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SENSITIVITY AND STRONG CONTROLLABILITY OF A NONLINEAR CHEMOSTAT MODEL

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Abstract. We investigate the sensitivity behaviour and the controllability for an aerobic wastewater model. The problem is formulated as a nonlinear dynamical system. Using the tools of nonsmooth analysis, we firstly analyse the positivity and dissipation of the model. On the other hand, through the Gronwell's inequality, we prove a sensitivity property of the model, quantified by the control parameters and initial conditions. This sensitivity leads to an error estimation between two trajectories. The strong controllability is investigated in a new setting: we assume that the recycle rate R, the residence time τ and the dissolved oxygen saturation concentration C_s are measurable time varying control functions. Hence, we reformulate the system as a nonlinear control problem. In this context and without linearising, we provide a strong controllability result with respect to the perturbations on initial conditions. As a consequence, we prove that an equilibrium point (when it exists) is locally controllable. Finally, we give some simulations illustrating our results.

Résumé. Dans ce travail, nous étudions la sensibilité et la contrôlabilité forte d'un modèle aérobie de traitement des eaux usées. Le problème est formulé sous forme d'un système dynamique non linéaire. En utilisant les outils d'analyse non lisse, nous nous intéressons, dans un premier temps, à l'analyse de la positivité et de la dissipation du modèle, puis à travers l'inégalité de Gronwall, nous mettons en évidence la sensibilité du modèle par rapport aux paramètres, (susceptibles d'être des outils de contrôle), et les conditions initiales. Cette sensibilité conduit à une estimation de l'erreur entre deux trajectoires. La contrôlabilité forte est étudiée dans un nouveau contexte: nous supposons que le taux de recyclage R, le temps de séjour τ et le taux de saturation d'oxygène dissous C_s sont des fonctions du temps mesurables. Dans cette optique, nous formulons le système comme un problème de contrôle non linéaire et nous fournissons un résultat de contrôlabilité forte, par rapport aux perturbations des conditions initiales. Par conséquent, nous démontrons qu'un point d'équilibre (s'il existe) est localement contrôlable. Finalement, nous donnons quelques simulations illustrant nos résultats. .

1. Introduction

We are interested in a model of wastewater treatment, namely, the activated sludge process. The using principle can be briefly described as follows (see for instance [4, 7, 17, 19]): the influent feeds into a tank, called the aerator. At this stage, one bacteria population degrades the pollutant by a biological oxidation of the substrate. The reaction creates an aerobic environment by consuming the oxygen. In a second phase the mixture is forwarded to a settler tank. Here, due to gravity, the solid components settle and concentrate at the bottom. Part of the sludge is recycled into the aerator to stimulate the oxidation.
The wastewater treatment problem was often investigated under several points of view: dynamic [1, 2, 17],

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observation [9,15], control [6,12,13,16]. But in most cases, the anaerobic aspect is considered, [1,2,10,16,17]. For the aerobic studies, there are not enough works dealing interested in control aspects, with several parameters at the same time. In this paper, we explore a model including three differential equations describing substrate, bacteria and oxygen evolutions. The specific growth kinetic $\mu(S,C)$ depends on two variables, the substrate S and the oxygen C, and obeys to the Monod law. The aim of this work is firstly, to provide an estimation error on the trajectories, which gives a tool to test the sensitivity of the model with respect to the parameters and the initial conditions. Secondly, it aims at using, at the same time, three parameters to control the model, namely : the recycle rate R for the energy; the residence time τ for the influent rate and the dissolved oxygen saturation concentration C_s for the aeration. Hence, in the context of the nonsmooth analysis, we prove the positivity and the invariance as well as the boundedness of the trajectories, using the characterization of the viability property throughout the Bouligand tangent cone. On the other hand, invoking the Gronwell's inequality, we investigate the sensitivity of the trajectories with respect to the initial conditions and the parameters R, τ and C_s. We provide an error estimate depending on those parameters. This error can be viewed as a Lipschitz property of the trajectories. In the control part, we relax the hypotheses by assuming that the control parameters $R(.)$, $\tau(.)$ and $C_s(.)$ are measurable varying time functions. The control, $u = (R, \tau, C_s)$ is not affine due to the presence of terms as $\frac{1}{\tau}$ and $\frac{R}{\tau}$. In this context and without linearising, we exhibit a strong controllability result, by exploiting the normality of the system and the properties of the Fréchet limiting normal cone. The controllability property can be used to control the trajectories, obtained in the presence of a disturbance on initial conditions, with respect to a referential trajectory. We apply this control feature to prove that the equilibrium, if it exists, is locally controllable. Finally, we give some numerical simulations to illustrate our theoretical results.

2. Mathematical model

The mathematical model is formulated as a nonlinear ordinary differential system in dimensionless form. Three phenomena are considered: the reaction kinetics in the aerator linked to microbial growth with recycling, the substrate degradation and finally the consumption of oxygen. The mass equilibrium of the various components around the aerator gives the following dynamic system

$$\frac{dS}{dt} = \frac{1}{\tau}(1-S) - X\frac{S}{K_s+S}\frac{C}{K_c+C}, \quad S(0) = S_0 > 0, \tag{1}$$

$$\frac{dX}{dt} = \frac{1}{\tau}X(R-1) + X\frac{S}{K_s+S}\frac{C}{K_c+C} - K_D X, \quad X(0) = X_0 > 0, \tag{2}$$

$$\frac{dC}{dt} = -X\frac{1}{\alpha_0}\frac{S}{K_s+S}\frac{C}{K_c+C} + \frac{1}{\tau}(1-C) + K_{La}(C_s - C), \quad C(0) = C_0 > 0. \tag{3}$$

The dimensionless specific growth function of bacteria obeys to Monod law with two variables, (see [14]).

$$\mu(S,C) = \frac{S}{K_s+S}\frac{C}{K_c+C}$$

where S is the substrate biomass, X the micro-organism biomass and C the oxygen concentration. The parameters can be described respectively as, the residence time τ, the recycle concentration rate for the settling unit on reactor R, the death coefficient K_D, the substrate saturation rate K_s, the oxygen saturation rate K_c, the oxygen transfer coefficient K_{La}, the dissolved oxygen saturation concentration C_s, the yield coefficient α_0.

2.1. Assumptions

H_1- The parameters τ, K_s, K_c, K_D, K_{La} and C_s are positive and constant.

H_2- The parameter R is such that $0 \leq R < 1$.

In the hypothesis H_2, $0 \leq R < 1$ means that we model the treatment with imperfect recycle ($0 < R < 1$) or without recycle ($R = 0$). The case of perfect recycle ($R = 1$) is not considered, since generally not all sludge are recycled to the aerator.

3. Dynamical Analysis

3.1. Invariance and dissipation

We reformulate the problem in a general nonlinear dynamical system. Consider $Z = (Z_1, Z_2, Z_3) := (S, X, C)$, and define

$$L(Z_1, Z_3) := L(S, C) = \frac{S}{K_s + S}\frac{C}{K_c + C}\ , \tag{4}$$

and $f = (f_1, f_2, f_3)$ such that

$$f_1(Z, R, \tau, C_s) = \frac{1}{\tau}(1 - Z_1) - Z_2 L(Z_1, Z_3), \tag{5}$$

$$f_2(Z, R, \tau, C_s) = \frac{1}{\tau} Z_2(R - 1) - K_D Z_2 + Z_2 L(Z_1, Z_3), \tag{6}$$

$$f_3(Z, R, \tau, C_s) = -Z_2 \frac{1}{\alpha_0} L(Z_1, Z_3) + \frac{1}{\tau}(1 - Z_3) + K_{La}(C_s - Z_3). \tag{7}$$

Therefore the system (1-3) can be rewriting as

$$\dot{Z} = f(Z, R, \tau, C_s), \ \ Z(0) = Z_0 > 0. \tag{8}$$

Consider the following subset, Ω, of $\mathbb{R}^3_+$

$$\Omega := \{Z \in \mathbb{R}^3_+ / \ Z \geq 0\}.$$

The inequality must be shown coordinate by coordinate.
The system (1-3) describes the evolution over time, of bacteria, substrate and oxygen. In order to give a physical meaningful of this system, it is natural to ask whether those trajectories are positively viable, that is, if there exists a trajectory, solution of the system which starts from Ω and remains in it thereafter, see [3].

Proposition 3.1. *Under the assumptions $H_1 - H_2$, the dynamical system (8) admits a unique solution and the set Ω is positively invariant by (1-3).*

Proof. We prove, firstly, the existence and uniqueness of the solution.
It's easy to see that $f(., R, \tau, C_s)$ is differentiable, let us prove that $f(., R, \tau, C_s)$ satisfies a linear growth condition. Indeed, since $L(Z_1, Z_3) \leq 1$, then

$$|f_1(Z, R, \tau, C_s)| < \frac{1}{\tau} + (\frac{1}{\tau} + 1)\|Z\|.$$

By the same way

$$|f_2(Z, R, \tau, C_s)| < (|\frac{R-1}{\tau} - K_D| + 1)\|Z\|,$$

and

$$|f_3(Z, R, \tau, C_s)| < (\frac{1}{\tau} + \frac{1}{\alpha_0} + K_{La})\|Z\| + (K_{La}C_s + \frac{1}{\tau}).$$

Then

$$\|f(Z, R, \tau, C_s)\|_\infty < M_1\|Z\| + M_2,$$

where $M_1 = sup\{(\frac{1}{\tau}+1),(|\frac{R-1}{\tau}-K_D|+1),(\frac{1}{\tau}+\frac{1}{\alpha_0}+K_{La})\}$ and $M_2 = (K_{La}C_s+\frac{1}{\tau})$.
We conclude hence that the system admits a unique solution on $\mathbb{R}^+$, (see [8], [18]).

On the other hand, since there exists a unique solution, the notions of invariance and viability coincide. It turns out to prove that the trajectory cannot violate the boundaries of Ω. Using the contingent cone at Ω on Z, $T_\Omega(Z)$, we can characterize the viability property (see [3]), as follows :
The set Ω is viable for the dynamic system (8), (that is there exists a trajectory $Y(t)$ starting at Z_0 and remaining in Ω thereafter), iff

$$f(Z,R,\tau,C_s)\in T_\Omega(Z),\ \ \forall\, Z\in\Omega, \tag{9}$$

where $d\in T_\Omega(Z)$; there exist $(d_k)_k\subset\mathbb{R}^3$, $(t_k)_k\subset\mathbb{R}$ such that $d_k\longrightarrow d$, $t_k\searrow 0$ and

$$Z+t_kd_k\in\Omega.$$

The inclusion (9) must hold for all $Z\in\Omega$, so, we can treat separately two cases:
The case of all $Z\in\mathring{\Omega}$ (the interior of Ω), in this case we have $T_\Omega(Z)=\mathbb{R}^3$ and hence $f(Z,R,\tau,C_s)\in T_\Omega(Z)$.

Now, $\forall Z\in\partial\Omega$ (boundary of Ω), there exists $i\in\{1,2,3\}$ such that $Z_i=0$. By noticing that Ω may be reformulated as a polyhedral set, we can write :

$$\Omega=\{Z\in\mathbb{R}^3\,/\,AZ\le 0\},$$

where the matrix $A := \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$.
Let $I(Z) := \{i\in\{1,2,3\}\,/\,(AZ)_i=0\}$, then using [3]

$$\begin{aligned} T_\Omega(Z) &= \{d\in\mathbb{R}^3\,/\,(Ad)_i\le 0,\,\forall\, i\in I(Z)\}, &(10)\\ &= \{d\in\mathbb{R}^3\,/\,d_i\ge 0,\,\forall\, i\in I(Z)\}, &(11)\end{aligned}$$

In order to obtain $f(Z,R,\tau,C_s)\in T_\Omega(Z)$, we must have

$$f_i(Z,R,\tau,C_s)\ge 0. \tag{12}$$

For more details, see the reference (10).
It is easy to show from (5-7) that, for each $i=1,2$ or 3 such that $Z_i=0$, the inequality (12) is true. Hence Ω is viable. □

The second property related to physical aspects of the system (8), concerns the boundedness of the trajectories.

Proposition 3.2. *Suppose that H_1-H_2 hold. For each initial condition, $Z_0\in\Omega$, the trajectory $Z(t,Z_0)$, solution of the system (8), is uniformly bounded in $\bar{\Omega}$ (the closure of Ω).*

Proof. Let $W := Z_1+2Z_2+\alpha_0Z_3$. So,

$$\dot{W}=\dot{Z}_1+2\dot{Z}_2+\alpha_0\dot{Z}_3=-\frac{1}{\tau}Z_1-2(K_D+\frac{1}{\tau}(1-R))$$

$$-\alpha_0(\frac{1}{\tau}+K_{La})Z_3+\gamma,$$

where $\gamma = \frac{1}{\tau} + \alpha_0(\frac{1}{\tau} + K_{La}C_s)$. Using a constant M which satisfies the following condition :

$$0 < M < \min(\frac{1}{\tau}, K_D + \frac{1}{\tau}(1-R), \frac{1}{\tau} + K_{La}),$$

we obtain

$$\begin{aligned} -\frac{1}{\tau}Z_1 - 2(K_D + \frac{1}{\tau}(1-R)) - \alpha_0(\frac{1}{\tau} + K_{La})Z_3 &< -MZ_1 - 2MZ_2 - \alpha_0 MZ_3, \\ &= -MW. \end{aligned}$$

It follows that

$$\dot{W} < -MW + \gamma.$$

Hence, using the differential inequality properties, we conclude that

$$\limsup(W) < \frac{\gamma}{M},$$

as required. □

3.2. **Sensitivity to parameters**

Our goal is to study the sensitivity of the model to the parameters R, τ and C_s which are used to control the system. The aim is to establish a Lipschitz dependence leading to an error estimation.
Consider the two trajectories of the system (8): $Y(t) = Y(t, R_1, \tau_1, C_s^1)$ starting at Y_0 and $Z(t) = Y(t, R_2, \tau_2, C_s^2)$ starting at Z_0.

Theorem 3.3. *There exist $M, N_1, N_2 > 0$ such that*

$$\begin{aligned} \|Y(t) - Z(t)\| &\leq \|Y_0 - Z_0\|e^{Mt} + (\frac{N_1}{M}|\tau_1 - \tau_2| \\ &+ \frac{N_2}{M}|R_1 - R_2| + \frac{K_{La}}{M}|C_s^1 - C_s^2|)e^{Mt}, \ \forall t \geq 0. \end{aligned}$$

M, N_1 and N_2 depend only on the parameters of the model.

The proof of the theorem needs the following Gronwell's lemma, see [18].

Lemma 3.4. *Consider $Z(t)$ the trajectory of the system starting at Z_0 and suppose that there exist γ, $c > 0$ such that the linear growth condition holds*

$$\|\dot{Z}(t)\| \leq \gamma\|Z(t)\| + c, \ \ \forall t \geq 0$$

Then

$$\|Z(t) - Z_0\| \leq \|Z_0\|(e^{\gamma t} - 1) + \frac{c}{\gamma}e^{\gamma t}, \ \ \forall t \geq 0.$$

Proof. **Proof of the previous theorem**
We prove that L given by (4), is Lipschitz:

$$\begin{aligned} |L(Y_1, Y_3) - L(Z_1, Z_3)| &= |\frac{Y_1}{K_s + Y_1}\frac{Y_3}{K_c + Y_3} - \frac{Z_1}{K_s + Z_1}\frac{Z_3}{K_c + Z_3}|, \\ &\leq \frac{Y_3}{K_c + Y_3}|\frac{Y_1}{K_s + Y_1} - \frac{Z_1}{K_s + Z_1}| + \frac{Z_1}{K_s + Z_1}|\frac{Y_3}{K_c + Y_3} - \frac{Z_3}{K_c + Z_3}|. \end{aligned}$$

Remark that $\dfrac{Y_3}{K_s+Y_3} < 1$ and $\dfrac{Z_1}{K_s+Z_1} < 1$, hence

$$|L(Y_1,Y_3) - L(Z_1,Z_3)| \leq \frac{K_s|Y_1-Z_1|}{(K_s+Y_1)(K_s+Z_1)} + \frac{K_c|Y_3-Z_3|}{(K_c+Y_3)(K_c+Z_3)}. \tag{13}$$

Remark also that $\dfrac{K_s^2}{(K_s+Y_1)(K_s+Z_1)} < 1$ and $\dfrac{K_c^2}{(K_c+Y_3)(K_c+Z_3)} < 1$, then

$$|L(Y_1,Y_3) - L(Z_1,Z_3)| \leq K(|Y_1-Z_1| + |Y_3-Z_3|), \tag{14}$$

where $K = \sup\{\dfrac{1}{K_s}, \dfrac{1}{K_c}\}$.
Returning now to the derivatives of trajectories, we have

$$\begin{aligned} \|\dot{Y} - \dot{Z}\| &= \|f(Y) - f(Z)\|, && (15)\\ &= \sum_{i=1}^{3} |f_i(Y) - f_i(Z)|. && (16) \end{aligned}$$

where $f(Z)$ denotes $f(Z,R,\tau,C_s)$ for more simplicity.

Proceeding index by index, we obtain :
▶ $i = 1$

$$\begin{aligned} |f_1(Y) - f_1(Z)| &= |\frac{1}{\tau_1}(1-Y_1) - Y_2L(Y_1,Y_3) - \frac{1}{\tau_2}(1-Z_1) + Z_2L(Z_1,Z_3)|,\\ &\leq \frac{|\tau_2-\tau_1|}{\tau_1\tau_2} + |\frac{Z_1}{\tau_2} - \frac{Y_1}{\tau_1}| + |Z_2L(Z_1,Z_3) - Y_2L(Y_1,Y_3)|,\\ &\leq \frac{|\tau_2-\tau_1|}{\tau_1\tau_2} + \frac{1}{\tau_1\tau_2}(\tau_1|Z_1-Y_1| + Y_1|\tau_1-\tau_2|)\\ &+ |Z_2L(Z_1,Z_3) - Y_2L(Z_1,Z_3)| + |Y_2L(Z_1,Z_3) - Y_2L(Y_1,Y_3)|,\\ &\leq \frac{1+Y_1}{\tau_1\tau_2}|\tau_2-\tau_1| + \frac{1}{\tau_2}|Z_1-Y_1|\\ &+ L(Z_1,Z_3)|Z_2-Y_2| + Y_2|L(Z_1,Z_3) - L(Y_1,Y_3)|. \end{aligned}$$

According to proposition (3.2), the trajectories of the system (8) are bounded. So, there exists $A > 0$ independent of the system's trajectories, (A can be calculated using $\dfrac{M}{\gamma}$ in the proof of the proposition (3.2)), such that $\|Y(t)\| \leq A,\ \forall t \geq 0$ and $\|Z(t)\| \leq A,\ \forall t \geq 0$. This fact together with $L(Z_1,Z_3) < 1$ and Lipschitz property imply that

$$\begin{aligned} |f_1(Y) - f_1(Z)| &\leq \frac{1+A}{\tau_1\tau_2}|\tau_2-\tau_1| + \frac{1}{\tau_2}|Z_1-Y_1| + |Z_2-Y_2|\\ &+ AK(|Y_1-Z_1| + |Y_3-Z_3|). \end{aligned}$$

It follows that

$$|f_1(Y) - f_1(Z)| \leq \frac{1+A}{\tau_1\tau_2}|\tau_2-\tau_1| + M_1\|Y-Z\|, \tag{17}$$

where $M_1 := \sup\{\frac{1}{\tau_2} + AK, 1\}$.

▶ $i = 2$

$$\begin{array}{rcl}|f_2(Y) - f_2(Z)| & = & |\frac{1}{\tau_1}Y_2(R_1 - 1) - K_D Y_2 + Y_2 L(Y_1, Y_3) - \frac{1}{\tau_2}Z_2(R_2 - 1) \\ & + & K_m Z_2 - Z_2 L(Z_1, Z_3)|, \\ & \leq & |\frac{R_1 Y_2}{\tau_1} - \frac{R_2 Z_2}{\tau_2}| + |\frac{Z_2}{\tau_2} - \frac{Y_2}{\tau_1}| + K_D|Z_2 - Y_2| \\ & + & |Y_2 L(Y_1, Y_3) - Z_2 L(Z_1, Z_3)|, \\ & \leq & \frac{1}{\tau_1\tau_2}|R_1\tau_2 Y_2 - R_2\tau_1 Z_2| + \frac{1}{\tau_2}|Z_2 - Y_2| + \frac{A}{\tau_1\tau_2}|\tau_1 - \tau_2| \\ & + & K_D|Z_2 - Y_2| + |Z_2 - Y_2| + AK(|Y_1 - Z_1| + |Y_3 - Z_3|), \\ & \leq & \frac{1}{\tau_1\tau_2}(R_1\tau_2|Y_2 - Z_2| + AR_1|\tau_2 - \tau_1| + A\tau_1|R_1 - R_2|) \\ & + & \frac{1}{\tau_2}|Z_2 - Y_2| + \frac{A}{\tau_1\tau_2}|\tau_1 - \tau_2| + K_D|Z_2 - Y_2| + |Z_2 - Y_2| \\ & + & AK(|Y_1 - Z_1| + |Y_3 - Z_3|).\end{array}$$

Hence

$$|f_2(Y) - f_2(Z)| \leq M_2\|Y - Z\| + \frac{A + AR_1}{\tau_1\tau_2}|\tau_2 - \tau_1| + \frac{A}{\tau_2}|R_1 - R_2|, \tag{18}$$

where $M_2 := \sup\{\frac{R_1\tau_2 + \tau_1}{\tau_1\tau_2} + K_D + 1, AK\}$.

▶ $i = 3$

$$\begin{array}{rcl}|f_3(Y) - f_3(Z)| & = & |-Y_2\frac{1}{\alpha_0}L(Y_1, Y_3) + \frac{1}{\tau_1}(1 - Y_3) + K_{La}(C_s^1 - Y_3) + Z_2\frac{1}{\alpha_0}L(Z_1, Z_3) \\ & - & \frac{1}{\tau_2}(1 - Z_3) - K_{La}(C_s^2 - Z_3)|, \\ & \leq & \frac{1}{\alpha_0}|Z_2 L(Z_1, Z_3) - Y_2 L(Y_1, Y_3)| + \frac{1}{\tau_1\tau_2}|\tau_2 - \tau_1| + |\frac{Z_3}{\tau_2} - \frac{Y_3}{\tau_1}| \\ & + & K_{La}|Z_3 - Y_3| + K_{La}|C_s^1 - C_s^2|, \\ & \leq & \frac{1}{\alpha_0}|Z_2 - Y_2| + \frac{AK}{\alpha_0}(|Y_1 - Z_1| + |Y_3 - Z_3|) + \frac{1}{\tau_1\tau_2}|\tau_2 - \tau_1| \\ & + & \frac{1}{\tau_2}|Z_3 - Y_3| + \frac{A}{\tau_1\tau_2}|\tau_1 - \tau_2| + K_{La}|Z_3 - Y_3| + K_{La}|C_s^1 - C_s^2|.\end{array}$$

We get

$$|f_3(Y) - f_3(Z)| \leq M_3\|Y - Z\| + \frac{A + 1}{\tau_1\tau_2}|\tau_1 - \tau_2| + K_{La}|C_s^1 - C_s^2|, \tag{19}$$

where $M_3 := \sup\{\frac{AK}{\alpha_0} + \frac{1}{\tau_2} + K_{La}, \frac{1}{\alpha_0}\}$.

So, equalities (15) and (16) lead to

$$\|\dot{Y} - \dot{Z}\| \leq M\|Y - Z\| + N_1|\tau_1 - \tau_2| + N_2|R_1 - R_2| + K_{La}|C_s^1 - C_s^2|, \tag{20}$$

where $M := M_1 + M_2 + M_3$, $N_1 := \dfrac{2 + 3A + AR_1}{\tau_1\tau_2}$ and $N_2 = \dfrac{A}{\tau_2}$.
Using the previous Gronwell's lemma (3.3), we obtain

$$\begin{aligned}\|Y(t) - Z(t)\| \quad &\leq \quad \|Y_0 - Z_0\|e^{Mt} + (\frac{N_1}{M}|\tau_1 - \tau_2| \\ &+ \quad \frac{N_2}{M}|R_1 - R_2| + \frac{K_{La}}{M}|C_s^1 - C_s^2|)e^{Mt},\ \forall\, t \geq 0,\end{aligned}$$

as required □

The above theorem gives a way to test the sensitivity of the model with respect to the initial conditions and the parameters R, τ and C_s, see figure (1). This result constitutes an error estimation between two trajectories Y and Z. Furthermore, if we write the trajectories as a function ϕ of the initial condition and parameters R, τ and C_s :

$$Y(t) = \phi(t, Y_0, R_1, \tau_1, C_s^1)$$
$$Z(t) = \phi(t, Z_0, R_2, \tau_2, C_s^2)$$

the theorem proves the Lipschitz-dependence of ϕ on those variables.

4. Strong controllability

The aim of this section is to establish a result of local strong controllability with respect to a given referential trajectory of the system (8). We begin by introduce some tools of nonsmooth analysis needed for our purpose, more details can be found in [5].
Let C be a closed subset of $\mathbb{R}^n$ and $c \in C$. The Fréchet Normal cone to C at c, $N^F(C, c)$, is defined by: $v \in N^F(C, c)$ if and only if: $\forall\, \varepsilon > 0$ there exists a neighbourhood U of c such that

$$< v, c' - c > \leq \varepsilon\|c' - c\| \ \ \forall\, c' \in C \cap U,$$

where $\langle y, z\rangle$ denotes the usual scalar product in $\mathbb{R}^n$.
and the limiting Fréchet normal cone is the set

$$N^L(C, c) = \limsup_{y \in C \to c} N^F(C, y)$$

Now consider $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, a lower semicontinuous (l.s.c.) function and $c \in \mathbb{R}^n$ be such that $\varphi(c) < +\infty$. The limiting Fréchet subdifferential of φ at c is given by

$$\partial\varphi(c) := \{\xi \in \mathbb{R}^n \,:\, (\xi, -1) \in N^L(epi\varphi; (c, \varphi(c)))\},$$

where $epi\varphi$ denotes the epigraph of φ.
We also define the distance function to C as

$$d(v; C) := \inf\{\|v - c\| \,:\, \forall\, c \in C\}.$$

Finally, we consider the characteristic function $\Psi_C(c) = \begin{cases} 0 \text{ if } c \in \mathcal{C} \\ +\infty \text{ otherwise} \end{cases}$,
Let us state now our controllability result. Suppose that we wish to control the system (8) with the parameters : recycle rate R, residence time τ and dissolved oxygen saturation concentration C_s. For that, let $T > 0$ and formulate the following new assumptions:

Assumptions

A_1: The parameters K_s, K_c, K_D and K_{La} are non null positive constants.
A_2: $R, \tau, C_s : [0,T] \to \mathbb{R}_*^+$ are Lebesgue measurable and there exist $\underline{R}, \overline{R} \in [0,1[$, $\underline{\tau}, \overline{\tau} \in \mathbb{R}_*^+$ and $\underline{C_s}, \overline{C_s} \in \mathbb{R}_*^+$ such that

$$\underline{R} < R(t) < \overline{R},\ \underline{\tau} < \tau(t) < \overline{\tau} \text{ and } \underline{C_s} < C_s(t) < \overline{C_s}, \text{ for all } t \in [0,T].$$

Consider $W^1 := AC([0,T], \mathbb{R}^3)$ the space of absolutely continuous functions and $L^1 := L^1([0,T], \mathbb{R}^3)$ the space of integrable functions. We define on L^1 and W^1, respectively, the norms

$$\|Z\|_1 = \int_0^T \|Z(t)\| dt \text{ and } \|Z\|_{W^1} = \|Z(0)\| + \int_0^T \|\dot{Z}\| dt.$$

where $\|\cdot\|$ denotes here the Euclidean norm on $\mathbb{R}^3$. Consider the set of controls

$$\mathcal{U} := \{(R, \tau, C_s) \in \mathbb{R}^3 : \underline{R} \leq R \leq \overline{R},\ \underline{\tau} \leq \tau \leq \overline{\tau},\ \underline{C}_s \leq C_s \leq \overline{C}_s\}.$$

Our aim is to study the controllability of the system (8) under the new assumptions $A_1 - A_2$:

$$\begin{aligned} \dot{Z}(t) &= f(Z(t), u(t))\ \forall t \in [0,T], && (21)\\ (Z(0), Z(T)) &\in S := \mathbb{R}^+ \times \mathbb{R}^+, && (22) \end{aligned}$$

where $u(t) := (R(t), \tau(t), C_s(t)) \in \mathcal{U}$ for all $t \in [0,T]$.
To this system we associate the following perturbed one:

$$\begin{aligned} \dot{Z}(t) &= f(Z(t), u(t))\ \forall t \in [0,T], && (23)\\ (Z(0) + a_0, Z(T) + b_0) &\in S, && (24) \end{aligned}$$

where $(a_0, b_0) = (a_{01}, a_{02}, a_{03}, b_{01}, b_{02}, b_{03}) \in \mathbb{R}^6$.

Remark 4.1. Regarding the first formulation (1-3) of the system (8), we remark that the control $u(t) := (R(t), \tau(t), C_s(t))$ is not affine, according to presence of the terms $\frac{1}{\tau}$ and $\frac{R}{\tau}$. So, our control problem is nonlinear in (Z, u) and not affine in terms of control variable u.

Consider now (Z, u) a solution of the system (21-22) with $u \in \mathring{\mathcal{U}}$ (the interior of $\mathcal{U}$). From [11], (see also [12]), we define the local strong controllability.

Definition 4.2. We say that the system (21-22) is strongly locally controllable at (Z, u) if there exist $\alpha > 0$ and $\beta > 0$ such that for all $(a_0, b_0) \in \mathbb{R}^6$, with $\|a_0\| + \|b_0\| \leq \beta$, there exist a trajectory Z_1 and a control u_1 for the system (23-24) such that

$$\|Z - Z_1\|_{W^1} \leq \alpha(\|a_0\| + \|b_0\|) \quad \text{and} \quad \|u - u_1\|_1 \leq \alpha(\|a_0\| + \|b_0\|).$$

This controllability concept invokes some tools of nonsmooth analysis. Jourani, in ([11]), establishes the relationship between the strong controllability and the normality of the trajectory Z as follows:

Definition 4.3. The system (21-22) is said to be normal at Z if the unique arc $p \in W^1$ satisfying

$$\begin{aligned} \dot{p}(t) \in co\{q : (q, 0) &\in \partial[-<p(t), f(.,.)> + \Psi_{\mathcal{U}}(.)](Z(t), u(t))\}\ a.e. && (25)\\ <p(t), f(t, Z(t), u(t))> &= \max_{w \in \mathcal{U}} <p(t), f(t, Z(t), w)>\ a.e. && (26)\\ (p(0), p(T)) &\in \partial d((Z(0), Z(T)); S), && (27) \end{aligned}$$

is the trivial one ($p = 0$).
Where "*co*" is the convex hull.

According to ([11]), we know that if the system (21-22) is normal at Z then it is strongly controllable at Z.

Theorem 4.4. *([11]) Suppose that*
- *$f(Y, w)$ is continuous in w for all Y.*
- *there exist $\varepsilon > 0$ and an integrable function $k : [0, T] \mapsto \mathbb{R}$ such that, for almost all $t \in [0, T]$, given two points $Z_1, Z_2 \in Z(t) + \varepsilon\mathbb{B}$, $q_1, q_2 \in u(t) + \varepsilon\mathbb{B}$ and $q \in \mathcal{U}$ we have*

$$|f(Z_1, q) - f(Z_2, q)| \leq k(t)\|Z_1 - Z_2\|,$$

$$|f(Z_1, q_1) - f(Z_2, q_2)| \leq k(t)[\|Z_1 - Z_2\| + \|q_1 - q_2\|],$$

where $\mathbb{B}$ is the unit ball in the appropriate space.
Then the normality of (21-22) at Z implies its local strong controllability.

Using this concept, we prove without linearizing, and in non-affine context, the controllability of the system at (Z, u).

Theorem 4.5. *Suppose that $A_1 - A_2$ hold. Then the system (21-22) is strongly locally controllable at (Z, u).*

Let us first give the following useful lemma

Lemma 4.6. *Under the assumption A_2, if $Z_2(0) > 0$ then*

$$Z_2(t) > 0, \ \forall t \geq 0.$$

The proof of the lemma uses the standard notions of differential equations.

Proof. **Proof of the previous theorem**
According to relations (5-7) and (8), its easy to show that $w \mapsto f(Y, w)$ is continuous for all $Y \in \mathbb{R}^3$. On the other hand, applying the inequality (20) for two trajectories Z_1 and Z_2 with the same control $q = (R, \tau, C_s)$, we obtain

$$\|f(Z_1, q) - f(Z_2, q)\| \leq M\|Z_1 - Z_2\|, \tag{28}$$

where M is given in (20).
Now the same inequality (20) applied to Z_1 and Z_2 with two controls $q_1 = (R_1, \tau_1, C_s^1)$ and $q_2 = (R_2, \tau_2, C_s^2)$ respectively , gives

$$\begin{aligned} \|f(Z_1, q_1) - f(Z_2, q_2)\| &\leq M\|Z_1 - Z_2\| + N_1|\tau_1 - \tau_2| + N_2|R_1 - R_2| + K_{La}|C_s^1 - C_s^2| & (29) \\ &\leq M_2[\|Z_1 - Z_2\| + \|q_1 - q_2\|]. & (30) \end{aligned}$$

Where $M_2 := \sup\{M; N_1; N_2; K_{La}\}$.
The relations (28) and (30) assure the required items of the theorem (4.4).
So, the assumptions of the theorem (4.4) are fulfilled and hence the system (21-22) is strongly locally controllable at (Z, u) if Z is normal as mentioned above. It suffice then, to prove the normality of the trajectory Z.
Let $p \in W^1$ and consider the function

$$h : (Y, u) \mapsto \langle p, f(t, \cdot)\rangle(Y, u), \ \ \forall \ (Y, u) \in \mathbb{R}^3 \times \mathcal{U}.$$

We will show that there is no trivial arc $p \in W^1$, ($p \equiv 0$), satisfying the relations (25- 27).
Firstly, in (3.1) we proved that the trajectories are positively invariant, then

$$d((Z(0), Z(T)); S) = 0,$$

and by the way

$$\partial d((Z(0), Z(T)); S) = \{(0,0)\}.$$

So, according to (27) we obtain that $p(0) = 0$ and $p(T) = 0$.
Now, exploiting the differential properties of f, the relation (25) becomes

$$\dot{p}(t) = -\nabla_Z h(Z(t), u(t)), \ a.e.\ t \in [0,T], \tag{31}$$

and according to nonsmooth analysis theory, see [5], we know that $\partial\Psi_{\mathcal{U}}(u(t)) = N^L(\mathcal{U}, u(t))$, so

$$\nabla_u h(Z(t), u(t)) \in N^L(\mathcal{U}, u(t)), \ a.e.\ t \in [0,T]. \tag{32}$$

On the other hand, since $u \in \mathring{\mathcal{U}}$, we have $N^L(\mathcal{U}, u(t)) = \{0\}$, and by the way, the equation (32) becomes

$$\nabla_u h(Z(t), u(t)) = 0.$$

Equivalently, using the fact that $u(t) = (R(t), \tau(t), C_s(t))$, the gradient of the function h leads to

$i)$ $p_2 Z_2 = 0$,

$ii)$ $(1 - Z_1)p_1 + (R-1)p_2 Z_2 + (1 - Z_3)p_3 = 0$,

$iii)$ $K_{La} p_3 = 0$,

and the relation (31) implies that

$i')$ $-\dot{p}_1(t) = (-\frac{1}{\tau} - \frac{Z_2 Z_3}{K_c + Z_3}\frac{K_s}{(K_s+Z_1)^2})p_1 + (\frac{Z_2 Z_3}{K_c + Z_3}\frac{K_s}{(K_s+Z_1)^2})p_2$
$\quad - (\frac{Z_2 Z_3}{\alpha_0(K_c + Z_3)}\frac{K_s}{(K_s+Z_1)^2})p_3$,

$ii')$ $-\dot{p}_2(t) = -L(Z_1, Z_3)p_1 + (\frac{R-1}{\tau} - K_D + L(Z_1, Z_3))p_2 - \frac{1}{\alpha_0}L(Z_1, Z_3)p_3$,

$iii')$ $-\dot{p}_3(t) = (-\frac{Z_2 Z_1}{K_s + Z_1}\frac{K_c}{(K_c+Z_3)^2})p_1 + (\frac{Z_2 Z_1}{K_s + Z_1}\frac{K_c}{(K_c+Z_3)^2})p_2$
$\quad - (\frac{Z_2 Z_1}{\alpha_0(K_s + Z_1)}\frac{K_c}{(K_c+Z_3)^2} + \frac{1}{\tau} + K_{La})p_3$.

Since $K_{La} > 0$, (assumption A_1), the relation (iii) implies that

$$p_3 = 0.$$

Now taking into account the lemma (4.6) and item (i) we obtain

$$p_2 = 0.$$

Relation (ii) leads to

$$p_1 = Z_1 p_1 0. \tag{33}$$

The relation (iii') implies that

$$0 = \frac{Z_2 Z_1}{K_s + Z_1}\frac{K_c}{(K_c+Z_3)^2})p_1.$$

It follows that

$$Z_1 Z_2 p_1 = 0.$$

According to (33) we obtain

$$Z_2 p_1 = 0. \tag{34}$$

Taking into account the relation (34), item (i') implies that

$$\dot{p}_1(t) = \frac{1}{\tau}p_1, \tag{35}$$

and relation (ii') implies that

$$L(Z_1, Z_3)p_1 = 0,$$

which leads to

$$Z_3 p_1 = 0. \tag{36}$$

according to (33). The derivation of this equation gives

$$\dot{Z}_3 p_1 + Z_3 \dot{p}_1 = 0.$$

So, using (35), we obtain

$$\dot{Z}_3 p_1 + \frac{1}{\tau} Z_3 p_1 = 0.$$

Hence,

$$\dot{Z}_3 p_1 = 0.$$

By substituting $\dot{Z}_3$ in its value, we get

$$(-Z_2 \frac{1}{\alpha_0} L(Z_1, Z_3) + \frac{1}{\tau}(1 - Z_3) + K_{La}(C_s - Z_3))p_1 = 0.$$

Then, using (34) and (36), we obtain

$$(\frac{1}{\tau} + K_{La}C_s)p_1 = 0.$$

So, from A_1 and A_2, it follows that

$$p_1 = 0.$$

As required □

It is well known that the change of climate and seasons has an impact on how does a wastewater treatment plant operates. In this context, the controllability property permits to control the system by means of three tools: recycling energy through R, influent flow energy through τ and aeration energy through C_s. We obtain, in the presence of disturbances on initial conditions, an associate trajectory which remains close to the reference one (see figures 2-4).

4.1. Equilibrium controllability

We now apply this purpose to an equilibrium point. Suppose that the system (8) admits an equilibrium point Z_e associated to a control $u_e \in \mathring{\mathcal{U}}$. It follows that

$$0 = f(Z_e, u_e),$$

Our hope is to prove the controllability of this equilibrium. That is, if we start in a neighbourhood of Z_e, we can find a control taking back the trajectory to Z_e at T. For this, consider a small real $\delta > 0$ and define the following problem,

$$\dot{Z}(t) = f(Z(t), u(t)), \ \forall\, t \in [0, T], \tag{37}$$

$$(Z(0), Z(T)) \in S := (Z_e + \delta\mathbb{B}) \times \{Z_e\}, \tag{38}$$

and the perturbed one

$$\dot{Z}(t) = f(Z(t), u(t)),\ \forall\, t \in [0, T], \tag{39}$$

$$(Z(0) + a_0, Z(T)) \in S. \tag{40}$$

Remark that we perturbed only the initial condition, since our hope is to obtain $Z(T) = Z_e$ for the perturbed problem. So, using the above analysis developed in the theorem (4.5), we get

Corollary 4.7. *Under assumptions $A_1 - A_2$, there exist $\alpha > 0$ and $0 < \beta \leq \delta$ such that for all $a_0 \in \mathbb{R}^3$, with $\|a_0\| \leq \beta$, there exist a trajectory Z_1 and a control u_1 solution of the problem (39-40) such that*

$$\|Z_e - Z_1\|_{W^1} \leq \alpha \|a_0\| \quad \textit{and} \quad \|u_e - u_1\|_1 \leq \alpha \|a_0\|.$$

and $Z_1(T) = Z_e$

5. Numerical simulations

In this section we give some numerical simulations to illustrate our results.

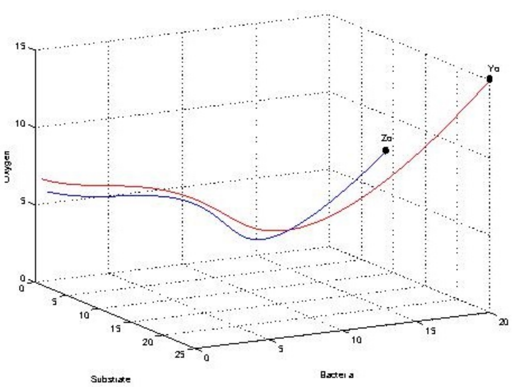

Figure 1. The Gronwell's error for two trajectories starting at Y_0 and Z_0 respectively.

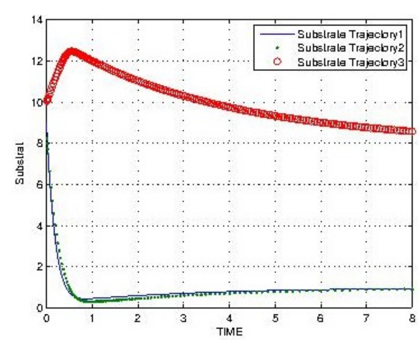

Figure 2. The influence of the disturbance on initial conditions on substrate trajectories.

Figure 1 shows the Gronwell's error estimation provided by the theorem (3.3), between two trajectories $Y(t, \tau_1, R_1, C_s^1)$ and $Z(t, \tau_2, R_2, C_s^2)$ starting respectively at Y_0 and Z_0.
Figures 2-4 illustrate the controllability property (section 4): in the presence of perturbations on the initial conditions, and taking into account a reference trajectory (trajectory 1: " $-$ "), two situations may occur. The trajectories starting far enough away from the trajectory 1, (trajectory 3: "o",) go away from this reference trajectory. Otherwise, the trajectories starting in a neighbourhood of trajectory 1, (trajectory 2: ".."), can be controlled by choosing an adequate control parameters (R, τ, C_s) to keep them near this reference trajectory.

6. Conclusion

In this work, we were interested in a non linear aerobic wastewater model considering two cases. Firstly, when the parameters are constant, we establish the positivity, invariance and boundedness of the trajectories together with the sensitivity leading to an error estimation. Secondly, when some parameters are functions of time, we investigate, without linearizing and in a non affine setting, the strong controllability of the model using

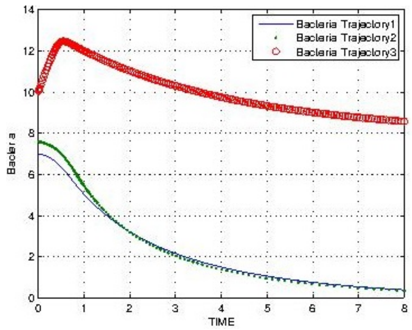

FIGURE 3. The influence of the disturbance on initial conditions on bacteria trajectories.

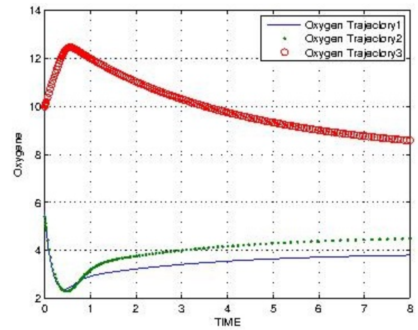

FIGURE 4. The influence of the disturbance on initial conditions on oxygen trajectories.

those parameters. We exploit this result to prove the local controllability of an equilibrium point, if it exists. The next step will be the optimal control of the model since the controllability involves the energy.

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