned}
\psi_\varepsilon^1(x) &= \int_{\mathbb{R}^d} \frac{1}{|x-y|^\alpha} \theta_\varepsilon(y) dy \\
&\le \int_{\{y : |y| < \frac{|x|}{2}\}} \frac{\theta_\varepsilon(y)}{|x-y|^\alpha} dy + \int_{\{y : |y| \ge \frac{|x|}{2}\}} \frac{\theta_\varepsilon(y)}{|x-y|^\alpha} dy \\
&\le \frac{2^\alpha \varepsilon}{|x|^\alpha} \int_{\mathbb{R}^d} \theta_\varepsilon(y) dy + \mathbf{1}_{\{|x| \le 2\varepsilon\}} \int_{\{y :\ \varepsilon \ge |y|\}} \frac{\theta_\varepsilon(y)}{|x-y|^\alpha} dy \\
&\le \frac{C}{|x|^\alpha} + \frac{C\varepsilon^\alpha}{|x|^\alpha} \int_{\{y :\ \varepsilon \ge |y|\}} \frac{\theta_\varepsilon(y)}{|x-y|^\alpha} dy \le \frac{C}{|x|^\alpha}.
\end{aligned} \tag{3.4}
$$

Thus we obtain

$$|\psi_\varepsilon^1(x) - \psi_\varepsilon^1(y)| \le \frac{C|x-y|}{\min(|x|,|y|)^{1+\alpha}}, \tag{3.5}$$

where C is independent of ε.

We now show the growth estimate for $d_1(f_\varepsilon(t), f_{\varepsilon'}(t))$.

Proposition 3.2. *Let f_ε and $f_{\varepsilon'}$ be two solutions of the system* (3.1)*. Then there exists C independent of ε and ε' such that*

$$\frac{d}{dt} d_1(f_\varepsilon(t), f_{\varepsilon'}(t)) \le C(d_1(f_\varepsilon(t), f_{\varepsilon'}(t)) + \varepsilon + \varepsilon'),$$

holds for all $t \ge 0$.

Proof. We first define flows $Z_\varepsilon := (X_\varepsilon, V_\varepsilon), Z_{\varepsilon'} := (X_{\varepsilon'}, V_{\varepsilon'}) : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ generated from (3.1) satisfying

$$\begin{cases} \dfrac{d}{dt} X_\varepsilon(t; s, x, v) = V_\varepsilon(t; s, x, v), \\ \dfrac{d}{dt} V_\varepsilon(t; s, x, v) = F_1^\varepsilon(f_\varepsilon)(Z_\varepsilon(t; s, x, v), t), \\ (X_\varepsilon(s; s, x, v), V_\varepsilon(s; s, x, v)) = (x, v), \end{cases} \tag{3.6}$$

and

$$\begin{cases} \dfrac{d}{dt} X_{\varepsilon'}(t; s, x, v) = V_{\varepsilon'}(t; s, x, v), \\ \dfrac{d}{dt} V_{\varepsilon'}(t; s, x, v) = F_1^{\varepsilon'}(f_{\varepsilon'})(Z_{\varepsilon'}(t; s, x, v), t), \\ (X_{\varepsilon'}(s; s, x, v), V_{\varepsilon'}(s; s, x, v)) = (x, v), \end{cases} \tag{3.7}$$

for all $s, t \in [0, T]$. Since $\psi_\varepsilon^1, \psi_{\varepsilon'}^1 \in \mathcal{C}^\infty$, (3.6) and (3.7) are well-defined for $s, t \in [0, T]$. We now choose an optimal transport map $\mathcal{T}^0 = (\mathcal{T}_1^0(x), \mathcal{T}_2^0(v))$ between $f_\varepsilon(t_0)$ and $f_{\varepsilon'}(t_0)$ for fixed $t_0 \in [0, T)$, i.e., $f_{\varepsilon'}(t_0) = \mathcal{T}^0 \# f_\varepsilon(t_0)$. It is known from [8] that such an optimal transport map exists when $f_\varepsilon(t_0)$ is absolutely continuous with respect to the Lebesgue measure. Then we apply the similar argument in [16, Lemma 5.5] to obtain $f_\varepsilon(t) = Z_\varepsilon(t; t_0, \cdot, \cdot) \# f_\varepsilon(t_0)$ and $f_{\varepsilon'}(t) = Z_\varepsilon(t; t_0, \cdot, \cdot) \# f_{\varepsilon'}(t_0)$ using the mass transportation (not necessarily optimal) notation of push-forward. More precisely, we obtain that for any $g \in \mathcal{C}_c^1(\mathbb{R}^d \times \mathbb{R}^d \times [0, T])$,

$$\begin{aligned} &\int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, v, t) f_\varepsilon(x, v, t)\, dx dv - \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, v, t_0) f_\varepsilon(x, v, t_0)\, dx dv \\ &\quad = \int_{t_0}^t \int_{\mathbb{R}^d \times \mathbb{R}^d} (\partial_s g + v \cdot \nabla_x g + F_1^\varepsilon(f_\varepsilon) \cdot \nabla_v g)\, f_\varepsilon(x, v, s)\, dx dv ds. \end{aligned} \tag{3.8}$$

We now choose

$$g(x, v, t) := h(X_\varepsilon(s; t, x, v), V_\varepsilon(s; t, x, v)), \quad \text{for a fixed } t,$$

where $h \in \mathcal{C}_c^1(\mathbb{R}^d \times \mathbb{R}^d)$. This makes the right hand side of (3.8) vanishing and

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} h(x, v) f_\varepsilon(x, v, t)\, dx dv = \int_{\mathbb{R}^d \times \mathbb{R}^d} h(X_\varepsilon(t_0; t, x, v), V_\varepsilon(t_0; t, x, v)) f_\varepsilon(x, v, t_0)\, dx dv. \tag{3.9}$$

Thus we conclude $f_\varepsilon(t) = Z_\varepsilon(t; t_0, \cdot, \cdot) \# f_\varepsilon(t_0)$. The same argument can be applied to get $f_{\varepsilon'}(t) = Z_\varepsilon(t; t_0, \cdot, \cdot) \# f_{\varepsilon'}(t_0)$. We also notice that

$$\mathcal{T}^t \# f_\varepsilon(t) = f_{\varepsilon'}(t), \quad \text{where} \quad \mathcal{T}^t = Z_{\varepsilon'}(t; t_0, \cdot, \cdot) \circ \mathcal{T}^0 \circ Z_\varepsilon(t_0; t, \cdot, \cdot).$$

By Definition 2.1, when $p = 1$, we obtain

$$d_1(f_\varepsilon(t), f_{\varepsilon'}(t)) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} |Z_\varepsilon(t;t_0,x,v) - Z_{\varepsilon'}(t;t_0,\mathcal{T}^0(x,v))| f_\varepsilon(x,v,t_0)dxdv.$$

Set

$$Q_{\varepsilon,\varepsilon'}(t) := \int_{\mathbb{R}^d\times\mathbb{R}^d} |Z_\varepsilon(t;t_0,x,v) - Z_{\varepsilon'}(t;t_0,\mathcal{T}^0(x,v))| f_\varepsilon(x,v,t_0)dxdv.$$

Then straightforward computations yield

$$\begin{aligned}
&\frac{d}{dt}Q_{\varepsilon,\varepsilon'}(t)\Big|_{t=t_0+}\\
&\le \int_{\mathbb{R}^d\times\mathbb{R}^d} |V_\varepsilon(t;t_0,x,v) - V_{\varepsilon'}(t;t_0,\mathcal{T}^0(x,v))| f_\varepsilon(x,v,t_0)dxdv\Big|_{t=t_0+}\\
&+ \int_{\mathbb{R}^d\times\mathbb{R}^d} \left| F_1^\varepsilon(f_\varepsilon)(Z_\varepsilon(t;t_0,x,v),t) - F_1^{\varepsilon'}(f_{\varepsilon'})(Z_{\varepsilon'}(t;t_0,\mathcal{T}^0(x,v)),t)\right| f_\varepsilon(x,v,t_0)dxdv\Big|_{t=t_0+}\\
&=: \mathcal{I} + \mathcal{J}.
\end{aligned}$$

For the estimate of $\mathcal{I}$, it is easy to find

$$\mathcal{I} = \int_{\mathbb{R}^d\times\mathbb{R}^d} |v - \mathcal{T}_2^0(v)| f_\varepsilon(x,v,t_0)dxdv \le C d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)). \tag{3.10}$$

For the estimate of $\mathcal{J}$, we notice that

$$\begin{aligned}
\mathcal{J} &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \bigg| \int_{\mathbb{R}^d\times\mathbb{R}^d} \psi_\varepsilon^1(x-y)(w-v)f_\varepsilon(y,w,t_0)dydw\\
&\quad - \int_{\mathbb{R}^d\times\mathbb{R}^d} \psi_{\varepsilon'}^1(\mathcal{T}_1^0(x)-y)(w-\mathcal{T}_2^0(v))f_{\varepsilon'}(y,w,t_0)dydw \bigg| f_\varepsilon(x,v,t_0)dxdv\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \bigg| \int_{\mathbb{R}^d\times\mathbb{R}^d} \psi_\varepsilon^1(x-y)(w-v)f_\varepsilon(y,w,t_0)dydw\\
&\quad - \int_{\mathbb{R}^d\times\mathbb{R}^d} \psi_{\varepsilon'}^1(\mathcal{T}_1^0(x)-\mathcal{T}_1^0(y))(\mathcal{T}_2^0(w)-\mathcal{T}_2^0(v))f_\varepsilon(y,w,t_0)dydw \bigg| f_\varepsilon(x,v,t_0)dxdv.
\end{aligned}$$

For notational simplicity, we omit the time dependency on t_0 in the rest of computations. We decompose $\mathcal{J}$ into two parts:

$$\mathcal{J} = \int_{\mathbb{R}^d\times\mathbb{R}^d} |\mathcal{J}_1 + \mathcal{J}_2| f_\varepsilon(x,v)dxdv,$$

where

$$\begin{aligned}
\mathcal{J}_1 &:= \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\psi_\varepsilon^1(x-y) - \psi_{\varepsilon'}^1\left(\mathcal{T}_1^0(x)-\mathcal{T}_1^0(y)\right)\right)(w-v)f_\varepsilon(y,w)dydw,\\
\mathcal{J}_2 &:= \int_{\mathbb{R}^d\times\mathbb{R}^d} \psi_{\varepsilon'}^1\left(\mathcal{T}_1^0(x)-\mathcal{T}_1^0(y)\right)\left((w-v) - \left(\mathcal{T}_2^0(w)-\mathcal{T}_2^0(v)\right)\right)f_\varepsilon(y,w)dydw.
\end{aligned}$$

For the estimates of $\mathcal{J}$, we divide it into two steps for the sake of the reader.

- In Step A, we show

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |\mathcal{J}_1| f_\varepsilon dxdv \le C\max(\|f_\varepsilon\|, \|f_{\varepsilon'}\|)d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)) + C\|f_\varepsilon\|^2(\varepsilon+\varepsilon'), \tag{3.11}$$

where C is a positive constant independent of ε, ε', and $\|f\| = \|f\|_{L^\infty(0,T;L^1\cap L^p)}$.

- In Step B, we show
$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |\mathcal{J}_2| f_\varepsilon dxdv \le C\|f_{\varepsilon'}\| d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)),$$
where C is a positive constant independent of ε and ε'.

Step A: By adding and subtracting, we find that

$$\begin{aligned}\mathcal{J}_1 \le &\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|(\psi^1_\varepsilon - \psi^1_{\varepsilon'})(x-y)\right| |w-v| f_\varepsilon(y,w)dydw \\ &+ \int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\psi^1_{\varepsilon'}(x-y) - \psi^1_{\varepsilon'}\left(\mathcal{T}^0_1(x) - \mathcal{T}^0_1(y)\right)\right| |w-v| f_\varepsilon(y,w)dydw.\end{aligned} \tag{3.12}$$

It follows from a similar estimate to (3.4) that

$$\begin{aligned}|\psi^1_\varepsilon(x) - \psi^1(x)| &\le \int_{\mathbb{R}^d} |\psi^1(x-y) - \psi^1(x)|\theta_\varepsilon(y)dt \\ &\le 2\int_{\mathbb{R}^d}\left(\frac{1}{|x|^{1+\alpha}} + \frac{1}{|x-y|^{1+\alpha}}\right)|y|\theta_\varepsilon(y)dy \\ &\le 2\varepsilon\int_{\{y:\ \varepsilon\ge|y|\}}\left(\frac{1}{|x|^{1+\alpha}} + \frac{1}{|x-y|^{1+\alpha}}\right)\theta_\varepsilon(y)dy \\ &\le \frac{C\varepsilon}{|x|^{1+\alpha}}.\end{aligned} \tag{3.13}$$

Then we use (3.13) to obtain

$$\begin{aligned}&\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left|(\psi^1_\varepsilon - \psi^1)(x-y)\right| |w-v| f_\varepsilon(y,w) f_\varepsilon(x,v)dxdvdydw \\ &\le 2R^v_0 \int_{\mathbb{R}^{2d}\times\tilde\Omega^2_0} \left|(\psi^1_\varepsilon - \psi^1)(x-y)\right| f_\varepsilon(y,w) f_\varepsilon(x,v)dxdvdydw \\ &\le C\varepsilon\int_{\mathbb{R}^{2d}\times\tilde\Omega^2_0} \frac{1}{|x-y|^{1+\alpha}} f_\varepsilon(y,w) f_\varepsilon(x,v)dxdvdydw \\ &\le C\varepsilon\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\int_{\{y:|x-y|<1\}\times\tilde\Omega_0} + \int_{\{y:|x-y|\ge1\}\times\tilde\Omega_0}\frac{1}{|x-y|^{1+\alpha}} f_\varepsilon(y,w)dydw\right) f_\varepsilon(x,v)dxdv \\ &\le C\varepsilon\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\left(\int_{\{y:|x-y|\le1\}}\frac{1}{|x-y|^{(1+\alpha)p'}}dy\right)^{\frac{1}{p'}} \|f_\varepsilon\|_{L^p} + \|f_\varepsilon\|_{L^1}\right) f_\varepsilon(x,v)dxdv \\ &\le C\varepsilon\|f_\varepsilon\|^2 \le C\varepsilon.\end{aligned} \tag{3.14}$$

Similarly, we get
$$\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left|(\psi^1_{\varepsilon'} - \psi^1)(x-y)\right| |w-v| f_\varepsilon(y,w) f_\varepsilon(x,v)dxdvdydw \le C\varepsilon'.$$

Thus we have
$$\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left|(\psi^1_\varepsilon - \psi^1_{\varepsilon'})(x-y)\right| |w-v| f_\varepsilon(y,w) f_\varepsilon(x,v)dxdvdydw \le C(\varepsilon+\varepsilon'). \tag{3.15}$$

Concerning the second term in the right hand side of (3.12), we employ (3.5) and use the change of variables $(x,v) \leftrightarrow (y,w)$ to find

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}\times\tilde{\Omega}_0^2} \left|\psi^1_{\varepsilon'}(x-y) - \psi^1_{\varepsilon'}\left(\mathcal{T}^0_1(x) - \mathcal{T}^0_1(y)\right)\right| |w-v| f_\varepsilon(x,v) f_\varepsilon(y,w)\,dxdvdydw \\
&\quad \le 2R^v_0 \int_{\mathbb{R}^{2d}\times\tilde{\Omega}_0^2} \left(\frac{|\mathcal{T}^0_1(x) - x|}{|\mathcal{T}^0_1(x) - \mathcal{T}^0_1(y)|^{1+\alpha}} + \frac{|\mathcal{T}^0_1(x) - x|}{|x-y|^{1+\alpha}} \right) f_\varepsilon(x,v) f_\varepsilon(y,w)\,dxdvdydw \\
&\quad =: \mathcal{K}_1 + \mathcal{K}_2.
\end{aligned}$$

By direct computations, we get

$$\begin{aligned}
\mathcal{K}_1 &= 2R^v_0 \int_{\mathbb{R}^d\times\mathbb{R}^d} |\mathcal{T}^0_1(x) - x| f_\varepsilon(x,v) \left(\int_{\mathbb{R}^d\times\Omega_0} \frac{1}{|\mathcal{T}^0_1(x) - y|^{1+\alpha}} f_{\varepsilon'}(y,w)\,dydw \right) dxdv \\
&\le C\|f_{\varepsilon'}\| \int_{\mathbb{R}^d\times\mathbb{R}^d} |\mathcal{T}^0_1(x) - x| f_\varepsilon(x,v)\,dxdv \le C\|f_{\varepsilon'}\| d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)),
\end{aligned}$$

where we used the same estimates in (3.14) to obtain

$$\int_{\mathbb{R}^d\times\tilde{\Omega}_0} \frac{1}{|\mathcal{T}^0_1(x) - y|^{1+\alpha}} f_{\varepsilon'}(y,w)\,dydw \le C\|f_{\varepsilon'}\|.$$

Similarly, we also obtain $\mathcal{K}_2 \le C\|f_\varepsilon\| d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0))$. This yields

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left|\psi^1_{\varepsilon'}(x-y) - \psi^1_{\varepsilon'}\left(\mathcal{T}^0_1(x) - \mathcal{T}^0_1(y)\right)\right| |w-v| f_\varepsilon(x,v) f_\varepsilon(y,w)\,dxdvdydw \\
&\qquad \le C\max(\|f_\varepsilon\|, \|f_{\varepsilon'}\|) d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)).
\end{aligned} \tag{3.16}$$

We now combine (3.15) and (3.16) to conclude our desired claim.

Step B: A straightforward computation yields that

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} |\mathcal{J}_2| f_\varepsilon\,dxdv \\
&\quad \le \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left|\psi^1_{\varepsilon'}\left(\mathcal{T}^0_1(x) - \mathcal{T}^0_1(y)\right)\right| \left|w - \mathcal{T}^0_2(w)\right| f_\varepsilon(x,v) f_\varepsilon(y,w)\,dxdydvdw \\
&\qquad + \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left|\psi^1_{\varepsilon'}\left(\mathcal{T}^0_1(x) - \mathcal{T}^0_1(y)\right)\right| \left|v - \mathcal{T}^0_2(v)\right| f_\varepsilon(x,v) f_\varepsilon(y,w)\,dxdydvdw \\
&\quad =: \mathcal{J}^1_2 + \mathcal{J}^2_2.
\end{aligned}$$

On the other hand, we can easily find that

$$\begin{aligned}
\mathcal{J}^1_2 &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\int_{\mathbb{R}^d\times\Omega_0} \left|\psi^1_{\varepsilon'}\left(\mathcal{T}^0_1(x) - \mathcal{T}^0_1(y)\right)\right| f_\varepsilon(x,v)\,dxdv \right) \left|w - \mathcal{T}^0_2(w)\right| f_\varepsilon(y,w)\,dydw \\
&\le C\|f_{\varepsilon'}\| \int_{\mathbb{R}^d\times\mathbb{R}^d} |w - \mathcal{T}^0_2(w)| f_\varepsilon(y,w)\,dydw \le C\|f_{\varepsilon'}\| d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)),
\end{aligned}$$

where we used the estimates in (3.14) again. Similarly, we get

$$\mathcal{J}^2_2 \le C\|f_{\varepsilon'}\| d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)),$$

and this deduces

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |\mathcal{J}_2| f_\varepsilon \, dxdv \le C\|f_{\varepsilon'}\| d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)). \tag{3.17}$$

We now combine (3.10), (3.11) and (3.17) to find

$$\frac{d}{dt} Q_{\varepsilon,\varepsilon'}(t)\Big|_{t=t_0+} \le C(d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)) + \varepsilon + \varepsilon').$$

We finally write the integral form, dividing $t - t_0$, and taking the limit $t \to t_0^+$ to conclude

$$\frac{d}{dt} d_1(f_\varepsilon(t), f_{\varepsilon'}(t))\Big|_{t=t_0^+} \le C(d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0) + \varepsilon + \varepsilon').$$

Since t_0 is arbitrary in $[0, T)$, this yields

$$\frac{d}{dt} d_1(f_\varepsilon(t), f_{\varepsilon'}(t)) \le C(d_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0) + \varepsilon + \varepsilon'),$$

where C is independent of ε and ε'. □

3.2. **Existence and uniqueness of weak solutions (limit as $\varepsilon \to 0$)**

It follows from Proposition 3.2 that $\{f_\varepsilon\}_{\varepsilon>0}$ is a Cauchy sequence in $\mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$, and this implies that there exists a limit curve of measure $f \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$, and $f \in L^\infty(0,T;(L^1_+ \cap L^p)(\mathbb{R}^d\times\mathbb{R}^d))$. Thus it only remains to show that f is a solution of the Cucker-Smale model (1.1). Choose a test function $\Psi(x,v,t) \in \mathcal{C}_c^\infty(\mathbb{R}^d\times\mathbb{R}^d\times[0,T])$, then f_ε satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} \Psi^0(x,v) f^0(x,v) dxdv \\
&\quad = \int_{\mathbb{R}^d\times\mathbb{R}^d} \Psi(x,v,T) f_\varepsilon(x,v,T) dxdv + \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} \partial_t \Psi(x,v,t) f_\varepsilon(x,v,t) dxdvdt \\
&\qquad - \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla_x \Psi)\cdot v f_\varepsilon dxdvdt - \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla_v \Psi)\cdot F_1^\varepsilon(f_\varepsilon) f_\varepsilon dxdvdt,
\end{aligned} \tag{3.18}$$

where $\Psi^0(x,v) = \Psi(x,v,0)$. We can easily show that the first, second, and third terms in the rhs of (3.18) converge to

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} \Psi(x,v,T) f(x,v,T) dxdv + \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} \partial_t \Psi(x,v,t) f(x,v,t) dxdvdt \\
&\qquad - \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla_x \Psi)\cdot v f dxdvdt \quad \text{as} \quad \varepsilon \to 0,
\end{aligned}$$

since $f_\varepsilon \to f$ in $\mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$. We also notice that

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} (\psi_\varepsilon^1 - \psi^1)(x-y)(\nabla_v \Psi)\cdot(w-v) f_\varepsilon(x,v) f_\varepsilon(y,w) dxdvdydwdt \right| \\
&\le C\varepsilon R_0^v \|\nabla_v \Psi\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d\times\mathbb{R}^d))} \int_0^T \int_{\mathbb{R}^{2d}\times\tilde{\Omega}_0^2} \frac{1}{|x-y|^{1+\alpha}} f_\varepsilon(x,v) f_\varepsilon(y,w) dxdvdydwdt \\
&\le C\varepsilon \|f_\varepsilon\|^2 \le C\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0\,,
\end{aligned}$$

where we used the decomposition in near and far fields as in (3.14). Thus in order to obtain

$$\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla_v \Psi)\cdot F_1^\varepsilon(f_\varepsilon) f_\varepsilon dxdvdt \to \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla_v \Psi)\cdot F_1(f) f dxdvdt,$$

it only remains to show

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \psi^1(x-y)(\nabla_v\Psi)\cdot(w-v)f_\varepsilon(x,v)f_\varepsilon(y,w)dxdvdydwdt\\ &\qquad \to \int_0^T \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \psi^1(x-y)(\nabla_v\Psi)\cdot(w-v)f(x,v)f(y,w)dxdvdydwdt,\end{aligned} \tag{3.19}$$

as $\varepsilon\to 0$. For this, we introduce a cut-off function $\chi_\delta \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that

$$\chi_\delta(x) = \begin{cases} 1 & \text{if} \quad |x|\le\delta \\ 0 & \text{if} \quad |x|\ge 2\delta \end{cases}.$$

Then since $(1-\chi_\delta(x-y))\psi^1(x-y)(w-v)\cdot\nabla_v\Psi$ is a Lipschitz function and $f_\varepsilon\to f$ in $\mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$, we find

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} (1-\chi_\delta)\psi^1(x-y)(\nabla_v\Psi)\cdot(w-v)f_\varepsilon(x,v)f_\varepsilon(y,w)dxdvdydwdt\\ &\qquad \to \int_0^T \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} (1-\chi_\delta)\psi^1(x-y)(\nabla_v\Psi)\cdot(w-v)f(x,v)f(y,w)dxdvdydwdt,\end{aligned} \tag{3.20}$$

as $\varepsilon\to 0$ for any $\delta>0$. On the other hand, the remaining term is estimated as follows:

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \chi_\delta(x-y)\psi^1(x-y)(\nabla_v\Psi)\cdot(w-v)f_\varepsilon(x,v)f_\varepsilon(y,w)dxdvdydwdt\\ &\le C\delta\int_0^T \int_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d:\ |x-y|\le 2\delta\}\times\tilde{\Omega}_0^2} \frac{1}{|x-y|^{1+\alpha}} f_\varepsilon(x,v)f_\varepsilon(y,w)dxdvdydwdt\\ &\le C\delta \to 0 \quad \text{as} \quad \delta\to 0,\end{aligned} \tag{3.21}$$

and similarly, we also have

$$\int_0^T \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \chi_\delta(x-y)\psi^1(x-y)(\nabla_v\Psi)\cdot(w-v)f(x,v)f(y,w)dxdvdydwdt \le C\delta\to 0, \tag{3.22}$$

as $\delta\to 0$ due to the fact that f has a compact support in velocity. Hence we conclude the convergence (3.19) combining (3.20), (3.21), and (3.22). Uniqueness of the weak solutions f_ε is just followed from Proposition 3.2. More specifically, let $f_1, f_2 \in L^\infty(0,T;(L^1_+\cap L^p)(\mathbb{R}^d\times\mathbb{R}^d))\cap\mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$ be the weak solutions to the system (1.1) with same initial data $f^0 \in (L^1_+\cap L^p)(\mathbb{R}^d\times\mathbb{R}^d)\cap\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d)$ satisfying the Case A. Then Proposition 3.2 yields that

$$\frac{d}{dt}d_1(f_1(t),f_2(t)) \le C\max(\|f_1\|,\|f_2\|)d_1(f_1(t),f_2(t)), \quad \text{for} \quad t\in[0,T].$$

This completes the proof for the Case A.

Proposition 3.3. *Let f be a weak solutions to* (1.1) *on the time-interval* $[0, T)$ *in the sense of Definition* 2.3. *Then f is determined as the push-forward of the initial density through the flow map generated by* $(v, F_1(f))$.

Proof. Consider the following flow map:

$$\begin{cases} \dfrac{d}{dt}X(t;s,x,v) = V(t;s,x,v), \\ \dfrac{d}{dt}V(t;s,x,v) = F_1(f)(X(t;s,x,v),V(t;s,x,v),t), \\ (X(s;s,x,v),V(s;s,x,v)) = (x,v), \end{cases} \tag{3.23}$$

for all $s, t \in [0, T]$. Then since f has compact support in v, the flow map (3.23) is well-defined using the same argument in the proof of Proposition 3.2. Moreover we can use the similar argument to (3.9) to have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} h(x,v)f(x,v,t)dxdv = \int_{\mathbb{R}^d\times\mathbb{R}^d} h(X(0;t,x,v),V(0;t,x,v))f^0(x,v)dxdv,$$

for $t \in [0, T]$. This yields that f is determined as the push-forward of the initial density through the flow map (3.23). □

Proof of Theorem 2.1 in the Case B. Similarly, we first regularize the nonlinear velocity coupling as $\nabla\phi_\varepsilon(v) := \frac{v}{|v|^{2-\beta_2}+\varepsilon}$, and define the f_ε by this regularized system. Then we easily find that the estimates of support of f in position and velocity, and first momentums using the same arguments in Lemma 3.1 and Remark 3.2. The remaining parts are obtained by using the similar arguments in Section 3. □

4. Dynamics of the generalized Cucker-Smale particle system

We recall our generalized Cucker-Smale particle system:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= v_i(t), \\ \frac{dv_i(t)}{dt} &= \frac{1}{N}\sum_{j=1}^N \psi(x_j(t)-x_i(t))\frac{v_j-v_i}{|v_j-v_i|^{2-\beta}}, \quad i=1,\cdots,N. \end{aligned} \tag{4.1}$$

Here, the communication weight is either ψ^1 or ψ^2 as defined in the introduction.

Note that we can choose an index i such that $\|v(t)\|_\infty = |v_i(t)|$ at any time t. Then it follows from [5, 15] that

$$\frac{d}{dt}\|v(t)\|_\infty^2 \le -C_0\psi(2\|x(t)\|_\infty)\|v(t)\|_\infty^\beta, \quad \text{for} \quad \beta \in (0,3),$$

where C_0 is a positive constant depending only on β. It is also clear to obtain $\left|\frac{d\|x\|_\infty}{dt}\right| \le \|v\|_\infty$. We now take the similar argument in [1, 16], and define two Lyanpunov type functionals $\mathcal{E}_\pm(x, v)$:

$$\mathcal{E}_\pm(x(t),v(t)) := \frac{1}{3-\beta}\|v(t)\|_\infty^{3-\beta} \pm \frac{C_0}{2}\Psi(2\|x(t)\|_\infty),$$

where $\Psi(\cdot)$ is a primitive of ψ.

We next present two lemmas that can be obtained by using the similar arguments in [1, 16].

Lemma 4.1. *Let* (x, v) *be any smooth solutions to the system* (4.1). *Then we have*

$$(i)\ \ \mathcal{E}_\pm(x(t), v(t)) \le \mathcal{E}_\pm(x_0, v_0).$$
$$(ii)\ \ \|v(t)\|_\infty^{3-\beta} + \frac{(3-\beta)C_0}{2}\left|\int_{2\|x_0\|_\infty}^{2\|x(t)\|_\infty} \psi(s)ds\right| \le \|v_0\|_\infty^{3-\beta}.$$

Lemma 4.2. *Let* (x, v) *be any smooth solutions to the system* (4.1). *If the initial data* (x_0, v_0) *satisfies*

$$\|x_0\|_\infty > 0, \quad \|v_0\|_\infty^{3-\beta} < \frac{(3-\beta)C_0}{2}\min\left\{\int_0^{2\|x_0\|_\infty}\psi(s)ds, \int_{2\|x_0\|_\infty}^{\infty}\psi(s)ds\right\},$$

then there exist positive constants $x_m, x_M > 0$ *such that*

$$\|x(t)\|_\infty \in [x_m, x_M], \quad \frac{d}{dt}\|v(t)\|_\infty^2 \le -C_0\psi(2x_M)\|v(t)\|_\infty^\beta,$$

where x_m *and* x_M *are defined by*

$$\|v_0\|_\infty^{3-\beta} = \frac{(3-\beta)C_0}{2}\int_{2x_m}^{\|2x_0\|_\infty}\psi(s)ds \quad \text{and} \qquad \|v_0\|_\infty^{3-\beta} = \frac{(3-\beta)C_0}{2}\int_{2\|x_0\|_\infty}^{2x_M}\psi(s)ds,$$

respectively.

Proof of Theorem 2.2. The inequalities for $\|v(t)\|_\infty$ are clearly obtained from the results in Lemma 4.2. Concerning the initial configuration for avoiding collisions between particles, a straightforward computation yields that for $\beta = 2$

$$\begin{aligned}|\eta_{m,X}(t) - \eta_{m,X}^0| &\le \left|\int_0^t \frac{d\eta_{m,X}(s)}{ds}ds\right| \\ &\le 2\|v_0\|_\infty\int_0^t e^{-\frac{C_0\psi(2x_M)s}{2}}ds \\ &\le \frac{\|v_0\|_\infty}{C_0\psi(2x_M)}.\end{aligned}$$

Thus we conclude that

$$\eta_{m,X}(t) \ge \eta_{m,X}^0 - |\eta_{m,X}(t) - \eta_{m,X}^0| \ge \eta_{m,X}^0 - \frac{\|v_0\|_\infty}{C_0\psi(2x_M)} > 0.$$

Similarly, for $\beta \in (0, 2)$, we have

$$|\eta_{m,X}(t) - \eta_{m,X}^0| \le \int_0^t \|v(s)\|_\infty ds \le T^*\|v_0\|_\infty,$$

and this deduces

$$\eta_{m,X}(t) \ge \eta_{m,X}^0 - T^*\|v_0\|_\infty > 0.$$

This completes the proof. □

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