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Samira EL YACOUBI, Larbi AFIFI, El Hassan ZERRIK, Abdessamad TRIDANE, Editors

DOMINATION OF DISCRETE DISTRIBUTED SYSTEMS

L. Afifi, M. Joundi, K. Lasri and E.M. Magri[1]

Abstract. The domination is an original way for systems classification, based on input and output operators. In this paper, we consider first a class of controlled discrete distributed parameter systems (DPS). We extend the domination concept to these systems. By duality, we consider a classification for discrete observed systems, based on output operators. We give various characterizations and the main properties in the general case, and then by means of the choice of actuators and sensors.

keywords: Distributed Parameter Systems, Discrete, Domination, Control, Actuators.

1. Introduction

Let us first consider, without loss of generality, a class of linear disturbed systems described by a discrete state equation as follows:

$$\left\{\begin{array}{lcl} z_{k+1} & = & \Phi z_k + f_k \quad ; \;\; 0 \leq k \leq N-1 \\ z_0 & \in & Z \end{array}\right. \tag{1}$$

where Z is a Hilbert state space, $\Phi \in \mathcal{L}(Z)$, $z_k \in Z$ and $f_k \in Z$ are respectively the state and the disturbance at step k; N is an integer ≥ 1.
The system (1) is augmented by the discrete output :

$$y_i = Cz_i \; ; \;\; 0 \leq i \leq N \tag{2}$$

where $C \in \mathcal{L}(Z, Y)$, Y is a Hilbert observation space.

The state of the system (1) and the observation at step N are respectively given by :

$$z_N = \Phi^N z_0 + \sum_{i=0}^{N-1} \Phi^{N-1-i} f_i \tag{3}$$

and

$$y_N = C\Phi^N z_0 + \sum_{i=0}^{N-1} C\Phi^{N-1-i} f_i \tag{4}$$

[1] University Hassan II Casablanca, B.P.5366-Maârif, Casablanca, Morocco
Presently at: Department of Mathematics and Computer Science Faculty of Sciences Ain Chock

The disturbance $f = (f_0,\ f_1,\ \cdots,\ f_{N-1})$ may be unknown. In this case, it is possible, with a convenient choice of the output operator C, to detect and reconstruct any disturbance using the corresponding observation only [4,5,8]. However, the knowledge of f is not sufficient to solve the problem because one has to compensate the effect of the disturbance on the system by bringing the observation to its normal state.

In order to study a possible space compensation (remediability) of any disturbance, we introduce control terms Bu_k. The system (1) becomes

$$\left\{\begin{array}{lcl} z_{k+1} & = & \Phi z_k + f_k + Bu_k; \quad 0 \leq k \leq N-1 \\ \\ z_0 & \in & Z \end{array}\right. \tag{5}$$

where $B \in \mathcal{B}(U, Z)$, $u_k \in U$; U is a control Hilbert space. It is shown in [9], that with a convenient choice of the input operator B (efficient actuators), it is possible to compensate at the final step N, the effect of any disturbance. The optimal control ensuring this compensation is also given.

A natural and practical question in systems theory consists to consider the case where the disturbances are due (voluntarily or accidentally) to actions given by

$$f_k = Gv_k$$

Such a form, is often used in control theory and describes the nature of the disturbance, its location, its spatial distribution and its intensity. The system (5) becomes

$$\left\{\begin{array}{lcl} z_{k+1} & = & \Phi z_k + Gv_k + Bu_k; \quad 0 \leq k \leq N-1 \\ \\ z_0 & \in & Z \end{array}\right. \tag{6}$$

with $G \in \mathcal{L}(W, Z)$; W is a Hilbert space.

The compensation problem and the choice of the control term will depend obviously on the operator G. In order to simplify, we consider the case where $C = I$. Hence, the remediability problem can be formulated as follows :

For $v = (v_0,\ v_1,\ \cdots,\ v_{N-1}) \in W^N$, does a control $u = (u_0,\ u_1,\ \cdots,\ u_{N-1}) \in U^N$ such that

$$H_N(G)v + H_N(B)u = 0 \tag{7}$$

where $H_N(G)$ and $H_N(B)$ are respectively the operators defined by

$$\begin{array}{rcccl} H_N(G) & : & W^N & \longrightarrow & Z \\ & & v & \longrightarrow & H_N(G)v = \displaystyle\sum_{k=0}^{N-1} \Phi^{N-k-1} G v_k \end{array}$$

and

$$\begin{array}{rcccl} H_N(B) & : & U^N & \longrightarrow & Z \\ & & u & \longrightarrow & H_N(B)u = \displaystyle\sum_{k=0}^{N-1} \Phi^{N-k-1} B u_k \end{array}$$

The condition (7) is equivalent to

$$Im[H_N(G)] \subset Im[H_N(B)] \tag{8}$$

This inclusion means that the operator B is stronger than G in the sense that it is able to compensate the effect of any action due to the operator G. Hence, if B and G correspond respectively to actuators $(\Omega_i, h_i)_{1\leq i\leq p}$ and $(D_j, h_j)_{1\leq j\leq q}$, then $(\Omega_i, h_i)_{1\leq i\leq p}$ are more efficient than $(D_j, h_j)_{1\leq j\leq q}$ and dominate them.

This is the origin of the notion of domination introduced and developed for continuous systems, first in the parabolic case and then in the hyperbolic one in [11,12,13]. It consists to study the possibility of comparison (or classification) of input operators (actuators), and by duality for output operators (sensors).

The choice of the number of actuators is not a sufficient condition for the domination. Indeed, one actuator may dominate several others. This notion is also extended to the asymptotic case. The regional aspect and other situations are equally examined.

Notice also that one can consider equivalently the domination (classification) of systems $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$ with the same dynamics Φ and respectively excited by two control operators B and G.

In this paper, we give a generalization to a class of discrete distributed systems $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$, with dynamics Φ_1 and Φ_2 which are not necessarily the same. We study the problem of domination and classification of such controlled systems, with respect to an output operator C. We give the main properties of this notion and we explore the case of discrete versions of diffusion processes as well as that of actuators and sensors. The results and the approach are extended by duality, to the case of observed systems and output operators.

2. Domination of controlled discrete systems

We consider the following linear distributed systems described by the following discrete state equations

$$(S_1) \begin{cases} z_{1,k+1} = \Phi_1 z_{1,k} + B_1 u_{1,k}; \quad 0 \leq k \leq N-1 \\ z_{1,0} \in Z \end{cases} \tag{9}$$

$$(S_2) \begin{cases} z_{2,k+1} = \Phi_2 z_{2,k} + B_2 u_{2,k}; \quad 0 \leq k \leq N-1 \\ z_{2,0} \in Z \end{cases} \tag{10}$$

where, for $i = 1, 2$, $\Phi_i \in \mathcal{L}(Z)$, $B_i \in \mathcal{L}(U_i, Z)$, $z_{i,k} \in Z$ and $u_{i,k} \in U_i$ are respectively the state of (S_i) and the control at step k; U_i is a control space and Z is the state space. Each system (S_i) is augmented with the discrete output equation:

$$y_{i,k} = C z_{i,k} \ ; \ \ 0 \leq k \leq N \tag{11}$$

where $C \in \mathcal{L}(Z, Y)$. The state of system (S_i) at the final step N, is given by :

$$\begin{aligned} z_{i,N} &= \Phi_i^N z_{i,0} + \sum_{k=0}^{N-1} (\Phi_i)^{N-k-1} B_i u_{i,k} \\ &= \Phi_i^N z_{i,0} + H_{i,N}\, u_i \end{aligned} \tag{12}$$

where $H_{i,N}$ is defined by

$$\begin{array}{rcccl} H_{i,N} & : & U_i^N & \rightarrow & Z \\ & & u_i = (u_{i,0}, \cdots, u_{i,N-1}) & \rightarrow & H_{i,N}\, u_i = \sum_{k=0}^{N-1} (\Phi_i)^{N-k-1} B_i u_{i,k} \end{array} \tag{13}$$

We consider the following definitions.

Definition 1. *We say that*

1. *The system (S_1) dominates the system (S_2) exactly, with respect to the operator C, if*
$$Im(CH_{2,N}) \subset Im(CH_{1,N}) \tag{14}$$
2. *The system (S_1) dominates the system (S_2) weakly, with respect to the operator C, if*
$$\overline{Im(CH_{2,N})} \subset \overline{Im(CH_{1,N})} \tag{15}$$

Obviously, the exact domination implies the weak one. The converse is not necessarily true. On an other hand, if $\Phi_1 = \Phi_2$, the domination (comparison or classification) concerns then the control operators B_1 and B_2, with respect to C.

The following result gives a characterization of the exact domination with respect to the output operator C.

Proposition 2. *The following properties are equivalent :*

(1) *The system (S_1) dominates the system (S_2) exactly, with respect to the operator C.*
(2) *For any $u_2 = (u_{2,0}, \cdots, u_{2,N-1}) \in U_2^N$, there exists $u_1 = (u_{1,0}, \cdots, u_{1,N-1}) \in U_1^N$ such that*
$$CH_{1,N}u_1 + CH_{2,N}u_2 = 0 \tag{16}$$
(3) *There exists $\gamma > 0$ such that for any $\theta \in Y'$, we have*
$$|| \sum_{k=0}^{N-1} B_2^*(\Phi_2^*)^{N-1-k} C^*\theta || \leq \gamma \, || \sum_{k=0}^{N-1} B_1^*(\Phi_1^*)^{N-1-k} C^*\theta || \tag{17}$$

Concerning the weak domination with respect to C, we have the following characterization.

Proposition 3. *The system (S_1) dominates the system (S_2) weakly, with respect to the operator C, if and only if*
$$Ker(H_{1,N}) \subset Ker(H_{2,N})$$

In this case, we note $(\Phi_2, B_2) \underset{C}{\lesssim} (\Phi_1, B_1)$. In the case of the exact domination, we note $(\Phi_2, B_2) \underset{C}{\leq} (\Phi_1, B_1)$.

It is clear that if the system (S_1) is exactly (respectively weakly) controllable, i.e. $Im(H_{1,N}) = Z$ (respectively $\overline{Im(H_{1,N})} = Z$), then (S_1) dominates (S_2) exactly (respectively weakly), with respect to any output operator C.

In the discrete case, the exact (as well as the weak) domination is reflexive and transitive, but not symmetric (neither antisymmetric).

In the particular case where $\Phi_1 = \Phi_2$, one can consider indifferently a single system

$$\begin{cases} z_{k+1} = \Phi z_k + B_1 u_{1,k} + B_2 u_{2,k} \; ; \quad 0 \le k \le N-1 \\ z_0 \in Z \end{cases}$$

The comparison concerns the operators B_1 and B_2, then we note simply $B_2 \underset{C}{\le} B_1$ (respectively $B_2 \underset{C}{\lesssim} B_1$) in the exact (respectively weak) case.
On an other hand, even if $B_1 = B_2$, the system (S_1) may dominate the system (S_2), this depends on the choice of Φ_1 and Φ_2. This situation is examined for continuous systems in [11, 12]. In the discrete case, it is illustrated in the following example.

Example 4. *Let us consider the case where (S_1) and (S_2) are the discrete versions of the following continuous systems*

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} = \dfrac{\partial^2 z(x,t)}{\partial x^2} + sin(n_0 \pi x) u(t) &]0,1[\times]0,T[\\ z(x,0) = z_0(x) &]0,1[\\ z(0,t) = z(1,t) = 0 &]0,T[\end{cases} \tag{18}$$

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} = \dfrac{\partial^2 z(x,t)}{\partial x^2} + \alpha z(x,t) + sin(n_0 \pi x) v(t) &]0,1[\times]0,T[\\ z(x,0) = z_0(x) &]0,1[\\ z(0,t) = z(1,t) = 0 &]0,T[\end{cases} \tag{19}$$

To simplify, we consider the case where $C = I_Z$, the identity operator of the state space $Z = L^2(0,1)$. The operator $A_\alpha = \dfrac{\partial^2}{\partial x^2} + \alpha I_Z$ generates on Z, the strongly continuous semi-group $(S_\alpha(t))_{t \ge 0}$ defined by

$$S_\alpha(t) w = 2 \sum_{n \ge 1} e^{(\alpha - n^2 \pi^2) t} < w, sin(n\pi.) >_Z sin(n\pi.)$$

In this case, we have

$$\Phi_1 = S_0(\tau) \text{ and } \Phi_2 = S_\alpha(\tau)$$

where $\tau = \dfrac{T}{N}$, N is an integer sufficiently large. Concerning the control operators B_1 and B_2, they are defined by

$$B_1 u_{1,k} = \int_0^\tau S_0(\tau - r) sin(n_0 \pi.) u_{1,k}(r) dr$$

and

$$B_2 u_{2,k} = \int_0^\tau S_\alpha(\tau - r) sin(n_0 \pi.) u_{2,k}(r) dr$$

*where $u_{1,k}$ and $u_{2,k}$ are respectively the restrictions of the control functions u and v to the interval $[t_k, t_{k+1}[$, with $t_k = k\tau$, $0 \le k \le N-1$.
For τ sufficiently small, and $0 \le k \le N-1$, $u_{1,k}$ and $u_{2,k}$ may be assumed to be constant*

on $[t_k, t_{k+1}[$, but this is not necessary.

Hence,

- *if $\alpha > 0$, then (S_1) dominates (S_2) exactly, and then weakly.*
- *if $\alpha < 0$, we have the converse, i.e. (S_2) dominates (S_1) exactly and weakly.*

We give hereafter a result concerning the remediability, but let us first recall this notion. For $i = 1, 2$, a discrete system

$$(\overline{S}_i) \begin{cases} z_{i,k+1} &= \Phi z_{i,k} + B_i u_{i,k} + f_{i,k} \ ; \quad 0 \leq k \leq N-1 \\ z_{i,0} &\in Z \end{cases}$$

augmented with the output equation

$$(\overline{E}_i) \ \ y_{i,k} = C z_{i,k} \ ; \ 0 \leq i \leq N$$

(or $(\overline{S}_i)$+$(\overline{E}_i)$) is said to be remediable :

1) exactly if for any disturbance $f_i = (f_{i,0}, f_{i,1}, \dots, f_{i,N-1}) \in Z^N$, there exists a control $u_i = (u_{i,0}, u_{i,1}, \dots, u_{i,N-1}) \in U_i^N$ such that

$$\sum_{k=0}^{N-1} \Phi^{N-1-k} B_i u_{i,k} + \sum_{k=0}^{N-1} \Phi^{N-1-k} f_{i,k} = 0$$

2) exactly if for any $\epsilon > 0$ and any $f_i = (f_{i,0}, f_{i,1}, \dots, f_{i,N-1}) \in Z^N$, there exists a control $u_i = (u_{i,0}, u_{i,1}, \dots, u_{i,N-1}) \in U_i^N$ such that

$$\left\| \sum_{k=0}^{N-1} \Phi^{N-1-k} B_i u_{i,k} + \sum_{k=0}^{N-1} \Phi^{N-1-k} f_{i,k} \right\|_Z \ < \ \epsilon$$

Characterizations, various properties and results, as well as applications and illustrations of this notion can be found in [6,7,9]. We have the following result.

Proposition 5. *Under the following conditions*

i) $(\overline{S}_1)$+$(\overline{E}_1)$ is exactly (respectively weakly) remediable.

ii) B_2 dominates B_1 exactly (respectively weakly).

the system $(\overline{S}_2)$+$(\overline{E}_2)$ is exactly (respectively weakly) remediable.

Hereafter, we examine the domination problem in the usual case of actuators, and then that where the observation is given by means of sensors.

3. Domination and actuators

In the case where te systems (S_1) and (S_2) are respectively excited by zone actuators $(\Omega_i, g_i)_{1 \leq i \leq p}$ and $(D_i, h_i)_{1 \leq i \leq q}$, we have $U_1 = \mathbb{R}^p$, $U_2 = \mathbb{R}^q$. We have

$$B_1 u_{1,k} = \sum_{i=1}^{p} g_i u_{1,k,i} \ ; \ B_2 u_{2,k} = \sum_{i=1}^{q} h_i u_{2,k,i}$$

where for $0 \le k \le N-1$, $u_{1,k} = (u_{1,k,1}, \cdots, u_{1,k,p})$ and $u_{2,k} = (u_{2,k,1}, \cdots, u_{2,k,q})$.

Hence, for $j = 1, 2$; u_j may be considered as a matrix, i.e.

$$u_1 \equiv (u_{1,k,l})_{k,l} \quad \text{with} \quad 1 \le k \le N-1 \ , \ 1 \le l \le p$$

and

$$u_2 \equiv (u_{2,k,l})_{k,l} \quad \text{with} \quad 1 \le k \le N-1 \ , \ 1 \le l \le q$$

In order to simplify, we consider without loss of generality[1], the case where $\Phi_1 = \Phi_2 = \Phi$ with

$$\Phi z = \sum_{n=1}^{\infty} e^{\lambda_n \tau} \sum_{j=1}^{r_n} < z, \varphi_{n,j} > \varphi_{n,j}$$

where $\tau > 0$ (sufficiently small), $(\varphi_{n,j})_{\substack{n\ge1 \\ 1\le j\le r_n}}$ is an orthonormal basis of eigenfunctions of the operator

$$Az = \sum_{n\ge1} \lambda_n \sum_{j=1}^{r_n} < z, \varphi_{n,j} > \varphi_{n,j}$$

corresponding to the eigenvalues $(\lambda_n)_{n\ge1}$, r_n is the multiplicity of λ_n.

Let us note that such systems may be a discrete version of a diffusion process (with an homogeneous Dirichlet boundary condition, for example).
In such a case, we have the following characterization result.

Proposition 6.

(1) *The system* (S_1) *dominates the system* (S_2) *exactly with respect to* C*, if and only if, there exists* $\gamma > 0$ *such that for any* $\theta \in Y'$*, we have*

$$\begin{aligned} &[\sum_{k=0}^{N-1} (\sum_{n\ge1} e^{\lambda_n k\tau} \sum_{j=1}^{r_n} < C^\star\theta, \varphi_{n,j} >< h_i, \varphi_{n,j} >)^2] \\ &\le \gamma \, [\sum_{k=0}^{N-1} (\sum_{n\ge1} e^{\lambda_n k\tau} \sum_{j=1}^{r_n} < C^\star\theta, \varphi_{n,j} >< g_i, \varphi_{n,j} >)^2] \end{aligned}$$

(2) *The system* (S_1) *dominates the system* (S_2) *weakly, with respect to* C*, if and only if*

$$[(< C^\star\theta, \varphi_{n,j} >)_j \in \ker(M_n); \forall n \in \mathbb{N}^\star] \Longrightarrow [(< C^\star\theta, \varphi_{n,j} >)_j \in \ker(G_n); \forall n \in \mathbb{N}^\star]$$

where M_n *and* G_n *are the corresponding controllability matrices defined respectively by :*

$$M_n = (< g_i, \varphi_{n,j} >)_{\substack{1\le i\le p \\ 1\le j\le r_n}} \text{ and } G_n = (< h_i, \varphi_{n,j} >)_{\substack{1\le i\le q \\ 1\le j\le r_n}}$$

If the system (S_1) dominates the system (S_2) exactly (respectively weakly), with respect to C, we say that the actuators $(\Omega_i, g_i)_{1\le i\le p}$ dominate $(D_i, h_i)_{1\le i\le q}$ with respect to C.

Now, in the practical case where the observation is given by means of sensors $(\omega_i, f_i)_{1\le i\le m}$, we have $Y = \mathbb{R}^m$ and the operator C is given by

$$Cz = \begin{pmatrix} < z, f_1 > \\ \vdots \\ < z, f_m > \end{pmatrix} \in \mathbb{R}^m$$

[1]The obtained results can be extended without major difficulties to more general situations

In this case, the exact and the weak dominations are equivalent. We deduce the following results.

Corollary 7. *The actuators* $(\Omega_i, g_i)_{1\leq i\leq p}$ *dominate* $(D_i, h_i)_{1\leq i\leq q}$ *with respect to the sensors* $(\omega_i, f_i)_{1\leq i\leq m}$*, if and only if*

$$\bigcap_{n\geq 1} \ker(M_n Q_n) \subset \bigcap_{n\geq 1} \ker(G_n Q_n)$$

where Q_n *is the observability matrix defined by*

$$Q_n = (< f_i, \varphi_{n,j} >)_{\substack{1\leq i\leq m \\ 1\leq j\leq r_n}}$$

Let us note the following :

1. One actuator may dominate p other actuators ($p > 1$) with respect to an output operator (sensors).
2. In the case of a discrete version of a diffusion process with a bounded and regular geometrical support $\Omega \subseteq \mathbb{R}^n$, then if $n = 1$, any operators B_1 and B_2 (and hence actuators) are comparable. This property is not true in a higher dimension $n \geq 2$ (for example in the case of a square $\Omega =]0,1[\times]0,1[$).

4. Domination with respect to output operators

In this section, we consider the problem of domination for observed discrete systems and output operators. We show that there is a duality between the two situations.

A dual problem : Let us first consider a dual problem where the control concerns the initial state z_0. More precisely, we consider the autonomous discrete system :

$$\begin{cases} z_{k+1} &= \Phi z_k \quad ; \; 0 \leq k \leq N-1 \\ z_0 &= B u_0 \end{cases} \tag{20}$$

where $\Phi \in \mathcal{L}(Z)$, $B \in \mathcal{L}(U, Z)$, $u_0 \in U$, z_k is the state of the system at step k. The system (20) is augmented with the output equation :

$$y_i = C_i z_i \quad ; \quad i = 1, 2$$

where for $i = 1, 2$; $C_i \in \mathcal{L}(Z, Y_i)$, Y_i is a Hilbert space; $z = (z_0,\ z_1,\ \cdots, z_{N-1})^{Tr}$ and $y_i = (y_{i,0},\ y_{i,1},\ \cdots, y_{i,N-1})^{Tr}$, with

$$y_{i,k} = C_i z_k \equiv C_i \Phi^k B u_0 \; ; \;\; 0 \leq k \leq N-1$$

We have

$$y_i \;=\; K_{i,N} u_0$$

where

$$K_{i,N} \;=\; C_i(\Phi^0, \Phi^1, \cdots, \Phi^{N-1})^{Tr} B \; ; \; \Phi^0 = I_Z$$

In order to establish a duality result, let $B_i = C_i^*$; for $i = 1, 2$ and $B = C^*$. We consider the dual systems :

$$(\mathcal{S}_i^*) \begin{cases} z_{i,k+1} &= \Phi^* z_{i,k} + B_i u_{i,k} \; ; \; 0 \leq k \leq N-1 \\ y_{i,k} &= C z_{i,k} \\ z_{i,0} &\in Z \end{cases}$$

and

$$(\widetilde{\mathcal{S}}_i)\begin{cases} z_{k+1} &= \Phi z_k \ ; \ 0 \leq k \leq N-1 \\ y_{i,k} &= C_i z_k \\ z_0 &= B u_0 \end{cases}$$

It is easy to show the following result establishing the relationship (duality) between the domination of these two systems.

Proposition 8.

i): *The controlled system* $(\mathcal{S}_1^*)$ *dominates the system* $(\mathcal{S}_2^*)$ *exactly, with respect to* C *if and only if*

$$Im[(K_{2,N})^*] \subset Im[(K_{1,N})^*]$$

ii): *The controlled system* $(\mathcal{S}_1^*)$ *dominates the system* $(\mathcal{S}_2^*)$ *weakly, with respect to* C *if and only if*

$$\overline{Im[(K_{2,N})^*]} \subset \overline{Im[(K_{1,N})^*]}$$

From this general result and the previous developments, we deduce similar results replacing B_i by C_i^*, actuators by sensors.

General controlled systems : Now, in the case of general systems of the form

$$\begin{cases} z_{k+1} &= \Phi z_k + B u_k \quad ; \ 0 \leq k \leq N-1 \\ z_0 &\in Z \end{cases}$$

augmented with the output equations

$$y_i = C_i z_i \quad ; \quad i = 1, 2$$

with the same hypothesis and notations. The domination of output operators is defined as follows.

Definition 9. *We say that the system with the output* C_1 *dominates the system with the output* C_2 *exactly (respectively weakly) if*

$$Im(C_2 H_N) \subset Im(C_1 H_N)$$

respectively

$$\overline{Im(C_2 H_N)} \subset \overline{Im(C_1 H_N)}$$

One can say simply that C_1 dominates C_2 exactly (respectively weakly) with respect to the pair (Φ, B).

Hence, if C_1 dominates C_2 (exactly or weakly), this traduces the fact that any observation (information) given by C_2, may be obtained (exactly or approximately) using C_1. That is to say that C_1 gives more rich information with respect to C_2, and observes better than C_2.

In this case, we note $C_2 \underset{(\Phi,B)}{\leq} C_1$ (respectively $B_2 \underset{(\Phi,B)}{\lesssim} B_1$).

Here also, we deduce easily analogous results and properties from those obtained on the domination of controlled systems and input operators.

In the case where the observations are given by means of sensors, the characterizations and the results are similar to those concerning the case of actuators.

Moreover, we have the following result concerning the domination of input and output operators. It is similar in the case of actuators and sensors.

Proposition 10. *If the following conditions hold:*

1) B_1 dominates B_2 exactly (respectively weakly) with respect to C_1.

2) C_1 dominates C_2 exactly (respectively weakly) with respect to the pair (Φ, B_2).

3) C_2 dominates C_1 exactly (respectively weakly) with respect to the pair (Φ, B_1).

then B_1 dominates B_2 exactly (respectively weakly) with respect to C_2.

5. Conclusion

In this paper, we have extended and characterized the notion of domination for a class of distributed discrete systems. We studied firstly the domination for controlled systems in connection with input operators, and then for observed systems and output operators. A duality between the two situations has been established. The case of actuators and sensors was explored and various properties are given. Definitions, different results and properties were concerned with the linear case, but one can consider more general situations and systems; and also regional and asymptotic aspects of this problem.

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