ESAIM: PROCEEDINGS AND SURVEYS, February 2015, Vol. 49, p. 11-22
Samira EL YACOUBI, Larbi AFIFI, El Hassan ZERRIK, Abdessamad TRIDANE, Editors

GLOBAL ADAPTIVE λ TRACKING OF A TEMPERATURE PROFILE IN TUBULAR REACTOR

N. Beniich[1], A. El Bouhtouri[2] and D. Dochain[3]

Abstract. This paper deals with the control design for a class of nonlinear distributed parameter systems, i.e. convection-diffusion-reaction systems, encountered under the form of non-isothermal tubular reactors in chemical engineering applications. More specifically the design concentrates on the boundary control of the temperature profile in an exothermic chemical reactor under input constraints. Our objective is to analyze the global stability of the closed-loop system by considering a simple control structure, i.e. an adaptive λ-tracking controller. It is shown that for all initial temperature and under simple feasibility assumptions, the tracking error tends asymptotically to a ball of arbitrary prescribed radius $\lambda > 0$, centered at the given temperature profile.

Résumé. Cet article traite un problème de conception de lois de commande pour une classe de systèmes non linéaires à paramètres répartis décrivant l'évolution de profil de température et de concentration d'un réacteur chimique tubulaire non isotherme. Notre objectif est la conception d'une loi de contrôle adaptative et globale basée sur l'approche de λ- tracking, satisfaisant les contraintes et qui permet de réguler la température du réacteur vers un voisinage de profil de température désiré. Il est montré que pour toute température initiale et sous des hypothèses simples, l'erreur de poursuite tend asymptotiquement vers une boule de centre 0 et de rayon arbitraire $\lambda > 0$.

Introduction

In this paper we consider an input constrained adaptive output feedback control for a class of nonlinear distributed parameter system, i.e. a non-isothermal tubular chemical reactor model in which the state variables are the temperature and the reactant concentration, the inputs are the coolant and the inlet temperatures and the reactant feedrate. Our objective is a reference profile control of the reactor temperature.
Recently, a constrained adaptive output feedback has been developed with the objective to regulate the temperature profile of exothermic tubular reactors, using a nonlinear distributed parameter model so that it converges to a prefixed neighborhood of a given reference profile [1]. It was shown that by acting only on the temperature equation, the convergence is local in the sense that the initial state is constrained to live in a prefixed set. In the present paper, we consider the same model as in [1] (axial dispersion exothermic tubular reactor model) and we show that under an extra control action via the reactant feedrate we can obtain the global convergence of the temperature profile towards a ball of arbitrary prefixed radius $\lambda > 0$ centered at the given temperature profile. The main difficulty for emphasizing the global convergence to the prefixed temperature profile without an extra coolant control action is due to the fact that when the initial state is too far from the desired profile,

[1] LINMA, Département de Mathématique Faculté des Sciences, BP 20, El Jadida, Morocco;e-mail: `Nadia.Beniich@gmail.com`
[2] LINMA, Département de Mathématique Faculté des Sciences, BP 20, El Jadida, Morocco;e-mail: `abdelmoula_elb@yahoo.fr`
[3] CESAME, Université catholique de Louvain 4-6 avenue Georges Lemaître, B-1348 Louvain-laNeuve, Belgium;
e-mail: `denis.dochain@uclouvain.be`

Article published online by EDP Sciences and available at http://www.esaim-proc.org or http://dx.doi.org/10.1051/proc/201549002

the control input is saturated, possibly at its lower bound. In such a situation, as the coolant temperature is constrained to be positive, an extra coolant action is needed, otherwise the temperature might converge to an undesirable equilibrium profile. The only approximation needed for the designed controller is to approximate the point input operators acting on the inlet temperature and the inlet reactant concentration by bounded ones, in line with the practical application viewpoint.

The paper is organized as follows. The basic dynamical model is presented in Section 2. We then develop a global constrained adaptive feedback controller of the reactor temperature and we end the section by explicitly formulating the hypothesis under which the tracking objective is achievable and by introducing a numerical simulation to justify the need for an extra control action (coolant temperature) to achieve the global convergence. In the third section, we state and prove our main results on the global λ-tracking. The paper is ended by numerical simulations for illustration purpose.

1. State space framework and statement of the control problem

Let us consider a non-isothermal reactor with the following chemical reaction:

$$A \longrightarrow bB$$

where $b > 0$, A and B are the stoichiometric coefficient of the reaction, the reactant and the product, respectively. The dynamics of the process in an exothermic tubular reactor with axial dispersion are readily obtained from energy and mass balances and are given by the following partial differential equations (e.g. [1] [5]):

$$\begin{aligned} \frac{\partial T(t,z)}{\partial t} &= D_1 \frac{\partial^2 T(t,z)}{\partial z^2} - v\,\frac{\partial T(t,z)}{\partial z} - \alpha f(T(t,z), C(t,z)) - k_0(T(t,z) - T_c(t,z)) \\ \frac{\partial C(t,z)}{\partial t} &= D_2 \frac{\partial^2 C(t,z)}{\partial z^2} - v\,\frac{\partial C(t,z)}{\partial z} - f(T(t,z), C(t,z)) \end{aligned} \tag{1}$$

with the boundary conditions :

$$-D_1 \frac{\partial T(t,0)}{\partial z} = v(T_{in}(t) - T(t,0)) \tag{2}$$

$$-D_2 \frac{\partial C(t,0)}{\partial z} = v(C_{in}(t) - C(t,0)) \tag{3}$$

$$\frac{\partial T(t,L)}{\partial z} = 0 \tag{4}$$

$$\frac{\partial C(t,L)}{\partial z} = 0 \tag{5}$$

In the above equations, t (>0) and z ($\in [0,L]$ with $L = 1 > 0$) hold for the time and the reactor length,respectively. $k_0 = \frac{4h}{\rho C_p d}$ (>0), $\alpha = \frac{-\Delta H}{\rho C_p}$ (>0), T, C, $D_1 > 0$, $D_2 > 0$, $v > 0$, $\Delta H < 0$, ρ, C_p, h, d, T_c , T_{in} and C_{in} are the temperature reactor, the reactant concentration, the energy and mass dispersion coefficients, the superficial fluid velocity, the heat of the reaction, the density, the specific heat, the wall heat transfer coefficient, the reactor diameter, the coolant temperature, the inlet temperature, and the inlet concentration, respectively. $f(T(t,z), C(t,z))$ is a nonlinear, positive and locally Lipshitz function that characterizes the reaction kinetics defined by $f(T(t,z), C(t,z)) = kC(t,z)e^{-\frac{E}{RT(t,z)}}$, with k, E and R the kinetic constant, the activation energy and the ideal gas constant, respectively.
In order to write the model (1)-(5) in an abstract semigroup formulation in Hilbert space, we consider the Hilbert space $H = L^2[0,1]$ endowed with the usual inner product and the usual partial order. The positive cone H^+ of H is defined by $H^+ = \{y \in H,\ s.t\ y \geq 0\}$. In this paper we consider that $T_{in}(t)$, $T_c(t)$ and $C_{in}(t)$

are system inputs, they appear in the boundary conditions, in order to obtain an appropriate boundary control problem in the abstract semigroup formulation given the system equations (2)-(5) are approximated as in [7] by:

$$\dot{x}_1(t) = A_1x_1(t) + \alpha f(x_1(t), x_2(t)) + v\Delta_\xi u_1(t) + k_0u_2(t) \tag{6}$$

$$\dot{x}_2(t) = A_2x_2(t) - f(x_1(t), x_2(t)) + v\Delta_\xi u_3(t) \tag{7}$$

with

$$x_1(t) = T(t, .),\ x_2(t) = C(t, .),\ u_1(t) = T_{in}(t),\ u_2(t) = T_C(t, .),\ u_3(t) = C_{in}(t)$$

and

$$A_1x = D_1\frac{\partial^2 x}{\partial z^2} - v\frac{\partial x}{\partial z} - k_0x, \quad x \in \mathcal{D}(A_1) \tag{8}$$

$$A_2x = D_2\frac{\partial^2 x}{\partial z^2} - v\frac{\partial x}{\partial z}, \quad x \in \mathcal{D}(A_2) \tag{9}$$

defined on:(for i=1,2)

$$\mathcal{D}(A_i) = \{x \in H : x, \frac{dx}{dz} \text{ absolutely continuous; } \frac{d^2x}{dz^2} \in H \text{ and } D_i\frac{dx}{dz}(0) - vx(0) = \frac{dx}{dz}(1) = 0\}$$

and

$$\Delta_\xi(z) = \begin{cases} \frac{1}{\xi} & 0 \le z \le \xi \\ 0 & \xi < z \le 1 \end{cases}$$

In the present paper we design and analyze a simple constrained adaptive controller with the objective to track a prefixed temperature profile, without violating the physical constraints and with global convergence properties. To this end, we recall the following results.
First of all, it is shown in [7] that A_1 and A_2 are Riesz spectral operators and infinitesimal generators of exponentially stable strongly continuous semigroups $(T_1(t))_{t\ge0}$ and $(T_2(t))_{t\ge0}$ of bounded operators in H. Besides, for all $x \in D(A_1)$ the inner product $\langle A_1x, x\rangle \le 0$, i.e. A_1 is dissipative.
Moreover, the following properties of $(T_i(t))_{t\ge0}$, (i =1, 2) are quoted from [7] [5] and will be used for the proof of our main results.

Lemma 1.1. *The following properties hold for T_i, i=1, 2 :*

i) $T_i(t)$ is a positive linear operator
ii) $(T_i(t)(M))(z) \le M$ for all $M > 0$ and $z \in [0, 1]$
iii) $(T_i(t)(M))(z) \le (T_i(t)M)(1)$ for all positive constant functions M and $z \in [0, 1]$

We consider the following initial value problem in H:

$$\begin{cases} \dot{x}(t) = Ax(t) + N(x(t)) \\ x(0) = x_0 \in D(A) \cap E \end{cases} \tag{10}$$

where A is the infinitesimal generator of a C_0-semigroup,E is a closed subset of H and N is a continuous function from E into H.
For $y \in H$, the distance from y to E is defined by $d(y; E) = \inf_{x\in E} \|y - x\|$

Lemma 1.2. *[5] Assume that:*

(i): *E is $T(t)$-invariant for all $t \ge 0$.*

(ii): *for all $x \in E$*

$$\lim_{h \longrightarrow 0^+} \frac{1}{h} d(x + hN(x); E) = 0$$

(iii): *N is continuous on E and there exists $l_n \in \mathbb{R}^+$ such that the operator $(N - l_n I)$ is dissipative on E, i.e $\langle (N - l_n I)(x) - (N - l_n I)(y), x - y \rangle \leq 0$ (I denotes the identity operator on H).*

Then the system (10) has a unique mild solution $x(t, x_0) \in E$ for all $x_0 \in E$ and $t \geq 0$. Moreover, if $S(t)$ is defined on E by $S(t)x_0 = x(t, x_0)$, for all $t \geq 0$ and $x_0 \in E$, it is a nonlinear semigroup on E, with $(A + N)$ as its generator.

Let us recall that we consider here the following input vector:

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = \begin{pmatrix} T_{in}(t) \\ T_c(t, z) \\ C_{in}(t) \end{pmatrix} \tag{11}$$

For physical reasons, we assume that u_i (i = 1, 3) are constrained as follows:

$$\underline{u}_1 \leq u_1(t) \leq \overline{u}_1, \ \underline{u}_2 \leq u_2(t) \leq \overline{u}_2, \ 0 \leq u_3(t) \leq \overline{u}_3 \quad \text{for all } t \geq 0 \tag{12}$$

where the inequalities hold in H and where $\underline{u}_i$, $\overline{u}_i$ (i = 1, 2), $\overline{u}_3$ are positive constants. We further consider the following assumptions:

(H_1) The positive cone $H^+ \times H^+$ is positively invariant under (6)(7) for all nonnegative control $u_i(.)$ (i = 1, 3) satisfying the above constraints.

(H_2) The desired profile $x^* \in H^+ \cap \mathcal{D}(A_1)$ and there exist $\rho > 0$, $\underline{x} \in H$, $\overline{x} \in H^+ \cap \mathcal{D}(A_1)$ with $0 < \underline{x} < x^* < \overline{x}$ such that for all x_1, x_2 in H satisfying $\underline{x} \leq x_1 \leq \overline{x}$ and $0 \leq x_2 \leq \overline{u}_3$:

$$\begin{cases} v\Delta_\xi \underline{u}_1 + k_0 \underline{u}_2 + \rho & \leq \quad k_0 x_1 - \alpha f(x_1, x_2) - A x^* \\ & \leq \quad k_0 \overline{u}_2 - \rho \\ A(\overline{x} - x^*) & \leq \quad 0, \end{cases} \tag{13}$$

where $A = A_1 + k_0 I$

(H_3) There exists a continuous function $\Phi_1 : H \mapsto H$ such that for all $y_1 \geq 0$ and all y_2 satisfying $0 \leq y_2 \leq \overline{u}_3$:

$$f(y_1, y_2) \ \leq \ \Phi_1(y_2) y_1 \ \text{ and } \ \lim_{y \to 0} \Phi_1(y) = 0 \tag{14}$$

(H_4) $0 < \lambda < \overline{x} - x^*$

For a given $\lambda > 0$, we define the tracking error by $e(t)$ and the control law $u_i(t)$ (i = 1, 2) as follows:

$$e(t) = x^* - x_1(t) \tag{15}$$

$$m(t) = \min_{z \ \in \ [0,1]} (e(t)(z)) \tag{16}$$

$$u_1(t) = sat_{[\underline{u}_1, \overline{u}_1]}(\beta_2(t) m(t) + u_1^*) \tag{17}$$

$$u_2(t) = sat_{[\underline{u}_2, \overline{u}_2]}(\beta_2(t) e(t) + u_2^*) \tag{18}$$

$$\dot{\beta}_i(t) = k_i \begin{cases} (\| e(t) \| - \lambda)^{l_i} & \text{if } \| e(t) \| > \lambda \quad i = 1, 2 \\ 0 & \text{if } \| e(t) \| \leq \lambda \end{cases} \tag{19}$$

$$\beta_i(0) = \beta_{0,i}, \ i = 1, 2 \tag{20}$$

As in [4], a possible choice of the extra control action u_3 is given by:

$$u_3(t) = \begin{cases} 0 & \text{if} \quad \beta_3(t)m(t) \in (-\infty, a] \\ g(\beta_3(t)m(t)) & \text{if} \quad \beta_3(t)m(t) \in (a, a+\delta] \\ \overline{u}_3 & \text{if} \quad \beta_3(t)m(t) \in (a+\delta, +\infty) \end{cases} \tag{21}$$

$$\beta_3(0) = \beta_{0,3} \tag{22}$$

with δ a sufficiently small positive constant and:

$$g(\beta_3(t)m(t)) = (\beta_3(t)m(t) - a)\frac{\overline{u}_3}{\delta} \quad \text{and} \quad a = \inf\{\underline{u}_1 - u_1^*, \underline{u}_2 - u_2^*\} \tag{23}$$

The saturation function $sat_{[\underline{u}_i, \overline{u}_i]}(y)$ is defined for all $y \in \mathbb{R}$ by:

$$sat_{[\underline{u}_i, \overline{u}_i]}(y) = \begin{cases} \overline{u}_i & \text{if} \quad y \geq \overline{u}_i \\ y & \text{if} \quad \underline{u}_i < y < \overline{u}_i \\ \underline{u}_i & \text{if} \quad y \leq \underline{u}_i \end{cases} \tag{24}$$

Remark 1.3.

- The feedback (17)-(21) has the following features : if $x_1 \geq \overline{x}$, we choose $\beta_{0.i}$ (i = 1, 2) so that $u_i(t) = \underline{u}_i$ and $u_3(t) = 0$ and hence we can obtain an upper bound of x_1 and x_2.
- The kinetics function has the following form [3] : $f(T, C) = K(T)\varphi(C)$ where $T \longrightarrow K(T)$ is positive, bounded and globally Lipschitz, and the function $C \longrightarrow \varphi(C)$ is nonnegative, continuous and vanishes if $C = 0$. This ensures the feasibility of (H_3).
- To illustrates the limitations in the performance of the controller with only two control actions u_1 and u_2, and with initial conditions far enough from the desired steady state profile, we consider same data under which the local convergence is obtained see [1]:

$$\begin{aligned} x^* &= 480, \ \bar{x} = 700, \ x_1(0) = 800, \ x_2(0) = 0.07, \ \beta_{0,1} = 4.2, \ \beta_{0,2} = 3.9 \\ \underline{u}_1 &= 155, \ u_1^* = 173, \ \overline{u}_1 = 273, \ \lambda = 2, \ \underline{u}_2 = 395, \ u_2^* = 450, \ \overline{u}_2 = 630, \ \xi = 0.3 \end{aligned}$$

The process parameters: $v = 1$ m/s, $D_1 = 0.25$ m^2/s, $D_2 = 0.25$ m^2/s, $E = 11916$ cal/mole, $k = 0.83$ s^{-1}, $\dfrac{4h}{\rho C_p d} = 13$ s^{-1} and $R = 1.986$ cal/(mole.K).

2. The global λ-tracking

To achieve the main control objective of this work, we consider two feedback strategies that will drive the temperature profile into a λ-neighborhood of the given reference temperature profile. Before considering the non adaptive version of the controller given by (17)(21) where for i = 1 to 3, $\beta_i : \mathbb{R}^+ \mapsto [\beta_i^*, +\infty)$ are a fixed continuous functions. The main result in this case is to show that there exists a $t' > 0$ such that for all $t \geq t'$, $\| e(t) \| \leq \lambda$.

Proposition 2.1. *Assume that (H_1) and (H_3) hold. Then the feedback control (17)(18)(21) applied to (6)(7) produces a unique global solution that satisfies:*

$$(x_1(.), x_2(.)) : I\!R^+ \longmapsto H^+ \times \{x_2 \in H : \ 0 \leq x_2 \leq \overline{u}_3\} \tag{25}$$

Proof. It follows from standard existence and uniqueness theorems on the solutions of ordinary differential equations in Hilbert space ([2], pp 56-57; [6], Theorem 1.4, pp 185-186) that for any initial condition $(x_1(0), x_2(0)) \in$

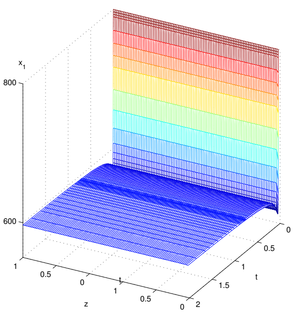

FIGURE 1. Control with only two control inputs

$H^+ \times \{x_2 \in H$ such that $0 \le x_2 \le \overline{u}_3\}$, there exists an unique maximal solution $(x_1(t), x_2(t))$ of (6)(7) with u_i, (i = 1 to 3) given by (17)(18)(21). This solution is defined on a maximal half-open interval $[0, w)$, $w \in (0, +\infty]$:

$$x_2(t) = T_2(t)x_2(0) + \int_0^t T_2(t-s)v\Delta_\xi u_3(\beta(s)e(s))ds - \int_0^t T_2(t-s)f(x_1(s), x_2(s))ds$$

The positivity of $T_2(t)$ and f implies that: $x_2(t) \le T_2(t)x_2(0) + \int_0^t T_2(t-s)\frac{v}{\xi}\overline{u}_3 ds$.
Consider the following system:
$\dot{y}(t) = A_2 y(t) + \frac{v}{\xi}\overline{u}_3\,; y(0) = x_2(0)$ and $E = \{x$ in H such that $x(z) \le \overline{u}_3$ for almost all $z \in [0,1]\}$.
and apply Lemma 1.1, we obtain $0 \le T_2(t)x \le T_2(t)\overline{u}_3 \le \overline{u}_3$; for all $x \in E$. Thus $T_2(t)E \subset E$, by Lemma [theorem 2.1 p 172 [5]] with $N(y) = \frac{v}{\xi}\overline{u}_3$, we have:

$$x_2(t) \le y(t) \le \overline{u}_3 \text{ for all } t \in [0, w).$$

For x_1, we have: $x_1(t) = T_1(t)x_1(0) + \int_0^t T_1(t-s)(\alpha f(x_1(s), x_2(s)) + v\Delta_\xi u_1(s) + k_0 u_2(s))ds$.
The positivity of $T_1(t)$ and $f(x_1(t), x_2(t)) \le \Phi_1(x_2(t))x_1(t))$ implies that:

$$x_1(t) \le T_1(t)x_1(0) + \int_0^t T_1(t-s)(\alpha\Phi_1(x_2(s))x_1(s)) + v\Delta_\xi u_1(s) + k_0 u_2(s))ds.$$

$T_1(t)$ is exponentially stable, then there exist a negative constant w_1 and $M \ge 1$ such that $\|T_1(t)\| \le Me^{\omega_1 t}$.
Consequently: $\|x_1(t)\| \le M\|x_0\| + M\int_0^t e^{\omega_1(t-s)}\|x_1(s)\| + M_1,\ \ M_1 > 0$.
The Gronwall's lemma implies that there exist $M_2 > 0$ such that: $\|x_1(t)\| \le M_2\ exp\ (M\int_0^t e^{\omega_1(t-s)}ds)$.
The boundedness of x_1, x_2 ensures that $w = +\infty$, we can conclude then that the closed-loop system has a global solution $(x_1(.), x_2(.))$ such that $x_1 \ge 0$ and $0 \le x_2 \le \overline{u}_3$ for all $t \ge 0$. □

The following proposition will be used in the sequel.

Proposition 2.2. *Assume that $(H_1)(H_2)(H_3)(H_4)$ hold and that for i= 1, 2,3 :*

$$\beta_i^* > \frac{u_i^* - \underline{u}_i}{\lambda} \tag{26}$$

$$((-A_1)^{-1}(\frac{v}{\xi}\underline{u}_1 + k_0\underline{u}_2))(1) < \overline{x} \tag{27}$$

Then there exists $t_1 > 0$ such that $x_1(t) \le \overline{x}$ for all $t \ge t_1$

Proof. 1) We first show that if there exists t_1 such that $x_1(t_1) \leq \overline{x}$, then $x_1(t) \leq \overline{x}$; $\forall t \geq t_1$, or equivalently $\Delta_1 = \{x \text{ in } H \text{ such that } x \leq \overline{x}\}$ is positively invariant with respect to (6).
From (H_3) we know that there exists a function Φ_1 such that $f(x_1, x_2) \leq \Phi_1(x_2)x_1$ for all $x_1 \geq 0$ and $0 \leq x_2 \leq \overline{u}_3$, this implies that $f(x_1, x_2) + v\Delta_\xi u_1 + k_0 u_2$ is upper bounded by a positive constant M.
Then $x_1(t) \quad \leq \quad T_1(t-t_0)x_1(t_0) \int_{t_0}^{t} T_1(t-t_0-s)Mds$
Let us consider the following system defined in Δ_1: $\dot{y} = A_1 y + M;\ y(t_0) = x_1(t_0)$.
It is easy to show that this system verifies the properties of Lemma 1.2. Consequently Δ_1 is positively invariant with respect to (6). Therefore $x_1(t_1) \leq \overline{x}$, then $x_1(t) \leq \overline{x}$ for all $t \geq t_1$.
2) Let us now show the existence of t_1 such that $x_1(t_1) \leq \overline{x}$.
Assume that $\forall\, t \geq 0$ there exists $z_t \in [0, L]$ such that $x_1(t)(z_t) = T(t, z_t) > \overline{x}(z_t)$. We then have from H_2 and (26): $\quad \forall\, 0 \leq s \leq t, \quad x_1(s)(z_t) > \overline{x}(z_t)$.
From (26) and (H_4) we obtain for i = 1, 2 and $s \in [0, t]$:

$$\beta_1(s)m(s) + u_1^* \leq \underline{u}_1 \text{ and } \beta_3(s)e(s)(z_t) + u_i^* \leq \beta_3(s)(x^* - x_1(s)(z_t)) + u_i^* \leq \underline{u}_i.$$

Then $\beta_3(s)m(s) \leq a$.
From(17)(18)(21), we can conclude that :

$$u_1(s) = \underline{u}_1, \ \ u_2(s)(z_t) = \underline{u}_2 \text{ and } u_3(s) = 0 \quad \forall s \in [0, t] \tag{28}$$

First we show that:

$$\lim_{\substack{t \longrightarrow +\infty \\ r \longrightarrow +\infty}} x_2(r)(z_t) = 0$$

(6)(7)(28) imply that :

$$x_2(r)(z_t) = (T_2(r)x_2(0))(z_t) - \int_0^r T_2(r-s)f(x_1(s), x_2(s))(z_t)ds$$

Since T_2 is positive, we have $x_2(r)(z_t) \leq (T_2(t)x_2(0))(z_t)$ $\forall r \in [0, t]$. By using the fact that $x_2(r) \leq \overline{u}_3$ and Lemma 1.2, we have :

$$(T_2(r)x_2(0))(z_t) \ \leq \ (T_2(r)\overline{u}_3)(z_t) \leq (T_2(r)\overline{u}_3)(1)$$

for all $r \in [0, t]$. Moreover T_2 is exponentially stable. Therefore :

$$\lim_{\substack{t \longrightarrow +\infty \\ r \longrightarrow +\infty}} (T_2(r)\overline{u}_3)(1) = 0$$

hence

$$\lim_{\substack{t \longrightarrow +\infty \\ r \longrightarrow +\infty}} (T_2(r)x_2(t))(z_t) = 0$$

Assumption (H_3) implies that there exists $t_0 > 0$ such that $\forall r \in [t_0, t]$ $\Phi_1(x_2(r)(z_t)) \leq \epsilon$. Hence

$$\begin{array}{rcl} x_1(t)(z_t) & \leq & (T_1(t-t_0)x_1(t_0))(z_t) + \int_{t_0}^{t} (T_1(t-s)(v\Delta_\xi \underline{u}_1 + k_0 \underline{u}_2))(z_t)ds \\ & & + \epsilon\alpha \int_{t_0}^{t} (T_1(t-s)x_1(s))(z_t)ds. \end{array}$$

Therefore $x_1(t)(z_t)$ is bounded and hence $\displaystyle\int_{t_0}^{t} (T_1(t-s)x_1(s))(z_t)ds$ is bounded. By taking the limit as ϵ goes to 0, we obtain :

$$x_1(t)(z_t) \leq (T_1(t-t_0)x_1(t_0))(z_t) + \int_{t_0}^{t} (T_1(t-s)(v\Delta_\xi \underline{u}_1 + k_0 \underline{u}_2))(z_t)ds$$

And from the exponential stability of T_1 and Lemma 1.2, we have :

$$\lim_{t\longrightarrow+\infty}(T_1(t-t_0)x_1(t_0))(z_t)=0$$

Therefore

$$\begin{aligned}\lim_{t\longrightarrow+\infty}\int_{t_0}^{t}(T_1(t-s)(\frac{v}{\xi}\underline{u}_1+k_0\underline{u}_2))(z_t) &\leq \lim_{t\longrightarrow+\infty}\int_0^{+\infty}(T_1(s)(\frac{v}{\xi}\underline{u}_1+k_0\underline{u}_2))(1)\\ &\leq \int_0^{+\infty}(T_1(s)(\frac{v}{\xi}\underline{u}_1+k_0\underline{u}_2)(1)\\ &=(A_1^{-1}(\frac{v}{\xi}\underline{u}_1+k_0\underline{u}_2))(1)\end{aligned}$$

And (27) leads to : $\lim_{t\longrightarrow+\infty}x_1(t)(z_t)\leq(A_1^{-1}(\frac{v}{\xi}\underline{u}_1+k_0\underline{u}_2))(1)$ which contradicts the fact that $x_1(t)(z_t)>\overline{x}(z_t)\ \forall t\geq 0$. Therefore there exists t_1 such that the proposition holds. □

Remark 2.3.
- Proposition 2.1 ensures the global existence and the positive invariance of $H^+\times\{x_2\in H$ such that$0<x_2<\overline{u}_3\}$ under (6)(7) and (17)(18)(21) while the Proposition 2.2 ensures that $x_1\leq\overline{x}$.
- In order to meet the assumption (27) of Proposition 2.2, the inverse operator of A_1 has to be computed. From Lemma 1.1 we can replace this assumption by the following one :

$$\frac{v}{\xi}\underline{u}_1+k_0\underline{u}_2<k_0\overline{x}$$

The following notations are of interest for the proof of our main result:

$$\begin{aligned}C_1 &= \max\{\|k_0x_1-\alpha f(x_1,x_2)-Ax^*-\tfrac{v}{\xi}\underline{u}_1+k_0u_2^*\|,\ (x_1,x_2)\in\Delta_1\times\Delta_2\}\\ C_2 &= max\{\|\overline{x}-x^*\|_\infty,\|x^*\|_\infty\},\quad \varepsilon=\frac{\rho\lambda^3}{4C_2^2C_1}.\end{aligned}$$

Where Δ_1 and Δ_2 are given by :

$$\begin{aligned}\Delta_1 &= \{y_1\in H \text{ such that } 0\leq\ y_1\leq\overline{x}\},\\ \Delta_2 &= \{y_2\in H \text{ such that } 0\leq y_2\leq\ \overline{u}_3\}\end{aligned}$$

Proposition 2.4. *Assume that $(H_1)(H_2)(H_3)(H_4)$ and (27) hold, and that :*

$$\begin{cases}\beta_1^* &\geq\max\left(\frac{u_1^*-\underline{u}_1}{\lambda};\frac{\overline{u}_1-u_1^*}{\lambda};\frac{u_1^*-\underline{u}_1}{\overline{x}-x^*}\right)\\ \beta_2^* &\geq\max\left(\frac{u_2^*-\underline{u}_2}{\lambda};\frac{\overline{u}_2-u_2^*}{\lambda};\frac{u_2^*-\underline{u}_2}{\overline{x}-x^*};\frac{2C_1}{k_0\lambda};\frac{\lambda C_1}{k_0\varepsilon^2}\right)\end{cases}\tag{29}$$

Then there exists $t'>0$ such that:

$$\|e(t)\|\leq\lambda\qquad\forall\ t\geq t^{'}$$

Proof. We know from Proposition 2.2 that there exists t_1 such that $x_1(t)\leq\overline{x}$ for all $t\geq t_1$. Let us define the function :

$$V_\lambda(t)=d_\lambda^2(e(t))\quad\forall t\geq\ t_1$$

where: $d_\lambda(f)=\max\{\|f\|-\lambda,0\}\qquad\forall f\in L^2[0,1]$.
If $\|e(t)\|\leq\lambda$, then $\frac{d}{dt}V_\lambda(t)=0$.
If $\|e(t)\|>\lambda$, then :

$$\frac{d}{dt}V_\lambda(t)\ =\ 2\frac{\sqrt{V_\lambda(e(t))}}{\|e(t)\|}\langle\dot{e}(t),e(t)\rangle$$

Since $\langle Ae, e\rangle \leq 0$, we have :

$$\begin{aligned}\frac{d}{dt}V_\lambda(t) &\leq 2\frac{\sqrt{V_\lambda(e(t))}}{\|e(t)\|}\langle k_0x_1(t)-\alpha f(x_1(t),x_2(t)) - v\Delta_\xi u_1(t) - k_0u_2(t) - Ax^*, e(t)\rangle \\ &\leq 2\frac{\sqrt{V_\lambda(e(t))}}{\|e(t)\|}(\sum_{i=1}^{3}\int_{G_i} e(t)(k_0x_1(t)-\alpha f(x_1(t),x_2(t)) - v\Delta_\xi u_1(t) - k_0u_2(t) - Ax^*)(z)dz)\end{aligned}$$

where: $G_1 = \{z \in [0,1] \setminus e(t)(z) > \lambda\}$, $G_2 = \{z \in [0,1] \setminus e(t)(z) < -\lambda\}$
and $G_3 = \{z \in [0,1] \setminus -\lambda \leq e(t)(z) \leq \lambda\}$
Let us show that there exists a positive constant c such that:

$$\frac{d}{dt}V_\lambda(t) < -c\sqrt{V_\lambda(t)}$$

If $z \in G_1$, then (29) implies that $u_2(t)(z) = \overline{u}_2$ and by (H_2) :

$$\int_{G_1} e(t)(k_0x_1(t)-\alpha f(x_1(t),x_2(t)) - v\Delta_\xi u_1(t) - k_0u_2(t) - Ax^*)(z)dz < -\rho\lambda \,\mathrm{mes}(G_1)$$

If $z \in G_2$, using (29) we obtain $u_1(t) = \underline{u}_1$ $u_2(t)(z) = \underline{u}_2$. Assumption (H_2) implies:

$$\int_{G_2} e(t)(k_0x_1(t)-\alpha f(x_1(t),x_2(t)) - v\Delta_\xi u_1(t) - k_0u_2(t) - Ax^*)(z)dz < -\rho\lambda \,\mathrm{mes}(G_2)$$

If $z \in G_3$, and without loss of generality, we suppose that $u_2(t)(z) = \beta_2(t)e(t) + u_2^*$.
In order to obtain an upper bound of III, two cases of $\mathrm{mes}(G_1) + \mathrm{mes}(G_2)$ are possible :

- $\mathrm{mes}(G_1) + \mathrm{mes}(G_2) < \dfrac{\lambda^2}{2C_2^2}$
- $\mathrm{mes}(G_1) + \mathrm{mes}(G_2) \geq \dfrac{\lambda^2}{2C_2^2}$

In the first case, by using Proposition 2.1 and (29) it easy to show that:

$$\int_{G_3} e(t)(k_0x_1(t)-\alpha f(x_1(t),x_2(t)) - v\Delta_\xi u_1(t) - k_0u_2(t) - Ax^*)(z)dz < 0.$$

Therefore there exists a positive constant θ_1 such that :

$$\frac{d}{dt}V_\lambda(t) < -\theta_1\frac{\sqrt{V_\lambda(e(t))}}{\| e \|}$$

In the second case we have :

$$\begin{aligned} &\int_{G_3} e(t)(k_0x_1(t)-\alpha f(x_1(t),x_2(t)) - v\Delta_\xi u_1(t) - k_0u_2(t) - Ax^*)(z)dz \\ \leq &\int_{\varepsilon\leq|e(t)(z)|\leq\lambda} |e(t)(z)|.|[k_0x_1(t) - \alpha f(x_1(t),x_2(t)) - v\Delta_\xi \underline{u}_1 - k_0u_2^* - Ax^*](z)| \\ &-k_0\beta_2(t)e^2(t)(z)dz + \int_{0\leq|e(t)(z)|\leq\varepsilon} |e(t)(z)|.|[k_0x_1(t) - \alpha f(x_1(t),x_2(t)) - v\Delta_\xi \underline{u}_1 \\ &-k_0u_2^* - Ax^*](z)|dz - \int_{0\leq|e(t)(z)|\leq\varepsilon} k_0\beta_2(t)e^2(t)(z)dz \\ \leq &\int_{\varepsilon\leq|e(t)(z)|\leq\lambda} (-k_0\beta_2^*\varepsilon^2 + \lambda C_1)dz + \int_{0\leq|e(t)(z)|\leq\varepsilon} (\varepsilon C_1)dz \end{aligned}$$

From (29), we can show that:

$$\int_{G_3} e(t)(k_0 x_1(t) - \alpha f(x_1(t), x_2(t)) - v\Delta_\xi u_1(t) - k_0 u_2(t) - Ax^*)(z)dz \quad \leq \quad \varepsilon C_1$$

Finally, there exists a positive constant θ_2 such that :

$$\frac{d}{dt}V_\lambda(t) < -\theta_2 \frac{\sqrt{V_\lambda(e(t))}}{\| e \|}$$

So in all cases, there exists a positive constant c such that the time derivative of V_λ satisfy for all $t \geq t_1$:

$$\begin{cases} \frac{d}{dt}V_\lambda(t) \leq -c\sqrt{V_\lambda(t)} & \text{if} \quad \|e(t)\| > \lambda \\ \frac{d}{dt}V_\lambda(t) = 0 & \text{if} \quad \|e(t)\| \leq \lambda \end{cases}$$

We can conclude that there exists $t' > t_1$ such that: $\forall\, t > t' \parallel e(t) \parallel \leq \lambda$

□

In the above proposition, we have shown that for a sufficiently large lower bound β_i^* of $\beta_i(.)$ (i=1, 2) there exists a finite time such that the norm of the error is smaller than λ. We now turn to the adaptive version of the controller (17)(18)(21). Our objective in this case is to emphasize the asymptotic convergence of the temperature into a $\lambda-$neighborhood of a given temperature x^* with arbitrary initial states $x_1(0) > 0$ and $x_2(0) \in E$

Theorem 2.5. *Assume that $(H_1)(H_2)(H_3)(H_4)$ and (27) hold. Then for $i = 1, 2, 3$, the closed loop system has the following properties :*

(1) $x_1(.), x_2(.), \beta_i(.) : \mathbb{R}^+ \longrightarrow H^+ \times \{x_2 \in H\, 0 \leq x_2 \leq \overline{u}_3\} \times \mathbb{R}^+$

(2) $\lim_{t \longrightarrow +\infty} \beta_i(t)$ *exists and is finite.*

(3) $\limsup_{t \longrightarrow +\infty} \parallel e(t) \parallel \leq \lambda$

Proof. Let us first note that the theory of ordinary differential equations in Hilbert space ensures the existence and uniqueness of the maximal solution in a maximal interval $[0, w)$. In Proposition 2.1 we have shown that $H^+ \times \{x_2 \in H$ such that $0 \leq x_2 \leq \overline{u}_3\}$ is positively invariant under (6)(7), u_i (i = 1, 2) and u_3 given by (17)(18)(21), and the adaptation gain in (19)(22) cannot exhibit a finite escape time on $[0, w)$ and so $w = +\infty$.

Let us show that $\beta_i(t)$ (i = 1, 2) is bounded. Suppose that $\beta_i(t)$ is unbounded. Then there exists $\widehat{t} \geq 0$ such that for i = 1, 2 :

$$\beta_i(t) \geq \max\Big(\frac{2C_1}{\lambda}; \frac{\lambda C_1}{\varepsilon^2}, \frac{u_i^* - \underline{u}_i}{\lambda}; \frac{\overline{u}_i - u_1^*}{\lambda}\Big) \qquad \forall t \geq \widehat{t}$$

Defining V_λ as in Theorem 2.4 for all $t \geq \widehat{t}$. We obtain:

$$\begin{array}{ll} \frac{d}{dt}V_\lambda(t) = 0 & \text{if} \quad \parallel e(t) \parallel \leq \lambda \\ \frac{d}{dt}V_\lambda(t) \leq -C_3\sqrt{V_\lambda} & \text{if} \quad \parallel e(t) \parallel > \lambda \end{array}$$

for some positive constant C_3. In summary we obtain for all $t \geq \widehat{t}$, $\frac{d}{dt}V_\lambda(t) \leq -C_3\sqrt{V_\lambda(t)}$, and so there exists $t' \geq \widehat{t}$ such that $\forall t \geq t' \;\parallel e(t) \parallel \leq \lambda$. Hence for i = 1 to 3, $\dot{\beta}_i(t) = 0\ \forall t \geq t'$, which contradicts the fact that β_i is unbounded. The result follows from the monotonicity of β_i.

Finally, it remains to prove the assertion (3) of the theorem, for $y \in H$. Let

$$d_\lambda(y) = \begin{cases} (\|y\| - \lambda)^l & \text{if} \quad \|y\| > \lambda \\ 0 & \text{if} \quad \|y\| \leq \lambda \end{cases}$$

By definition of the adaptation gain $\beta_3(.)$, we have :

$$\forall t \geq 0, \quad k_1 \int_0^t d_\lambda(e(t))^l = \beta_3(t) - \beta_{0.3}$$

then

$$\lim_{t \longrightarrow +\infty} dist(\|e(t)\|; [0, \lambda]) = 0$$

This completes the proof of the theorem. □

3. Numerical simulation results

Let us now illustrate the above results in numerical simulations. Let us consider the model equations (6)(7) with reaction kinetics modelled by third order with respect to the reactant concentration C and the Arrhenius law in temperature :

$$f(x_1, x_2) = kx_2^3 e^{-E/Rx_1} \tag{30}$$

and the same parameter values given below.

Let us consider an uniform temperature setpoint : $x^* = 480$ K, and the following constraints for the inputs u_i (i = 1, 2) are chosen as follows :

$$\underline{u}_1 = 51,\ u_1^* = 90,\ \overline{u}_1 = 91,\ \underline{u}_2 = 359,\ u_2^* = 450,\ \overline{u}_2 = 630 \tag{31}$$

It is routine to check that the feasibility assumptions $(H2)$,(H_3) and (H_4) are fulfilled in this case if :

$$\overline{x} = 700,\ \underline{x} = 400,\ \rho = 0.2,\ \bar{u}_3 = 0.07 \tag{32}$$

Let us also consider the following control parameters :

$$\lambda = 2,\ \delta = 0.1,\ \beta_{0.1} =,\ \beta_{0,2} = 3.9,\ \beta_3 = 4,\ \xi = 0.1,\ k_1 = k_2 = k_3 = 4,\ l_1 = l_2 = l_3 = 2$$

Note that the temperature in the reactor tends asymptotically to a ball centered at the reference signal x^* = 480 K and of radius $\lambda = 2$, along with the convergence of the adaptation gains.

4. Conclusion

In this paper we have considered a modification of the λ-tracking control approach developed in an earlier paper for a nonlinear distributed parameter exothermic chemical reaction in tubular reactor with axial dispersion in presence of input constraints. The aim of this modified version is to obtain global convergence properties. This gain of performance is obtained at the price of the introduction of an extra control action. The controller structure remains rather simple and its convergence properties are global in the sense that the initial temperature is only positive. Under simple feasibility assumptions, we have shown that the tracking objective is achievable. The performance of the controller are illustrated in numerical simulation.

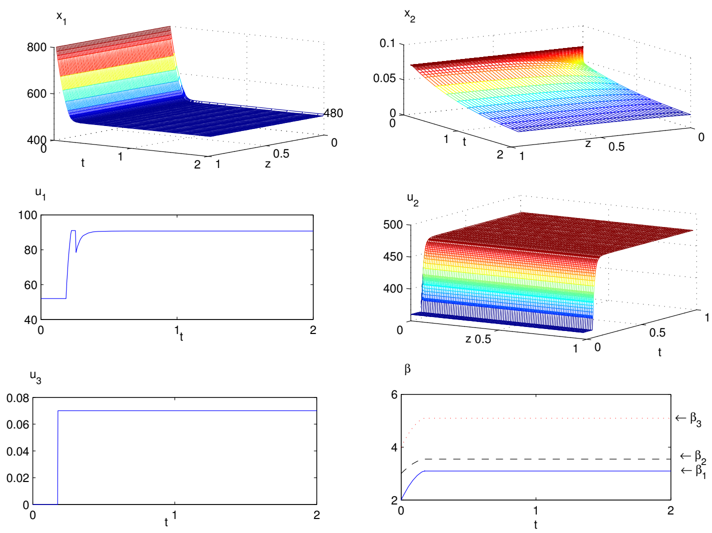

Figure 2. Numerical simulation of the global closed loop system

References

[1] Beniich N., A. El Bouhtouri and D.Dochain, Input constrained adaptive local tracking for a nonlinear distributed parameter tubular reactor, Int. J. Adapt. Control Signal Process V 24 (2010) pp 249-260.

[2] Cazenave T. and A. Haraux, An Introduction to Semilinear Evolution Equations, Oxford Lecture Series in Mathematics and its Applications Oxford University Press(1998). Dubljevic S., P. Mhaskar, N. H. El-Farra and P. Christofides, Predictive control of transport-reaction processes, Computers & Chemical Engineering. 29(11-12),(2005), 2335-2345.

[3] Jadot F., *Dynamics and Robust Nonlinear PI Control of Stirred Tank Reactors*, Doctoral thesis,(1996), Université catholique de Louvain, Belgium.

[4] Ilchmann A., M. Thuto and S. Townley, *Input constrained adaptive tracking with applications to exothermic chemical reactions*, SIAM J. Control. Opt, 43(2004), 154-173.

[5] Laabissi M., M.E. Achhab, J.J. Winkin and D. Dochain, *Trajectory analysis of nonisothermal tubular reactor nonlinear models*, Systems & Control Letters, 42(3)(2001), 169-184.

[6] Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York (1983).

[7] Winkin J., D. Dochain and P. Ligarius, *Dynamical analysis of disstributed parameter tubular reactors*, Automatica, 36, (2000), 349-361