ESAIM: PROCEEDINGS AND SURVEYS, February 2015, Vol. 49, p. 102-114
Samira EL YACOUBI, Larbi AFIFI, El Hassan ZERRIK, Abdessamad TRIDANE, Editors

SOME NEW STANDPOINTS IN THE DESIGN OF ASYMPTOTIC FUNCTIONAL LINEAR OBSERVERS.

DIE EULE DER MINERVA BEGINNT ERST MIT DER EINBRECHENDEN DÄMMERUNG IHREN FLUG.

F. Rotella and I. Zambettakis[1]

Abstract. *The owl of Minerva spreads its wing only with the falling of the dusk.* The aim of this work is to provide some new trends in the observation for linear systems. In the general framework of designing linear functional observers for linear systems the necessary and sufficient existence conditions are well known. Whether in the O'Reilly textbook or in the recently published ones on this topic, and, roughly speaking, the design methods can be categorized in two kinds. The first one is based on the solution of a Sylvester equation and a projection of the observed linear functional. The second one, based on the recent notion of functional observability, starts from the Darouach criterion which is an Popov-Belevitch-Hautus type one. Nevertheless, the main drawback of the deduced methods is that they cannot be used for linear time-varying systems. These models are of primary importance, for instance with linearization about a trajectory. Consequently, we cope with this problem by considering a new point of view for the design of linear functional observers. We see also that Darouach observers or Cumming-Gopinath observers are particular cases of the proposed methodology. For simplicity sake we suppose the system has no unknown inputs and is not described by a distributed parameters model as well. Nevertheless, these cases can be thought as possible extensions of the presented standpoints.

Résumé. *La chouette de la connaissance ne vole qu'à la nuit tombée.* L'objet de cet article est de proposer des pistes pour l'observation dans le cadre des systèmes linéaires. De façon plus générale, en considérant l'observation de fonctions linéaires de l'état d'un modèle linéaire, les conditions nécessaires et suffisantes d'existence de ces observateurs sont connues depuis longtemps. Que ce soit dans l'ouvrage de base de O'Reilly ou dans les ouvrages plus récents qui traitent de ce sujet, les différentes méthodes de conception d'un observateur de fonctionnelle linéaire peuvent, schématiquement, être rassemblées en deux groupes. Celles qui demandent la résolution d'une équation de Sylvester et d'une équation de projection et celles basées sur l'extension du critère d'existence de Darouach, ces dernières utilisant un critère de type Popov-Belevitch-Hautus. Ainsi les méthodes déduites de ces critères peuvent être difficilement étendues aux systèmes linéaires non stationnaires. De façon à contourner cet inconvénient majeur lorsque l'on songe à la linéarisation autour d'une trajectoire, nous proposons une technique pratique et systématique de conception d'un observateur. On montre par exemple que les observateurs de Darouach ou de Cumming-Gopinath n'en sont que des cas particuliers. Bien sûr, pour ne pas compliquer inutilement la présentation nous n'envisagerons que des modèles linéaires stationnaires de dimension finie à entrées toutes connues, les autre situations pouvant être envisagées à titre d'extension.

[1] Laboratoire de Génie de Production, 65016, Tarbes CEDEX, France.
rotella@enit.fr, izambettakis@iut-tarbes.fr

Introduction

The subtitle we have chosen for this paper may seem peculiar at first sight, and calls for explanations. We claim no authorship. This sentence is quoted from the German philosopher Georg Wilhelm Friedrich Hegel's book *Grundlinien der Philosophie des Rechts*, Nicolai, Berlin, 1820. Starting our presentation with such a quotation is our way of paying homage to all the researchers who adopted a new approach and opened new prospects to their subject, Automatic Control in particular, when their discipline was turning into academic routine. Abdelhacq El Jai is definitely one of the Elect. Allow us to point out that Hegel uses the phrase *Eule der Minerva* (Minerva's Owl), synonymous with wisdom in mythology. But as another such researcher claimed (he might recognize his statement), of all subjects, Automatic Control being the one we are the least ignorant of, we will stick to it.

What comes before is not the only reason why we have chosen such a title as we are now going to focus on observers of linear functional of the state of a linear system. Indeed, some interesting variables in a system are not measured. The owl's nocturnal sight leads to a piece for an analogy with the observation process, namely, to design a system to estimate hidden variables in the darkness of information. The observation of a linear functional of the state is motivated by the facts that, most of the time to know all the variables can be useless, or, the order of the observer must be as small as possible. For instance, the implementation of a static state feedback control $u(t) = v(t) - Kx(t)$ where K is a matrix gain, $x(t)$ the state vector and $v(t)$ the reference trajectory, by estimating directly $Kx(t)$ allows to get a smaller observer than a Cumming-Gopinath observer which is an entire state observer. Nevertheless, the observation of a linear functional of the state can be useful for other applications in control systems as diagnostic, fault detection and isolation [20]. From this viewpoint, an observer is used to generate residues that allow to take a decision inside a supervision and diagnostic level of the system when disturbances or faults occur on the process. Indeed, these variables influence the system but cannot be measured. The aim is then to build residues that will be, as the case may be, sensitive to faults or insensitive to disturbances. So the purpose is here to give a systematic design method for a linear observer of a linear functional in case of a linear model.

Since initial works of [10, 17, 18] two kinds of design techniques of such observers can be distinguished. On the one hand, we find those who start with the solution of the Sylvester equation which appears in the existence conditions of such an observer. Let us mention for example [30, 33]. On the other hand, we have the methods as those proposed in [6, 7] which build, by using the result of [5], the observer of a bigger size functional, which includes of course the functional to observe. Finely, notice that a lot of solutions need the initial model to be on a canonical form (e.g. [16]) and often consider the dynamics of the observer to be fixed at the outset as well. Nevertheless, whatever approach may be chosen, these techniques didn't lead, until now, to major progress for the design of a minimum size stable observer neither for the possible extension to other class of models. The reader can refer to [15, 22, 31] for more details on the different techniques considered to build a linear functional observer. The main interest of our approach is to avoid drawbacks of usual methods to design, quite directly, an asymptotic observer of a linear functional of the state of a linear system.

The origin of the developments we propose starts, for their principle, with the paper [9] of Michel Fliess where the design of a state observer is based on the notion of derivation. Two main ideas come out here. First, the design is not based on pole placement. These one being fixed *a priori,* their choice is not easy without any other constraint. Secondly, this technique can be extended to the case of linear time varying models, without too much difficulties, if the coefficients are enough derivable time functions. Moreover, this extension has been proposed first in [1], then by mean of the flat model of the system in [8] and in [29] using the algebraic approach [21] to estimate signals derivatives. In the first place necessary and sufficient conditions are outlined for the existence of an asymptotic observer of a linear time invariant system. These conditions need to solve a Sylvester equation inside which the linear transformation between the state of the system and the state of the observer is unknown. This Sylvester equation is not easy to solve because some design parameters of the observer (order, poles, etc.) are unknown *a priori*, and moreover, the solution is linked to the decomposition of the linear functional to observe. That's why the conditions of compatibility are difficult to verify [11, 27]. To avoid this problem we propose in a second part an original way to design an observer, based on the use of successive derivatives of the measured outputs and of the functional to observe as well. An usual realization method allows then to get the structure of the observer. The last point of this part deals with the stability analysis of the so-build observer which allows to conclude the design. A part is then devoted to some particular cases. Finally we show, in the appendix, how the method we propose induces a solution to the Sylvester equation related to the observer. This allows to show that the presented method generalizes the Darouach and Cumming-Gopinath observers.

1. Observation of a state linear functional

We consider a system modelized, after linearization around the set point [3, 23], by the state model :

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ m(t) &= Mx(t), \end{aligned} \tag{1}$$

where, for every t in $\mathbb{R}$, $x(t)$, is the n-dimensional state vector, $u(t)$, the p-dimensional control supposed to be accessible, and $m(t)$, the m-dimensional measure. Notice that these variables have to be distinguished from the variables to drive. A, B and C are constant matrices with convenient dimensions. The aim is to estimate, at least asymptotically, from measurements on the system, that is to say inputs $u(t)$ and measures $m(t)$ from every sensors, a vector described by :

$$v(t) = Lx(t), \tag{2}$$

where L is an $(l \times n)$ matrix whose size is fixed by the user. The observation of $v(t)$ can be carried out with the design of a linear functional observer or Luenberger observer [19] :

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gu(t) + Hm(t), \\ w(t) &= Pz(t) + Vm(t), \end{aligned} \tag{3}$$

whre $z(t)$ is its q-dimensional state vector and $w(t)$ an l-dimensional vector. Constant matrices F, G, H, P, V and order q are determined such that $\lim_{t\to\infty}(v(t) - w(t)) = 0$. In order to simplify the design of the observer we can suppose, without loss of generality, that :

$$\text{rank}\left(\begin{bmatrix} M \\ L \end{bmatrix}\right) = m + l.$$

In a first step we look for conditions ensuring, at least asymptotically, a linear relationship between $x(t)$ and $z(t)$ as $z(t) = Tx(t)$. Let us denote $\varepsilon(t) = z(t) - Tx(t)$, we get :

$$\begin{aligned} \dot{\varepsilon}(t) &= \dot{z}(t) - T\dot{x}(t), \\ &= Fz(t) + Gu(t) + Hm(t) - TAx(t) - TBu(t), \\ &= F\varepsilon(t) + (G - TB)\,u(t) + (FT - TA + HM)\,x(t). \end{aligned}$$

so if :

$$\begin{aligned} G &= TB, \\ TA - FT &= HM, \end{aligned}$$

when F is a Hurwitz matrix we have $\lim_{t\to\infty}\varepsilon(t) = 0$.

Now, the estimation error $e(t) = v(t) - w(t)$, can be written :

$$\begin{aligned} e(t) &= Lx(t) - Pz(t) - Vm(t), \\ &= (L - PT - VM)\,x(t) - P\varepsilon(t). \end{aligned}$$

If $L = PT + VM$, the conditions which lead to $\lim_{t\to\infty}\varepsilon(t) = 0$ ensure $\lim_{t\to\infty} e(t) = 0$.

So we prove here that if there exists a matrix T such that the following linear equations hold :

$$\begin{aligned} &G = TB, \\ &TA - FT = HM, \\ &L = PT + VM, \end{aligned} \tag{4}$$

with F a Hurwitz matrix, then an asymptotic estimation of the linear functional $v(t) = Lx(t)$ can be get by a Luenberger observer (3) whose parameters are determined to ensure the existence of matrices T and F. A lot of methods exist to find, in some particular cases, a minimum q-order observer with the eighenvalues of matrix F fixed at the outset [22, 31]. But the general problem of the design of an asymptotic observer which order is as small as possible has not found a complete solution yet.

2. Design of a Luenberger observer

The main feature of the design technique we present here is to start from the solution of a linear consistent system. The principle of the design of the observer uses successive derivatives of $v(t)$. We get then, for $k = 0$, 1, . . . :

$$v^{(k)}(t) = LA^k x(t) + \sum_{i=0}^{k-1} LA^{k-1-i} Bu^{(i)}(t).$$

Suppose there exist an index ν and matrices $F_{L,i}$, $i = 0$ to $\nu - 1$, and $F_{M,i}$, $i = 0$ to ν, such that :

$$LA^\nu = \sum_{i=0}^{\nu-1} F_{L,i} LA^i + \sum_{i=0}^{\nu} F_{M,i} MA^i, \tag{5}$$

then :

$$v^{(\nu)}(t) = \sum_{i=0}^{\nu-1} F_{L,i} LA^i x(t) + \sum_{i=0}^{\nu} F_{M,i} MA^i x(t) + \sum_{i=0}^{\nu-1} LA^{\nu-1-i} Bu^{(i)}(t). \tag{6}$$

2.1. Construction of matrices F to V.

The second step is to eliminate the state $x(t)$ until $v(t)$, $y(t)$ and their derivatives come into view. We use then the derivatives of $v(t) = Lx(t)$ and of $m(t) = Mx(t)$. It leads to :

- for $i = 1$ to $\nu - 1$: $v^{(i)}(t) = LA^i x(t) + \sum_{j=0}^{i-1} LA^j Bu^{(i-1-j)}(t)$, so :

$$LA^i x(t) = v^{(i)}(t) - \sum_{j=0}^{i-1} LA^{i-1-j} Bu^{(j)}(t);$$

- for $i = 1$ to ν : $m^{(i)}(t) = MA^i x(t) + \sum_{j=0}^{i-1} MA^j Bu^{(i-1-j)}(t)$, so :

$$MA^i x(t) = m^{(i)}(t) - \sum_{j=0}^{i-1} MA^{i-1-j} Bu^{(j)}(t).$$

Putting these expressions in (6) we get :

$$v^{(\nu)}(t) = \sum_{i=0}^{\nu-1} F_{L,i} v^{(i)}(t) + \sum_{i=0}^{\nu} F_{M,i} m^{(i)}(t) + \sum_{i=0}^{\nu-1} G_i u^{(i)}(t), \tag{7}$$

where matrices G_i can be calculated by means of the data. Indeed, after some algebraic manipulations we get $G_{\nu-1} = (L - F_{M,\nu}M)\,B$ and for $\nu \geq 2$ and $j = 0$ to $\nu - 2$:

$$G_j = \left(LA^{\nu-1-j} - \sum_{i=j+1}^{\nu-1} F_{L,i} LA^{i-1-j} - \sum_{i=j+1}^{\nu} F_{M,i} MA^{i-1-j} \right) B. \tag{8}$$

The third step consists of building a first order state order equation related to relation (7). Among all the realization techniques for usual differential equations [14,23] the most appropriate one is based on factorization, after operational coding, of the differential equation (7). Using the Heaviside coding of the time derivative operator p, so the coding of the time integration operator p^{-1} as well, this differential equation can be written :

$$\begin{aligned} v(t) \;= F_{M,\nu}m(t) + \; & p^{-1}\left[F_{L,\nu-1}v(t) + F_{M,\nu-1}m(t) + G_{\nu-1}u(t)\right. \\ & +p^{-1}\left[F_{L,\nu-2}v(t) + F_{M,\nu-2}m(t) + G_{\nu-2}u(t)\right. \\ & \vdots \\ & +p^{-1}\left[F_{L,1}v(t) + F_{M,1}m(t) + G_1 u(t)\right. \\ & +p^{-1}\left[F_{L,0}v(t) + F_{M,0}m(t) + G_0 u(t)\right]\right] \cdots \left.\right]. \end{aligned}$$

Let us define the following vectors :

$$\begin{aligned} z_0(t) &= p^{-1}\left[F_{L,0}v(t) + F_{M,0}m(t) + G_0u(t)\right], \\ z_1(t) &= p^{-1}\left[F_{L,1}v(t) + F_{M,1}m(t) + G_1u(t) + z_0(t)\right], \\ z_2(t) &= p^{-1}\left[F_{L,2}v(t) + F_{M,2}m(t) + G_2u(t) + z_1(t)\right], \\ &\vdots \\ z_{\nu-1}(t) &= p^{-1}\left[F_{L,\nu-1}v(t) + F_{M,\nu-1}m(t) + G_{\nu-1}u(t) + z_{\nu-2}(t)\right]. \end{aligned}$$

As $v(t) = z_{\nu-1}(t) + F_{M,\nu}m(t)$ we get :

$$\begin{aligned} z_0(t) &= p^{-1}\left[F_{L,0}z_{\nu-1}(t) + H_{M,0}m(t) + G_0u(t)\right], \\ z_1(t) &= p^{-1}\left[F_{L,1}z_{\nu-1}(t) + H_{M,1}m(t) + G_1u(t) + z_0(t)\right], \\ z_2(t) &= p^{-1}\left[F_{L,2}z_{\nu-1}(t) + H_{M,2}m(t) + G_2u(t) + z_1(t)\right], \\ &\vdots \\ z_{\nu-1}(t) &= p^{-1}\left[F_{L,\nu-1}z_{\nu-1}(t) + H_{M,\nu-1}m(t) + G_{\nu-1}u(t) + z_{\nu-2}(t)\right], \end{aligned}$$

where, for $i = 0$ to $\nu - 1$, $H_{M,i} = F_{M,i} + F_{L,i}F_{M,\nu}$. The vector $z(t)$ defined by :

$$z(t) = \begin{bmatrix} z_0(t) \\ z_1(t) \\ \vdots \\ z_{\nu-1}(t) \end{bmatrix}.$$

is then the state of the Luenberger observer (3) with :

$$F = \begin{bmatrix} & & & & F_{L,0} \\ I_l & & & & F_{L,1} \\ & \ddots & & & \vdots \\ & & I_l & & F_{L,\nu-2} \\ & & & I_l & F_{L,\nu-1} \end{bmatrix}, \; G = \begin{bmatrix} G_0 \\ G_1 \\ \vdots \\ G_{\nu-2} \\ G_{\nu-1} \end{bmatrix}, \; H = \begin{bmatrix} H_{M,0} \\ H_{M,1} \\ \vdots \\ H_{M,\nu-2} \\ H_{M,\nu-1} \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & \cdots & 0 & I_l \end{bmatrix} \text{ et } V = F_{M,\nu}.$$

When F is a Hurwitz matrix, we have then get an asymptotic observer of the linear functional $Lx(t)$. When the so-obtained observer is stable or either if it has not the right eigenvalues, a good idea is to increase index ν and to carry on again the realization for a higher order. Moreover, we can notice that the designed observer is observable.

2.2. **Fundamental linear system**

We can remark that all this design is based on relation (5). Let us define the matrix :

$$\Sigma_\nu = \begin{bmatrix} MA^\nu \\ LA^{\nu-1} \\ MA^{\nu-1} \\ \vdots \\ LA \\ MA \\ L \\ M \end{bmatrix},$$

then (5) shows that :

$$\text{rank } \left(\begin{bmatrix} LA^\nu \\ \Sigma_\nu \end{bmatrix} \right) = \text{rank } (\Sigma_\nu),$$

which means that the system :

$$LA^\nu = \Phi\Sigma_\nu, \tag{9}$$

where Φ is an unknown matrix, is compatible. The set of solutions of this system provides then the matrices $F_{M,i}$ et $F_{L,i}$ by :

$$\begin{bmatrix} F_{M,\nu} & F_{L,\nu-1} & F_{M,\nu-1} & \cdots & F_{L,0} & F_{M,0} \end{bmatrix} = LA^\nu \Sigma_\nu^{\{1\}} + \Omega \left(I_\rho - \Sigma_\nu \Sigma_\nu^{\{1\}} \right), \tag{10}$$

where $\rho = m + \nu(m+l)$, Ω is an arbitrary $l \times \rho$-matrix and $\Sigma_\nu^{\{1\}}$ any generalised inverse of Σ_ν, that is to say a matrix of the set [2] :

$$\left\{ X \in R^{n\times\rho},\ \Sigma_\nu X \Sigma_\nu = \Sigma_\nu \right\}.$$

Of course in case where rank $(\Sigma_\nu) = \rho$ this solution is unique and independent of the choice of a particular generalized inverse. Anyway the solution of system (9) provides matrices $F_{L,i}$, for $i = 0$ to $\nu - 1$, which allow to build matrix F for which the stability remains to be verified. We know then if we can carry on with the design of the observer or if we have to increase ν.

2.3. **Stabilizability of F**

We suppose here that rank $(\Sigma_\nu) = r < \rho$. For sake of simplicity, let us choose, from the singular values decomposition of Σ_ν [12] :

$$\Sigma_\nu = U_\nu S_\nu V_\nu^\top,$$

where U_ν and V_ν are two orthonormal matrices with size $(\rho \times \rho)$ and $(n \times n)$ respectively, and S_ν is the $\rho \times n$ diagonal matrix of singular values of Σ_ν :

$$S_\nu = \mathrm{diag}\left\{\sigma_1, \ldots, \sigma_r, 0, \ldots, 0\right\},$$

with $\sigma_1 \geq \cdots \geq \sigma_r > 0$, its pseudo-inverse [2] as generalized inverse of Σ_ν is given by :

$$\Sigma_\nu^{\{1\}} = \Sigma_\nu^\dagger = V_\nu S_\nu^{-\top} U_\nu^\top,$$

where $S_\nu^{-\top} = \mathrm{diag}\left\{\sigma_1^{-1}, \ldots, \sigma_r^{-1}, 0, \ldots, 0\right\}$. Then we get :

$$I_\rho - \Sigma_\nu \Sigma_\nu^\dagger = I_\rho - U_\nu \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U_\nu^\top = U_\nu \begin{bmatrix} 0 & 0 \\ 0 & I_{\rho-r} \end{bmatrix} U_\nu^\top.$$

Let us denote $U_{2,\nu}^\top$ the $\rho - r$ last lines of $U_\nu^\top$, Γ_2, the $\rho - r$ last columns of the arbitrary matrix $\Gamma = \Omega U_\nu$ and :

$$\Phi^b = \begin{bmatrix} F_{M,\nu}^b & F_{L,\nu-1}^b & F_{M,\nu-1}^b & \cdots & F_{L,0}^b & F_{M,0}^b \end{bmatrix} = LA^\nu \Sigma_\nu^\dagger.$$

Thus :

$$\begin{bmatrix} F_{M,\nu} & F_{L,\nu-1} & F_{M,\nu-1} & \cdots & F_{L,0} & F_{M,0} \end{bmatrix} = \Phi^b + \Gamma_2 U_{2,\nu}^\top.$$

After partitioning $U_{2,\nu}^\top$ according to :

$$U_{2,\nu}^\top = \begin{bmatrix} \Upsilon_{M,\nu} & \Upsilon_{L,\nu-1} & \Upsilon_{M,\nu-1} & \cdots & \Upsilon_{L,0} & \Upsilon_{M,0} \end{bmatrix},$$

where the size of matrices $\Upsilon_{M,i}$ and $\Upsilon_{M,i}$ are $((\rho - r) \times m)$ and $((\rho - r) \times l)$ respectively, we can write F as follows :

$$F = \begin{bmatrix} & & & & F_{L,0}^b + \Gamma_2 \Upsilon_{L,0} \\ I_l & & & & F_{L,1}^b + \Gamma_2 \Upsilon_{L,1} \\ & \ddots & & & \vdots \\ & & I_l & & F_{L,\nu-2}^b + \Gamma_2 \Upsilon_{L,\nu-2} \\ & & & I_l & F_{L,\nu-1}^b + \Gamma_2 \Upsilon_{L,\nu-1} \end{bmatrix}. \tag{11}$$

So the asymptotic observer exists if it is possible to find Γ_2 such that (11) is a Hurwitz matrix.

2.4. **Summary**

The previous result is summed up in the following theorem.

Theorem 1. *If there exists Ω such that matrices given by (10) lead to a Hurwitz matrix :*

$$F = \begin{bmatrix} & & & & F_{L,0} \\ I_l & & & & F_{L,1} \\ & \ddots & & & \vdots \\ & & I_l & & F_{L,\nu-2} \\ & & & I_l & F_{L,\nu-1} \end{bmatrix},$$

then the Luenberger observer (3) with :

$$G=\begin{bmatrix} G_0\\ G_1\\ \vdots\\ G_{\nu-2}\\ G_{\nu-1}\end{bmatrix},\ H=\begin{bmatrix} F_{M,0}\\ F_{M,1}\\ \vdots\\ F_{M,\nu-2}\\ F_{M,\nu-1}\end{bmatrix}+\begin{bmatrix} F_{L,0}\\ F_{L,1}\\ \vdots\\ F_{L,\nu-2}\\ F_{L,\nu-1}\end{bmatrix}F_{M,\nu},$$
$$P=\begin{bmatrix} 0 & \cdots & 0 & I_l\end{bmatrix} \text{ and } V=F_{M,\nu},$$

where matrices G_j, $j=0$ to $\nu-1$, are defined in (8), is an asymptotic observer of $Lx(t)$.

The proof follows from previous arguments added to the determination of a matrix T. From the expressions (8) for the matrices G_j and the relationship $G=TB$, we can induce the form of following($l\nu\times n$) matrix T :

$$T=\begin{bmatrix} LA^{\nu-1}-\sum_{i=1}^{\nu-1}F_{L,i}LA^{i-1}-\sum_{i=1}^{\nu}F_{M,i}MA^{i-1}\\ \vdots\\ LA^{\nu-j}-\sum_{i=j}^{\nu-1}F_{L,i}LA^{i-j}-\sum_{i=j}^{\nu}F_{M,i}MA^{i-j}\\ \vdots\\ L-F_{M,\nu}M\end{bmatrix}.$$

It can be readily seen, with some teddious calculations, that this matrix fulfills the necessary existence conditions (4).

When this theorem is satisfied, the dimension of the designed observer is νl. If rank $(T)<\nu l$, an easy last step to reduce this dimension is still necessary. Indeed, using the maximal rank factorization of T, we quickly get an observer of order $q=$ rank (T), with $(\nu-1)\,l<q\leq\nu l$. For sake of shortness this step is not detailed here. Finally, if ν is the smallest integer such that F is stable, we get the minimum order stable observer of $Lx(t)$.

3. Some interesting but well-known particular cases

3.1. single linear functional observer

This approach has allowed to propose stable Luenberger observers with minimal order to estimate a linear form of the state in [24]. It has been shown there that increasing index ν in the fundamental relation (5) allows to design a stable observer for which more and more poles can be fixed. The technique we used is generalized in this paper through the relations of the previous section but we can notice the two following features. Firstly, there is no step to reduce the order of the observer because each time just one component is added to the state of the observer. Secondly, the stability of F is expressed as the output feedback stabilisation of a particular system. Indeed, for $l=1$, we get :

$$
\begin{aligned}
F &= \begin{bmatrix} & & & & F^b_{L,0}+\Gamma_2\Upsilon_{L,0} \\ I_l & & & & F^b_{L,1}+\Gamma_2\Upsilon_{L,1} \\ & \ddots & & & \vdots \\ & & I_l & & F^b_{L,\nu-2}+\Gamma_2\Upsilon_{L,\nu-1} \\ & & & I_l & F^b_{L,\nu-1}+\Gamma_2\Upsilon_{L,\nu-1} \end{bmatrix}, \\
&= \begin{bmatrix} & & & & F^b_{L,0}+\Upsilon^\top_{L,0}\Gamma^\top_2 \\ I_l & & & & F^b_{L,1}+\Upsilon^\top_{L,1}\Gamma^\top_2 \\ & \ddots & & & \vdots \\ & & I_l & & F^b_{L,\nu-2}+\Upsilon^\top_{L,\nu-2}\Gamma^\top_2 \\ & & & I_l & F^b_{L,\nu-1}+\Upsilon^\top_{L,\nu-1}\Gamma^\top_2 \end{bmatrix}, \\
&= \begin{bmatrix} & & & & F^b_{L,0} \\ I_l & & & & F^b_{L,1} \\ & \ddots & & & \vdots \\ & & I_l & & F^b_{L,\nu-2} \\ & & & I_l & F^b_{L,\nu-1} \end{bmatrix} + \begin{bmatrix} \Upsilon^\top_{L,0} \\ \Upsilon^\top_{L,1} \\ \vdots \\ \Upsilon^\top_{L,\nu-2} \\ \Upsilon^\top_{L,\nu-1} \end{bmatrix} \Gamma^\top_2 \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

This implies that if an output feedback can stabilize the system :

$$
\begin{aligned}
\dot{\xi} &= \begin{bmatrix} & & & & F^b_{L,0} \\ I_l & & & & F^b_{L,1} \\ & \ddots & & & \vdots \\ & & I_l & & F^b_{L,\nu-2} \\ & & & I_l & F^b_{L,\nu-1} \end{bmatrix} \xi + \begin{bmatrix} \Upsilon^\top_{L,0} \\ \Upsilon^\top_{L,1} \\ \vdots \\ \Upsilon^\top_{L,\nu-2} \\ \Upsilon^\top_{L,\nu-1} \end{bmatrix} \upsilon, \\
\eta &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \xi,
\end{aligned}
$$

then Γ_2 exists such that F is a Hurwitz matrix.

3.2. Darouach observer

It is a Luenberger observer (3) for which $q=l$ and $P=I_l$. The existence condition for this observer, proved in [5], is :

$$
\forall s \in \mathbb{C},\ \Re e(s) \geq 0,\ \text{rank} \begin{bmatrix} M \\ MA \\ L(sI_n - A) \end{bmatrix} = \text{rank} \begin{bmatrix} MA \\ L \\ M \end{bmatrix} = \text{rank} \begin{bmatrix} LA \\ MA \\ L \\ M \end{bmatrix}. \tag{12}
$$

This condition can be split in two different conditions with different analysis. The last part of (12) :

$$
\text{rank} \begin{bmatrix} MA \\ L \\ M \end{bmatrix} = \text{rank} \begin{bmatrix} LA \\ MA \\ L \\ M \end{bmatrix},
$$

is related to the existence of structure (3) with $q=l$, so it leads immediately to the consistency of system (9) if $\nu=1$. After treatment this relation provides the decomposition, possibly non unique :

$$
LA = F_{M,1}MA + F_{L,0}L + F_{M,0}M. \tag{13}
$$

Now, let us use this decomposition in the first part of the Darouach condition :

$$\forall s \in \mathbb{C},\ \Re e(s) \geq 0,\ \text{rank} \begin{bmatrix} M \\ MA \\ L(sI_n - A) \end{bmatrix} = \text{rank} \begin{bmatrix} MA \\ L \\ M \end{bmatrix}. \tag{14}$$

It comes :

$$\begin{aligned}
\text{rank} \begin{bmatrix} M \\ MA \\ L(sI_n - A) \end{bmatrix} &= \text{rank} \begin{bmatrix} M \\ MA \\ sL - F_{M,1}MA - F_{L,0}L - F_{M,0}M \end{bmatrix}, \\
&= \text{rank} \begin{bmatrix} M \\ MA \\ sL - F_{L,0}L \end{bmatrix} = \text{rank} \begin{bmatrix} M \\ MA \\ (sI_l - F_{L,0})\, L \end{bmatrix}, \\
&= \text{rank} \left\{ \begin{bmatrix} & & I_m \\ I_m & & \\ & sI_l - F_{L,0} & \end{bmatrix} \begin{bmatrix} MA \\ L \\ M \end{bmatrix} \right\}.
\end{aligned}$$

Finally condition (14) gives

$$\forall s \in \mathbb{C},\ \Re e(s) \geq 0,\ \text{rank} \begin{bmatrix} sI_l - F_{L,0} \end{bmatrix} = l,$$

which prove that $F_{L,0}$ is a Hurwitz matrix, so the Luenberger observer is asymptotically stable. Therefore, theorem 1 is a generalization of the Darouach result. Nevertheless, we can notice the following difference between these two results. While our procedure is based on matrix calculus, the Darouach approach is based on a frequency criterion of Popov-Belevitch-Hautus type [14], criterion which is difficult to use out of the linear time invariant systems class.

3.3. **Cumming-Gopinath observer**

This observer, called reduced observer as well, is an $n-m$-order observer which allows to estimate, for an observable model, the whole state of system (1) [4, 13]. We show here how our approach allows to find this structure as well.

Let us suppose now that $M = \begin{bmatrix} I_m & 0 \end{bmatrix}$ and consequently $L = \begin{bmatrix} 0 & I_{n-m} \end{bmatrix}$, so we want to estimate the $n-m$ last components of the state of (1) which are not measured. These partitions lead to the state partition :

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where $x_1(t) = y(t)$ regroups the m first components of $x(t)$. We define also the matrix partitions :

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ et } B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

with $A_{11}(m \times m)$ and $B_1(m \times r)$.

3.3.1. *Reduced observer*

From relations :

$$\begin{aligned}
\dot{y}(t) &= A_{11}y(t) + A_{12}x_2(t) + B_1u(t), \\
\dot{x}_2(t) &= A_{21}y(t) + A_{22}x_2(t) + B_2u(t), \\
y(t) &= x_1(t),
\end{aligned}$$

well-tested methods lead to the Luenberger observer :

$$\begin{array}{rcl}\dot{z}(t) & = & (A_{22}-\Lambda A_{12})z(t)+(B_2-\Lambda B_1)u(t)\\ & & +(A_{21}-\Lambda A_{11}+A_{22}\Lambda-\Lambda A_{12}\Lambda)y(t),\\ \hat{x}(t) & = & \begin{bmatrix} y(t)\\ z(t)+\Lambda y(t)\end{bmatrix}=\begin{bmatrix}0\\ I_{n-m}\end{bmatrix}z(t)+\begin{bmatrix}I_m\\ \Lambda\end{bmatrix}y(t),\end{array}$$

where Λ is a $((n-m)\times m)$-matrix called the output innovation gain.

It is well known that when the pair (A,M) is detectable, which is equivalent to the pair (A_{22},A_{12}) is detectable, we can always find a matrix Λ such that matrix $A_{22}-\Lambda A_{12}$ is a Hurwitz matrix.

When a variable change has be done by means of a non singular matrix Π to satisfy hypothesis on L and M, the last relation has just to be changed into :

$$\hat{x}(t)=\Pi^{-1}\begin{bmatrix}0\\ I_{n-m}\end{bmatrix}z(t)+\Pi^{-1}\begin{bmatrix}I_m\\ \Lambda\end{bmatrix}y(t).$$

The link with the Luenberger observer (3) is then ensured by :

$$\begin{array}{rr}F=A_{22}-\Lambda A_{12}, & G=B_2-\Lambda B_1,\\ H=A_{21}-\Lambda A_{11}+A_{22}\Lambda-\Lambda A_{12}\Lambda, & P=\Pi^{-1}\begin{bmatrix}0\\ I_{n-m}\end{bmatrix},\\ V=\Pi^{-1}\begin{bmatrix}I_m\\ \Lambda\end{bmatrix}. & \end{array}$$

3.3.2. *Direct design of this observer*

This section is devoted to the design of the Cumming-Gopinath observer with the derivative-based presented method. Let us take $L=\begin{bmatrix}0 & I_{n-m}\end{bmatrix}$ and $M=\begin{bmatrix}I_m & 0\end{bmatrix}$, we get :

$$LA=\begin{bmatrix}A_{21} & A_{22}\end{bmatrix} \text{ and } MA=\begin{bmatrix}A_{11} & A_{12}\end{bmatrix}.$$

as $\operatorname{rank}\left(\begin{bmatrix}L\\ M\end{bmatrix}\right)=n$, it comes :

$$\operatorname{rank}\left(\begin{bmatrix}LA\\ MA\\ L\\ M\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix}MA\\ L\\ M\end{bmatrix}\right).$$

Thus, we have $\nu=1$ here. On the one hand, the unique decompositions :

$$\begin{array}{rcl}LA & = & A_{21}M+A_{22}L,\\ MA & = & A_{11}M+A_{12}L,\end{array}$$

and, on the other hand :

$$LA=\Phi\begin{bmatrix}MA\\ L\\ M\end{bmatrix},$$

where the unknown matrix is $\Phi = \begin{bmatrix} F_{M,1} & F_{L,0} & F_{M,0} \end{bmatrix}$, is consistent. The matrices $F_{M,1}$, $F_{L,0}$ and $F_{M,0}$ fulfill :

$$\begin{aligned} A_{21}M + A_{22}L &= F_{M,1}\left(A_{11}M + A_{12}L\right) + F_{L,0}L + F_{M,0}M, \\ &= \left(F_{M,0} + F_{M,1}A_{12}\right)M + \left(F_{L,0} + F_{M,1}A_{11}\right)L. \end{aligned}$$

The LA and MA unique decompositions lead to :

$$\begin{aligned} A_{22} &= F_{L,0} + F_{M,1}A_{12}, \\ A_{21} &= F_{M,0} + F_{M,1}A_{11}. \end{aligned}$$

The first one gives :

$$F_{L,0} = A_{22} - F_{M,1}A_{12}.$$

Thus, if the pair (A_{22}, A_{12}) is detectable, there exists a matrix $F_{M,1}$ such that $A_{22} - F_{M,1}A_{12}$ is a Hurwitz matrix. A particular choice for $F_{M,1}$, which ensures stability of $F_{L,0}$, gives also $F_{M,0} = A_{21} - F_{M,1}A_{11}$.

As $\nu = 1$, the calculus leads to :

$$\begin{aligned} F &= F_{L,0} = A_{22} - F_{M,1}A_{12}, \\ G &= G_0\left(L - F_{M,1}M\right)B = B_2 - F_{M,1}B_1, \\ H &= H_{M,0} = F_{M,0} + F_{L,0}F_{M,1}, \\ &= A_{21} - F_{M,1}A_{11} + A_{22}F_{M,1} - F_{M,1}A_{12}F_{M,1}, \\ P &= I_{n-m}, \\ V &= F_{M,1}. \end{aligned}$$

Letting $F_{M,1} = \Lambda$, we get the Cumming-Gopinath observer directly.

4. Conclusion

In this work we have proposed a constructive procedure to design a linear functional observer in a linear framework. The used method allows to extend some previous results on this topic. Particularly, we generalize the Darouach existence condition [5]. Nevertheless, the main interest, due to the fact that we don't use frequency-based methods or eigenvalues-based methods, is the possibility to consider linear time-varying models. This point has been presented in [25, 26]. These models are of primary importance due to some recent advances in automatic control, for instance in trajectory tracking [8, 28] with a flatness-based control strategy and time-varying linearization.

References

[1] Ayadi, M., Contribution à la commande des systèmes linéaires plats de dimension finie, Ph. D. Dissertation, INP-Toulouse, 2002.

[2] Ben-Israel, A., Greville, T. N., *Generalized inverses*, Springer, 2003.

[3] Chen, C.-T., *Linear system theory and design*, Oxford University Press, 1999.

[4] Cumming, S.D.G., Design of observers of reduced dynamics, *Electron. Lett.*, vol. 5, n. 10, pp. 213–214, 1969.

[5] Darouach, M., Existence and design of functional observers for linear systems, *IEEE Trans. Aut. Control*, AC-45, n. 5, pp. 940–943, 2000.

[6] Fernando, T., Trinh, H., Jennings, L., Functional observability and the design of minimum order linear functional observers, *IEEE Trans. Aut. Control*, vol. 55, n. 5, pp. 1268–1273, 2010.

[7] Fernando, T., Trinh, A procedure for designing linear functional observers, *Applied Mathematics Letters*, vol. 26, n. 2, pp. 240–243, 2013.

[8] Fliess, M., Lévine, J., Martin, P., Rouchon, P., Flatness and defect of nonlinear systems : introductory theory and examples, *Int. J. Control*, vol. 61, n. 6, pp. 1327–61, 1995.

[9] Fliess, M., Sur des pensers anciens faisons des vers nouveaux, Proceedings CIFA'2000, pp. 26-36, 2000.

[10] Fortmann, T.E., Williamson, D., Design of low-order observers for linear feedback control laws, *IEEE Trans. Aut. Control*, vol. AC-17, n. 2, pp. 301–308, 1972.

[11] Gerrish, F., Ward, A.J.B., Sylvester's matrix equation and Roth's removal rule, *The Mathematical Gazette*, vol. 82, n. 495, pp. 423–430, 1998.

[12] Golub, G.H., Van Loan, C.F., *Matrix computations*, John Hopkins, 1996.

[13] Gopinath, On the control of linear multiple input-output systems, *Bell Syst. Tech. J.*, vol. 50, pp. 1063–1081, 1971.

[14] Kailath T., *Linear systems*, Prentice Hall, 1980.

[15] Korovin, S.K., Fomichev, V.V., *State observers for linear systems with uncertainty*, in russian, Fitzmatgiz, 2007, english transl., Walter de Gruyter, 2009.

[16] Korovin, S.K., Fomichev, V.V., Minimum-order functional observers, *Computational Mathematics and Modeling*, vol. 21, n. 3, pp. 275–296, 2010.

[17] Luenberger, D.G., Determining the state of a linear system with observers of low dynamic order, Ph. D. Dissertation, Standford University, 1963.

[18] Luenberger, D.G., Observing the state of a linear system, *IEEE Trans. on Military Electronics*, MIL-8, n. 2, pp. 74–80, 1964.

[19] Luenberger, D.G., Observers for multivariable Systems, *IEEE Trans. Aut. Control*, vol. 11, n. 2, pp. 190–197, 1966.

[20] Maquin, D., Ragot, J., *Diagnostic des systèmes linéaires*, Hermès, 2000.

[21] M. Mboup, C. Join, M. Fliess, Numerical differentiation with annihilators in noisy environment, *Numerical Algorithms*, vol. 50, n. 4, pp. 439-467, 2009.

[22] O'Reilly, J., *Observers for linear systems*, Academic Press, 1983.

[23] Rotella, F., Zambettakis, I., *Automatique élémentaire*, Hermes-Lavoisier, 2008.

[24] Rotella, F., Zambettakis, I., Minimal single linear functional observers for linear systems, *Automatica*, vol. 47, n. 1, pp. 164–169, 2011.

[25] Rotella, F., Zambettakis, I., On functional observers for linear time-varying systems, *IEEE Trans. Aut. Control*, vol. 58, n. 5, pp. 1354-1360, 2013.

[26] Rotella, F., Zambettakis, I., A design procedure for a single time-varying functional observer, Proc. 52th IEEE Conference on Decision and Control, pp. *, 2013*.

[27] Roth, W.E., The equations $AX - YB = C$ and $AX - XB = C$ in matrices, *Proc. Amer. Soc.*, vol. 3, n. 3, pp. 392–396, 1952.

[28] Sira-Ramirez, H., Agrawal, S.K., *Differentially flat systems*, Marcel Dekker, 2004.

[29] Tian, Y., Floquet, T., Perruquetti, W., Fast state estimation in linear time-varying systems: an algebraic approach, Proc. 47th IEEE Conference on Decision and Control, pp. 2539–2544, 2008.

[30] Trinh, H., Nahavandi, S., Tran, T.D., Algorithms for designing reduced-order functional observers of linear systems, *Int. J. Innovative Computing, Information and Control*, vol. 4, n. 2, pp. 321–333, 2008.

[31] Trinh, H., Fernando, T., *Functional observers for dynamical systems*, Springer, 2012.

[32] Tsui C.C., A new algorithm for the design of multifunctional observers, *IEEE Trans. Aut. Control*, vol. AC-30, n. 1, pp. 89–93, 1985.

[33] Tsui C.C., An overview of the applications and solution of a fundamental matrix equation pair, *Jour. of Franklin Institute*, vol. 341, pp. 465–75, 2004.