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COLLISIONS IN MAGNETISED PLASMAS *

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Abstract. Approximations for closing the kinetic equation for the one particle distribution function are calculated by using propagators. These provide the formal structure of the collision term in the Landau approximation. The method allows one to investigate the effect of inhomogeneities at the Debye scale and to analyse magnetised collisions, when the Larmor radius is smaller than the Debye length. This method also allows one developing a simple renormalisation scheme to derive the Lenard-Balescu collision operator.

Résumé. Une méthode de propagateurs est utilisée pour fermer l'équation cinétique de la fonction de distribution à une particule. Celle-ci donne accès à la structure formelle de l'opérateur de collision dans l'approximation de Landau. Elle permet alors de calculer l'effet d'inhomogénéité à l'échelle de la longueur de Debye et d'analyser les collisions magnétisées, lorsque le rayon de Larmor est inférieur à la longueur de Debye. Cette méthode permet également de développer une méthode simple de renormalisation pour calculer l'opérateur de collision de Lenard-Balescu.

Introduction

The physics of burning plasma that will be investigated experimentally in ITER [1] require a first principle simulation support to prepare and analyse the experiments. In particular simulations of the turbulent transport must be addressed to understand and monitor the confinement performance of the device. At the contemplated density and thermal energy, the ITER plasmas will be characterised by a very small collisionality, in particular due to the plasma density. The latter is such that the typical distance between particles ranges from 10^{-6} m to 10^{-7} m. In this regime weak binary collisions are shown to prevail. One then benefits from these large inter-particle distances since they allow one to consider the collisions from a classical point of view and derive analytically first principle collision operators. The small collisionality of these plasmas leads to many unexpected features regarding plasma transport compared to neutral fluids. One of the most important is the requirement to address transport properties from the kinetic point of view.

In the present paper we review many known aspect of plasma collisions in Section 1. This allows one to introduce in a systematic fashion key aspects that will be used in the following Sections. In Section 2 we analyse the derivation of the kinetic equation for the one particle distribution function. Here we follow the 1987 "Thèse d'Etat" work [2] yet unpublished. The starting point is the Klimontovich density [3] but instead

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of using the BBGKY expansion [3] we rather follow an averaging procedure, as proposed in [4], based on the ability of differentiating various states to determine a state probability that is assumed to be continuous. In Section 3, the derivation of the Landau collision operator is undertaken based on a quasilinear approach to calculate the correlation function of the fluctuating distribution functions of interacting particles. This provides an elegant formalism where the physical insight presented in Section 1 is used as a guideline. The step to the Lenard-Balescu collision operator is then straightforward by further expanding the correlation function and the propagator formalism, Section 3. A simple renormalisation scheme then allows one to sum-up the contributions that readily appear as an expansion of the plasma permittivity. The issue of distribution function gradients at the Debye scale is addressed in Section 4 together with the collision operator for magnetised plasmas. The latter issue is more developed in [2] but requires to be revisited in the framework of modern gyrokinetics [5] and for optimum application for gyrokinetic codes such as the GYSELA code [6]. Finally the conservation laws are addressed together with the entropy production due to the collision operator, Section 5. A short conclusion closes the paper.

1. Collisions and particle trajectories in magnetised plasmas

1.1. Gyration motion in the magnetic field

The key aspect of magnetic confinement is the charged particle gyration motion in the magnetic field. The Newton law for such a motion is:

$$m_a \frac{d\boldsymbol{v}}{dt} = q_a \boldsymbol{v} \times \boldsymbol{B}. \tag{1}$$

where m_a, q_a are the mass and charge respectively of the particle of species a. This equation is homogeneous with respect to the velocity $\boldsymbol{v}$ and exhibits the characteristic time $m_a/(q_a B) = 1/\Omega_a$, hence the inverse of the Larmor gyration frequency Ω_a, which depends on the magnitude of the magnetic field. For ions with $q_a/m_a \approx e/m_p \approx 10^8\ C\ /\ kg$, e is the unit charge and m_p the proton mass, one then finds that $\Omega_a \approx 5.10^8\ s^{-1}$ for the magnetic field foreseen on ITER, $B \approx 5\ T$. In fusion plasmas, the Larmor period $1/\Omega_a$ is a very small and shorter than most time scales of interest which allows one to introduce a time scale separation.

The gyration motion characterised by Eq.(1) exhibits two symmetries, first $\boldsymbol{v} \cdot \dot{\boldsymbol{v}} = 0$, which ensures the conservation of the kinetic energy proportional to v^2, second, for a constant magnetic field in space and time, $\boldsymbol{B} \cdot \dot{\boldsymbol{v}} = 0$, which yields that the velocity component parallel to the magnetic field is constant, $v_\parallel(t) = v_\parallel(t_0)$ at all times. As a consequence, of these two first relations, one also has te conservation of $v_\perp^2$ where $v_\perp$ is the modulus of the velocity component transverse to the magnetic field.

One can notice from Eq.(1) that one can readily obtain a first integral of this equation:

$$m_a \frac{d\boldsymbol{x}}{dt} = q_a \Big(\boldsymbol{x} - \boldsymbol{x}_G\Big) \times \boldsymbol{B} + m_a v_\parallel(t_0) \boldsymbol{B}/B, \tag{2}$$

where $\boldsymbol{x}_G$ and $v_\parallel(t_0)$ are the three integration constants. In such an expression the position $\boldsymbol{x}_G$ is that of the cylindrical symmetry axis of the particle motion, the latter being aligned with $\boldsymbol{b} = \boldsymbol{B}/B$. Considering the transverse motion with velocity $\boldsymbol{v}_\perp$ and Larmor radius $\boldsymbol{\rho} = \boldsymbol{x} - \boldsymbol{x}_G$, one thus obtains:

$$\boldsymbol{v}_\perp = \boldsymbol{\rho} \times \boldsymbol{\Omega}_a \quad ; \quad \boldsymbol{\rho} = \boldsymbol{b} \times \frac{\boldsymbol{v}_\perp}{\Omega_a} \quad ; \quad \boldsymbol{\Omega}_a = \frac{q_a}{m_a}\boldsymbol{B}. \tag{3}$$

Even for very large ion velocities, typically $10^6\ m\ s^{-1}$ one finds that the ion Larmor radius is small, typically in the millimetre range, hence much smaller than the device size and consequently the scale of variation of the magnetic field in metres. In the framework of this analysis one thus finds that the motion imposed by the large

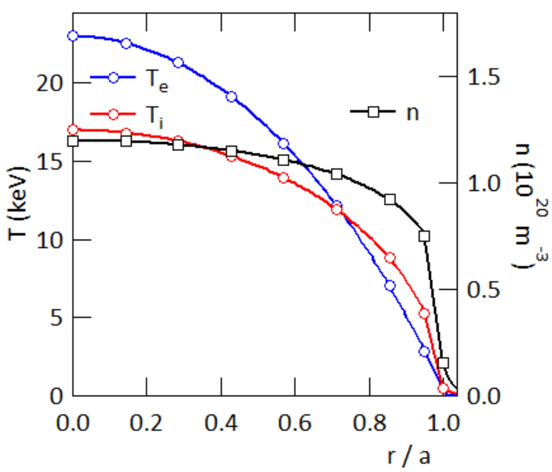

Figure 1. Typical ITER temperature and density profiles.

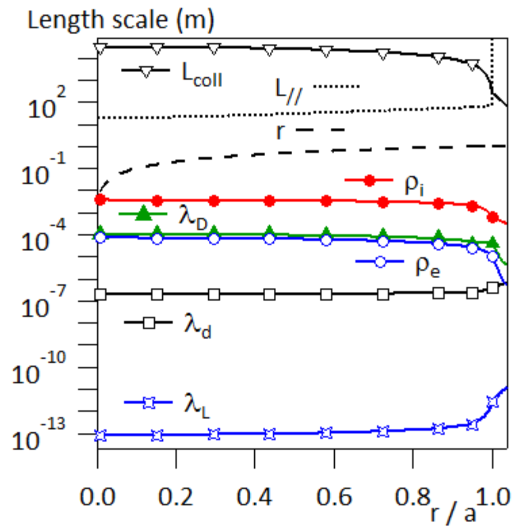

Figure 2. Profiles of plasma characteristic length scales.

magnetic field to the charged particles is at lowest order characterised by a cylindrical symmetry.

In order to illustrate the order of magnitude of the various scales introduced in this paper, we consider ITER [7] characteristic profiles for the ion and electron temperature as well as the density fig.(1). The ITER tokamak has a toroidal geometry with major radius $R = 6\ m$ and minor radius $a = 2\ m$. On the profiles, one readily identifies three regions: the core plasma extending typically from $r/a = 0$ to $r/a = 0.95$, the pedestal region with sharp gradients from $r/a = 0.95$ to $r/a = 1$, and the SOL region with $r/a \geq 1$ which is a thin boundary layer where the temperature profiles and the density decay exponentially to zero. Given ITER temperature characteristic profiles fig.(1), one can determine that of the Larmor radius, fig.(2), for the ions (closed circles) and for the electrons (open circles). These transverse scales are much smaller than the minor radius of the plasma, r (dashed line), fig.(2).

1.2. Maxwell equations: charge conservation, charge screening

Starting from the Maxwell equations and introducing the electric and vector potential allows one to split this set of equations into geometrical features that determine the relationship between the electromagnetic field and the electromagnetic potential on the one hand and to equations governing the source terms, space charge and currents for the electromagnetic field on the other hand.

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad ; \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \tag{4}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B} \qquad ; \qquad \boldsymbol{E} = -\boldsymbol{\nabla} U - \frac{\partial}{\partial t}\boldsymbol{A}, \tag{5}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad ; \qquad \frac{\partial}{c\,\partial t}\frac{\partial}{c\,\partial t}\boldsymbol{U} - \boldsymbol{\nabla}^2 \boldsymbol{U} = \frac{\rho}{\epsilon_0}, \tag{6}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{\partial}{c^2\,\partial t}\boldsymbol{E} \qquad ; \qquad \frac{\partial}{c\,\partial t}\frac{\partial}{c\,\partial t}\boldsymbol{A} - \boldsymbol{\nabla}^2 \boldsymbol{A} = \mu_0 \boldsymbol{j}. \tag{7}$$

Accordingly, the definition of the potentials is completed by the Lorenz gauge:

$$c\boldsymbol{\nabla}\cdot\boldsymbol{A}+\frac{\partial}{c\,\partial t}\boldsymbol{U}=0. \tag{8}$$

Combining the Maxwell equation relating the electromagnetic potentials, U and $\boldsymbol{A}$, to charge density ρ Eq.(6) and current density $\boldsymbol{j}$ Eq.(7) and the Lorenz gauge Eq.(8), one readily recovers the fundamental charge conservation equation:

$$\frac{\partial}{\partial t}\boldsymbol{\rho}+\boldsymbol{\nabla}\cdot\boldsymbol{j}=0. \tag{9}$$

The latter readily stipulates that if no current is flowing out of a given volume the charge density within that volume is the initial one, hence that charge neutrality in that volume is conserved. This is a general form of the quasi-neutrality equation since it postulates that at a given scale, so that the electric current vanishes, the system appears as neutral. Normalising the time space scales with τ and L and introducing $U^*=e\ U\ /\ \bar{T}$, $A^*=A\ L\ /\ \bar{B}$, $\rho^*=\rho/(e\ \bar{n})$ and $j^*=j\ L\ \bar{B}/(\bar{n}\bar{T})$ where $\bar{n}$ and $\bar{T}$ are the characteristic particle density and temperature of the plasma respectively and e is the electric charge. The normalising current is defined in terms of the normalised magnetic field having in mind the MHD force balance between the pressure gradient and the Laplace force $\boldsymbol{j}\times\boldsymbol{B}$. One then obtains:

$$\left(\frac{L}{c\ \tau}\right)^2\frac{\tau\partial}{\partial t}\frac{\tau\partial}{\partial t}\boldsymbol{U}^*-\Big(L\boldsymbol{\nabla}\Big)^2\boldsymbol{U}^*=\frac{e^2\ \bar{n}\ L^2}{\epsilon_0\ \bar{T}}\rho^* \quad = \quad \left(\frac{L}{\lambda_D}\right)^2\rho^*, \tag{10}$$

$$\left(\frac{L}{c\ \tau}\right)^2\frac{\tau\partial}{\partial t}\frac{\tau\partial}{\partial t}\boldsymbol{A}^*-\Big(L\boldsymbol{\nabla}\Big)^2\boldsymbol{A}^*=\frac{\bar{n}\ \bar{T}}{\bar{B}^2/\ \mu_0}\boldsymbol{j}^* \quad = \quad \beta\ \boldsymbol{j}^*, \tag{11}$$

$$\Big(L\boldsymbol{\nabla}\Big)\cdot\boldsymbol{A}^* \quad = \quad -\left(\frac{D_B}{\tau c^2}\right)\frac{\tau\partial}{c\ \partial t}\boldsymbol{U}^*. \tag{12}$$

The right-hand side of this set of equations exhibits three dimensionless parameters that depend on well known plasma parameters. The Debye scale $\lambda_D^2=\epsilon_0\ T/(e^2 n)$ is a measure of the importance of charge separation at scale L in generating the electromagnetic field Eq.(10). One thus finds that at large scales, very small charge densities associated to polarisation effects generate the electromagnetic fields. The plasma is then considered as quasineutral, $\rho\approx n(\lambda_D/L)^2$. Consequently, with $L\gg\lambda_D$, one finds the quasineutrality condition $\rho\to 0$. Regarding Eq.(11), the dimensionless parameter is the MHD β parameter which characterises the generation of magnetic fields by the plasma. In Eq.(12), one finds the Bohm diffusion coefficient $D_B=e\ T\ /\ B$, and therefore the departure from the Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$. The condition can also be expressed in terms of the Larmor radius and frequency Eq.(3), as $D_B/(\tau c^2)=\Big(\rho^2/(c\tau)^2\Big)(\Omega\tau)$. This parameter tends to zero in most cases of interest in plasma physics.

The profile of the Debye scale for ITER typical parameters is plotted on fig.(2) (closed triangles). One finds that $\lambda_D\geq\rho_e$, hence that regarding the electrons, the Coulomb interactions close to the screening limit are magnetised. This feature is also clear on fig.(4) where the ratio λ_D/ρ_e is larger than unity over the whole profile and increases sharply in the SOL region.

1.3. **Coulomb collision mean free path**

Various scales characterise the collisions. First one can introduce the Landau scale λ_L such that the kinetic energy balances the Coulomb potential energy. In the plasma media the characteristic particle kinetic energy

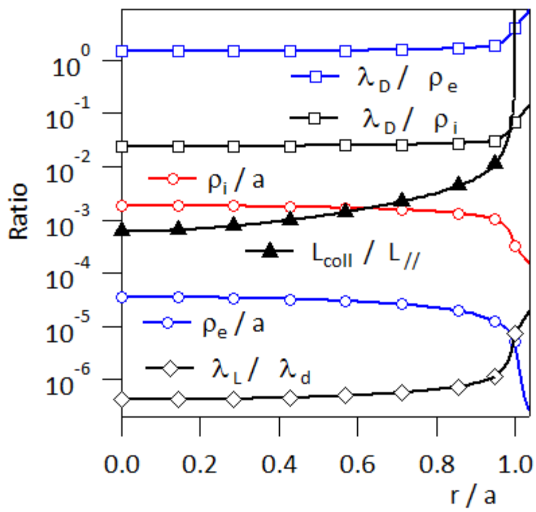

FIGURE 3. Profiles of the ratios of plasma characteristic length scales.

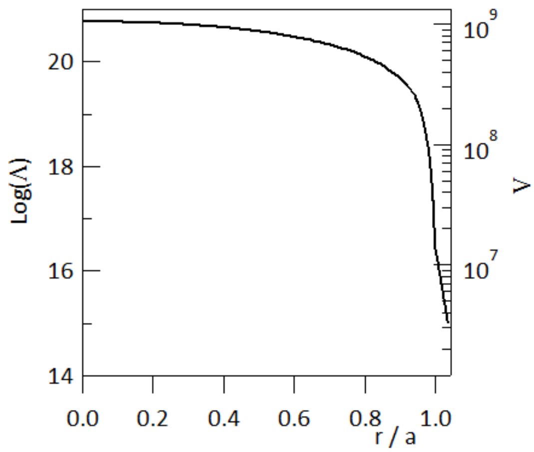

FIGURE 4. Plasma parameter profile.

is given by the thermal energy T. The Landau scale is therefore defined as:

$$T\lambda_L = \frac{e^2}{4\pi\epsilon_0} = \alpha \quad ; \quad \lambda_d^3\, n = 1. \tag{13}$$

In burning plasmas, this scale is such that $\lambda_L \leq 10^{-13}\ m$ so that a sufficiently large number of fusion events occur. This threshold effect is equivalent to the threshold on the plasma thermal energy $T \geq 10\ keV$. The constant $\alpha = e^2/(4\pi\varepsilon_0)$ will be used in the following to simplify the notations. Another relevant scale is the typical distance between the particles, the Loschmidt scale λ_d Eq.(13). In all magnetic fusion plasmas $\lambda_L/\lambda_d \ll 1$, see fig.(3). This means that the average kinetic energy of the particles, the thermal energy, is always much larger than the typical Coulomb potential energy. The collisions are therefore a weak effect and particles are mostly free-streaming and only constrained by the electromagnetic field, in particular the large magnetic field.

Given λ_L and λ_d one can readily estimate the Debye length Eq.(10).

$$\lambda_D = \left(\frac{\epsilon_0 T}{e^2 n}\right)^{1/2} = \lambda_d \frac{1}{2\sqrt{\pi}} \left(\frac{\lambda_d}{\lambda_L}\right)^{1/2}. \tag{14}$$

One thus finds $\lambda_D \gg \lambda_d$ when $\lambda_d \gg \lambda_L$, which means that the screening effect at the Debye scale occurs in regions of very small interaction energy via the collisions compared to the kinetic energy.

When analysing the effect of weak binary collisions, one finds that the most important effect is the change in direction of the relative velocity of the interacting particles during the collision, the deflection angle ε_b. One can expect this angle to depend on the impact parameter b, see figure (5). The deflection of the trajectory of a given particle is a random function with positive and negative deflection events. On average one thus finds $\langle\varepsilon_b\rangle = 0$ and therefore that the deflection angles exhibits a diffusive process. The collision mean free path L_{coll} is then be determined by the number of weak collisions $N(L_{coll})$ required to reach the efficiency of a strong interaction, hence a deflection of the order of π, hence:

$$\int db\ 2\pi\ b\ L_{coll}\ n\ \frac{\varepsilon_b^2}{\pi^2} \approx 1. \tag{15}$$

The key parameter to determine the collision mean free path is therefore the typical deflection angle during a binary collision. In the following we will calculate this angle, first in a simplified way that underlines the key aspects of weak Coulomb collisions and with an exact calculation. However, before proceeding to the calculation, one can estimate $\varepsilon_b = \lambda_L/b$ from a simple dimensional point of view, retaining the fact that the larger the impact parameter, the smaller the deflection. With this approximation one readily obtains:

$$Ł_{coll} \approx \frac{\pi}{2}\frac{\lambda_d^3}{\lambda_L^2}\left(\int\frac{db}{b}\right)^{-1} = \lambda_d\ \frac{\pi}{2}\left(\frac{\lambda_d}{\lambda_L}\right)^2\frac{1}{Log(\Lambda)}. \tag{16}$$

In typical systems with neutral particles, the collision mean free path is comparable to the Loschmidt scale. However, in plasmas this distance is much larger due to the parameter $(\lambda_L/\lambda_d)^{-2}$, fig.(3). The effect of the Coulomb logarithm $Log(\Lambda)$ stemming from the integration over the impact parameter b does not balance the former effect, fig(4).

The cut-off introduced to bound the logarithmic divergence of the integrated Coulomb potential energy is twofold, towards the smallest scale the Landau scale separates the weak and strong collisions is thus sets a lower bound. Towards the largest scales, the Debye scale λ_D is an upper bound governed by the screening properties. The Coulomb logarithm $Log(\Lambda)$ depends on the plasma parameter Λ, which is then defined by the chosen cut-offs:

$$\Lambda = \frac{\lambda_D}{\lambda_d}\frac{\lambda_d}{\lambda_L} = \frac{1}{2\sqrt{\pi}}\left(\frac{\lambda_d}{\lambda_L}\right)^{3/2} =\ 4\pi\ \lambda_D^3\ n. \tag{17}$$

One thus readily finds that the plasma parameter Λ is a large number, large enough to yield a Coulomb logarithm $Log(\Lambda)$ exceeding 10, fig(4) .

Regarding the strong binary collisions, one can readily estimate the mean free path associated to this effect by considering that all collisions in the volume $\pi\lambda_L^2\ L_{coll}^S$ to be large, hence given the density:

$$Ł_{coll}^S \approx \lambda_d\frac{1}{\pi}\frac{\lambda_d^2}{\lambda_L^2} = L_{coll}\left(\frac{2}{\pi^2}\ Log(\Lambda)\right). \tag{18}$$

The Coulomb logarithm set the ratio between the two collision mean free paths, $Ł_{coll}^S \gg L_{coll}$ provided that $Log(\Lambda) \gg 1$. Given this condition, one can then work in the approximation of weak binary collisions.

1.4. **Weak Coulomb collisions**

Let us consider the calculation of the deflection angle due to weak Coulomb collisions considering two charged particles labelled 1 and 2. We separate the system in the motion of the mass centre and the relative motion. For isolated binary collisions, the momentum of the mass centre is conserved and one can thus concentrate on the relative motion with the reduced mass $m = m_1m_2/M$, where $M = m1 + m_2$ and the central force $q_1q_2\alpha\hat{\boldsymbol{r}}/r^2$ where $4\pi\epsilon_0\alpha = e^2$, $q_i = e_i/e$ is the normalised charge of particle i of charge e_i. The distance to the mass centre is r and $\hat{\boldsymbol{r}}$ is the unit vector from the mass centre to the relative particle position. The plane where the collision deflection takes plane is then defined by the origin, the mass centre, the vector $\hat{\boldsymbol{r}}$ and the relative velocity $\boldsymbol{v}$ at time $t \to -\infty$ prior to the collision. The relative motion conserves the total energy, translation symmetry with respect to changing the origin of time, and angular momentum, axial symmetry with respect to the vector orthogonal to the plane where the deflection takes place. Let us consider that the axis x is the axis of symmetry of the deflection so that the relative velocity at $t \to -\infty$ is oriented along the y-axis, $\boldsymbol{v} = -v_\infty \, \hat{\boldsymbol{y}}$ and such that the particle reaches the x-axis, $y = 0$, at time $t = 0$. We further introduce the impact parameter b as the distance between the knock on axis (b=0) and the parallel axis defined by the position and velocity of the particle at $t \to -\infty$, see fig.(5).

In the simplified approach, we consider that the motion along the y-axis is unperturbed so that $y = v_\infty t$ at all times. The deflection will then be determined by the momentum transfer along x compared to mv_∞ and thus determined by the integration of the Coulomb force projected on x from $t \to -\infty$ to $t \to +\infty$, hence:

$$\varepsilon_b(t) = \frac{\Delta(m\ v_x)}{mv_\infty} = 2\int_0^t dt' \ \frac{q_1\ q_2\ \alpha}{r^2} = 2\ \frac{2\ q_1\ q_2\ \alpha}{m\ v_\infty^2\ b} \ \int_0^{tv_\infty/b} du \frac{1}{(1+u^2)^{3/2}} = \frac{\lambda_L}{b}\frac{t\ v_\infty/b}{\left(1+(t\ v_\infty/b)^2\right)^{1/2}}. \tag{19}$$

At the limit $t \to +\infty$, one thus recovers the expression used in Section (1.3), the trajectory during the collision remaining unaffected in the y-direction. This model also allows one to determine where most of the deflection

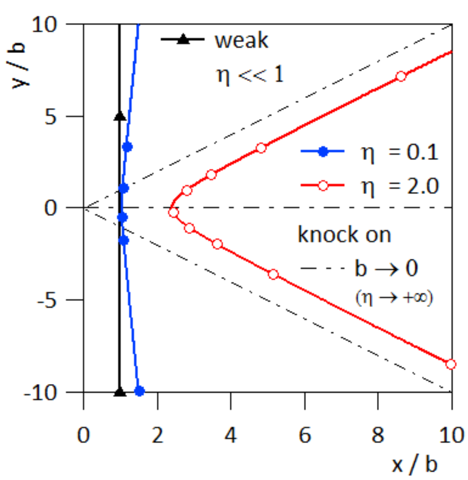

FIGURE 5. Relative trajectories during Coulomb collisions for different values of the parameter $\eta = \frac{\lambda_L}{b}$.

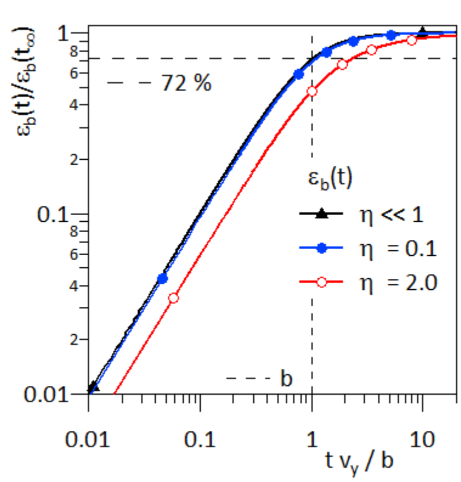

FIGURE 6. Normalised deflection of the trajectory starting from the symmetry axis, figure (5).

occurs, fig.(6). One finds that within this approximation, labelled by $\eta \ll 1$ in fig.(6), more than 70 % of the deflection occurs at a distance equal to the impact parameter from the mass centre. Hence, during a typical collision at the Loschmidt scale, the binary interaction is predominantly governed by one neighbouring particle.

The complete model for the Rutherford collision is readily derived from the angular momentum Eq.(20) and energy Eq.(21) conservation laws. In cylindrical coordinates (r, θ), this yields:

$$\frac{d\tau}{d\theta} = u^{-2}, \tag{20}$$

$$\frac{d\theta}{du} = -\left(\frac{1}{1-\eta\ u - u^2}\right)^{1/2}, \tag{21}$$

where $u = b/r$, b is the impact parameter, τ is the normalised time, $\tau = v_\infty t/b$ and $\eta = \lambda_L/b$. Changing variable in the second equation with $z = (u + \eta/2)/(1 + \eta^2/4)^{0.5}$ and $z = \cos(\theta)$, hence:

$$\cos(\theta) \quad = \quad \frac{u + \eta/2}{(1 + \eta^2/4)^{1/2}} \ . \tag{22}$$

Given the chosen symmetry, see fig.(5), one obtains the minimum value of r at $\theta = 0$. The deflection angle ϵ_b is then determined by:

$$\sin(\epsilon_b) = sin(2\theta_\infty) = \frac{\eta}{1 + \eta^2/4}. \tag{23}$$

One thus recovers $\epsilon_b = \eta = \lambda_L/b$ as used for the calculation of the mean free path. The deflection angle computed here is that of the relative motion. It must be multiplied by a mass ratio to determine the actual deflection for the particles. For like-particles collisions or for electrons impacting ions this does not change the order of magnitude. However for ions impacting electrons this has a large impact by reducing the deflection angle according to the mass ratio m_e/m_i.

2. Kinetic equation

2.1. Klimontovich density and Liouville probability

We follow here a standard approach [3] starting from the Klimontovich density $\mathcal{F}_a(Z, t)$ for species a at point $Z = (\boldsymbol{X}, \boldsymbol{P})$ of the phase space and at time t.

$$\mathcal{F}_a(Z, t) = \sum_{i_a} \delta\Big(Z - Z_{i_a}(t)\Big) = \sum_{i_a} \delta\Big(\boldsymbol{X} - \boldsymbol{X}_{i_a}(t)\Big)\delta\Big(\boldsymbol{P} - \boldsymbol{P}_{i_a}(t)\Big). \tag{24}$$

The Klimontovich density $\mathcal{F}_a(Z, t)$ accounts for point particles labelled by i_a. The trajectory for a given particle in phase space is thus defined by $Z_{i_a}(t) = \Big(\boldsymbol{X}_{i_a}(t), \boldsymbol{P}_{i_a}(t)\Big)$ and is determined by the Hamiltonian H such that:

$$\frac{d\boldsymbol{X}_{i_a}}{dt} \quad = \quad \frac{\partial \mathcal{H}}{\partial \boldsymbol{P}_{i_a}} = -\left[\mathcal{H}, \boldsymbol{X}_{i_a}\right]_{X,P} , \tag{25}$$

$$\frac{d\boldsymbol{P}_{\boldsymbol{i_a}}}{dt} \quad = \quad -\frac{\partial \mathcal{H}}{\partial \boldsymbol{X}_{i_a}} = -\left[\mathcal{H}, \boldsymbol{P}_{i_a}\right]_{X,P} , \tag{26}$$

where the Hamiltonian is the sum of the contribution of each kinetic energy and of the interaction coulomb potentials.

$$\mathcal{H} = \sum_a \sum_{i_a} \left(H_a^0 + \frac{1}{2} \sum_b \sum_{j_b} \frac{\alpha \ q_a q_b}{|\boldsymbol{X}_{i_a} - \boldsymbol{X}_{j_b}|} \right) = \sum_a \sum_{i_a} \mathcal{H}_a(Z_{i_a}, t). \tag{27}$$

The Coulomb interaction potential only depends on the distance between particles in physical space $|\boldsymbol{r}_{ij}| = |\boldsymbol{X}_i - \boldsymbol{X}_j|$. The summation over a and b are summations over species and the summation over i_a and j_b are summations over the particles of a given species a and b respectively. The mass m_a and charge number Z_a are only species dependent, hence labelled by a for species a, with $q_a = e_a/|e|$ e_a being the charge of species a. The interaction constant of the Coulomb potential is $\alpha = e^2/(4\pi\varepsilon_0)$. Note that for identical particles, hence identical species and identical particle label, one has $|\boldsymbol{r}_{ij}| = 0$, however the divergence is removed because the interaction potential between a particle and itself is equal to zero. Implicitly one thus assumes that only one particle can be located at a given location in physical space $\boldsymbol{X}$. The Hamiltonian H_a^0 can include the external fields.

The phase space integration of the Klimontovich density for species a yields the number of particles of species a, N_a. The phase space integral accounts for the whole space in $\boldsymbol{P}$ and a volume $\mathcal{V}$ for the physical space.

$$\int dZ\mathcal{F}_a(Z,t) = \int d\boldsymbol{X} \int d\boldsymbol{P}\ \mathcal{F}_a(Z,t) = N_a. \tag{28}$$

Assuming that particles are not created nor destroyed, one can then consider the particle conservation equation, which yields:

$$\frac{d\mathcal{F}_a(Z,t)}{dt} = \frac{\partial\mathcal{F}_a(Z,t)}{\partial t} - \Big[\mathcal{H}\mathcal{F}_a(Z,t)\Big]_{X,P} = 0. \tag{29}$$

The Poisson brackets are defined as:

$$\Big[F,G\Big]_{X,P} = \frac{\boldsymbol{\partial} F}{\partial \boldsymbol{X}}\cdot\frac{\boldsymbol{\partial} G}{\partial \boldsymbol{P}} - \frac{\boldsymbol{\partial} G}{\partial \boldsymbol{X}}\cdot\frac{\boldsymbol{\partial} F}{\partial \boldsymbol{P}}. \tag{30}$$

The subscript X, P in the Poisson bracket specifies which variables are used and the order of the derivatives. It will be dropped at a later stage to simplify the notation. The extension of the Liouville equation Eq.(29) to a case with particle creation or destruction can readily be done by implementing a non vanishing right-hand side, yielding the creation/destruction rate.

Introducing the definition Eq.(24) in Eq.(29), one obtains:

$$\frac{\partial\mathcal{F}_a(Z,t)}{\partial t} + \sum_{i_a}\left(\dot{\boldsymbol{X}}_{i_a}\cdot\frac{\boldsymbol{\partial}\mathcal{F}_a(Z,t)}{\partial \boldsymbol{X}_{i_a}} + \dot{\boldsymbol{P}}_{i_a}\cdot\frac{\boldsymbol{\partial}\mathcal{F}_a(Z,t)}{\partial \boldsymbol{P}_{i_a}}\right) = 0. \tag{31}$$

Given the definition of the Klimontovich density in terms of δ functions, one then readily obtains:

$$\frac{\partial\mathcal{F}_a(Z,t)}{\partial t} + \left(\dot{\boldsymbol{X}}\cdot\frac{\boldsymbol{\partial}\mathcal{F}_a(Z,t)}{\partial \boldsymbol{X}} + \dot{\boldsymbol{P}}\cdot\frac{\boldsymbol{\partial}\mathcal{F}_a(Z,t)}{\partial \boldsymbol{P}}\right) = \frac{\partial\mathcal{F}_a(Z,t)}{\partial t} - \Big[\mathcal{H},Z\Big]_{X,P}\cdot\frac{\partial\mathcal{F}_a(Z,t)}{\partial Z} = 0. \tag{32}$$

In this result, the two fields $\dot{\boldsymbol{X}}$ and $\dot{\boldsymbol{P}}$ are extended to regions where particles are not necessarily located.

Let us now introduce the Liouville probability $\mathcal{P}_L\Big(\{z_i\},t\Big)$ for a elementary-state where particle i is located in phase space number i $\Big(\boldsymbol{x}_i,\boldsymbol{p}_i\Big)$ at $(\boldsymbol{X}_i(t),\boldsymbol{P}_i(t)) = Z_i(t)$ at time t, and similarly for all particles. Each particle is then located in its own specific $6D$ phase space. Note that for the Liouville probability the species index is dropped because this probability encompasses all particles and species of the plasma in defining elementary-state for the whole plasma in the volume $\mathcal{V}$. Using a Hartree-Fock structure, hence multiplicative probabilities as for independent events, one then defines the probability of the given elementary-state as:

$$\mathcal{P}_L\Big(\{z_i\},t\Big) = \prod_{a,\ i_a}\delta\Big(z_i - Z_{i_a}(t)\Big) = \prod_{a,\ i_a}\delta\Big(\boldsymbol{x}_i - \boldsymbol{X}_{i_a}(t)\Big)\delta\Big(\boldsymbol{p}_i - \boldsymbol{P}_{i_a}(t)\Big), \tag{33}$$

where $\{z_i\}$ is the coordinate ensemble for all species in the $N = \sum_a Na$ phase-spaces of dimension $6D$. As defined in Eq.(33), the Liouville is the probability to have particle i located at the chosen position z_i and similarly for each particle. This means that one has access to a huge information on this elementary-state of the system. In practise, the knowledge we have, and that is actually useful, is not that large. It is in particular bounded by what can be measured and the measurement precision. In the case of plasmas, one can consider that an effective micro-state is characterised by a given electromagnetic field known within an uncertainty range. Such a micro-state is then the mean of all the elementary-states of the form (33) that yield the same electromagnetic field. The probability for the micro-state is then $P_L\Big(\{z_i\},t\Big) =< \mathcal{P}_L\Big(\{z_i\},t\Big) >_{\mu state}$ that we enforce to be continuous in phase space [8]. This procedure thus allows one to avoid the BBGKY hierarchy to directly address the physics of the one particle distribution function. When addressing the collisions, the equation for the two particle distribution function is then built starting from the one particle distribution. This leads to developing an inverse BBGKY hierarchy, where only the two first equations are relevant to address the collisions in fusion plasmas. More mathematical settings of such a procedure exist, for example [9]. However such works remain within the BBGKY standard framework. One then defines the one particle distribution function of species a as the average Klimontovich density for species a where the weight in the averaging process is the Liouville probability of the micro-state. This Liouville average can also be seen as the projection of P_L on $\mathcal{F}_a(Z,t)$.

$$F_a(Z,t) = \prod_{i_a=1}^{N_a} \int dz_{i_a} \mathcal{F}_a(Z,t) \left(\prod_{b,\ i_b} \int dz_{i_b} \mathcal{P}\Big(\{z_i\},t\Big) \right) = \langle \mathcal{F}_a(Z,t) \rangle_{P_L} . \tag{34}$$

2.2. **Average and fluctuation separation with the Liouville average**

Given the Liouville average defined in Eq.(34), one can separate the Klimontovich density as the sum of its average $F_a(Z,t)$ and fluctuating parts $f_a(Z,t)$, hence:

$$\mathcal{F}_a(Z,t) = F_a(Z,t) + f_a(Z,t). \tag{35}$$

Dropping the subscript X, P in the Poisson bracket, the kinetic equation Eq.(31) can then be written as the evolution of the mean and fluctuating parts of the Klimontovich density:

$$\frac{\partial\Big(F_a(Z,t) + f_a(Z,t)\Big)}{\partial t} - \Big[H + h, F_a(Z,t) + f_a(Z,t)\Big] = 0. \tag{36}$$

In this equation we also separate the Hamiltonian into its mean and fluctuating part so that: $\mathcal{H} = H + h$. The equation can then be split into the mean equation obtained by averaging Eq.(36):

$$\frac{\partial F_a(Z,t)}{\partial t} - \Big[H, F_a(Z,t)\Big] - \Big\langle \Big[h, f_a(Z,t)\Big] \Big\rangle_{P_L} = 0, \tag{37}$$

and the fluctuation part which is the obtained by subtracting Eq.(37) from Eq.(36).

$$\frac{\partial f_a(Z,t)}{\partial t} - \Big[H, f_a(Z,t)\Big] - \Big[h, F_a(Z,t)\Big] - \left(\Big[h, f_a(Z,t)\Big] - \Big\langle \Big[h, f_a(Z,t)\Big] \Big\rangle_{P_L} \right) = 0. \tag{38}$$

These steps are standard in all quasilinear approaches. Regarding collisions, one can replaces H and h by a more explicit dependence on the Coulomb potential starting from Eq.(27)

$$\begin{aligned} \mathcal{H}_a(Z_{i_a},t) &= H_a^0(Z_{i_a},t) + \sum_b \sum_{j_b} \frac{\alpha\ q_a q_b}{|\boldsymbol{X}_{\boldsymbol{i_a}} - \boldsymbol{X}_b|} \delta\Big(\boldsymbol{X}_b - \boldsymbol{X}_{j_b}\Big) \\ &= H_a^0(Z_{i_a},t) + \sum_b \frac{\alpha\ q_a q_b}{|\boldsymbol{X}_{i_a} - \boldsymbol{X}_b|} \sum_{j_b} \delta\Big(Z_b - Z_{j_b}(t)\Big). \end{aligned} \tag{39}$$

This expression thus depends explicitly on the Klimontovich density $\mathcal{F}_b(Z_b,t)$

$$\mathcal{H}_a(Z_a,t) = H_a^0(Z_a,t) + \sum_b \int dZ_b \frac{\alpha\ q_a q_b}{|\boldsymbol{X}_a - \boldsymbol{X}_b|} \mathcal{F}_b(Z_b,t), \tag{40}$$

so that:

$$H_a(Z_a,t) = H_a^0(Z_a,t) + \sum_b \int dZ_b \frac{\alpha\ q_a\ q_b}{|\boldsymbol{X}_a - \boldsymbol{X}_b|} F_b(Z_b,t), \tag{41}$$

$$h_a(Z_a,t) = \sum_b \int dZ_b \frac{\alpha\ q_a\ q_b}{|\boldsymbol{X}_a - \boldsymbol{X}_b|} f_b(Z_b,t). \tag{42}$$

These expressions then lead to:

$$\frac{\partial F_a(Z_a,t)}{\partial t} - \Big[H_a(Z_a,t), F_a(Z_a,t)\Big] - \sum_b \frac{\boldsymbol{\partial}}{\boldsymbol{\partial P}_a} \cdot \left\langle \int dZ_b \frac{\boldsymbol{\partial}}{\boldsymbol{\partial X}_a} \left(\frac{\alpha\ q_a\ q_b}{|\boldsymbol{X}_a - \boldsymbol{X}_b|} \right) f_b(Z_b,t) f_a(Z_a,t) \right\rangle_{P_L} = 0. \tag{43}$$

The source term in the evolution of the distribution function is the third term of Eq.(43) which explicitly depends on the correlation function $C_{a,b}(Z_a, Z_b, t) =< f_a(Z_a,t) f_b(Z_b,t) >_{P_L}$.

$$\frac{\partial F_a(Z_a,t)}{\partial t} - \Big[H_a(Z_a,t), F_a(Z_a,t)\Big] - \frac{\boldsymbol{\partial}}{\boldsymbol{\partial P}_a} \cdot \sum_b \int dZ_b \frac{\boldsymbol{\partial}}{\boldsymbol{\partial X}_a} \left(\frac{\alpha\ q_a\ q_b}{|\boldsymbol{X}_a - \boldsymbol{X}_b|} \right) C_{a,b}(Z_a, Z_b, t) = 0. \tag{44}$$

The next steps in the calculation of the collision operator then aims at reducing the correlation function $C_{a,b}(Z_a, Z_b, t)$ to a function of the single particle distribution functions so that one obtains an equation of the form Eq.(44) that only depends of the distribution function, thus closing the kinetic equation of the one particle distribution function.

3. Collision operators: expanding the correlation function

3.1. Expansion of the correlation function in terms of the one-particle and two-particle distribution functions

The correlation function $C_{a,b}$ is related to the Klimontovich densities:

$$C_{a,b}(Z_a, Z_b, t) = \langle \mathcal{F}_a(Z_a,t) \mathcal{F}_b(Z_b,t) \rangle_P - F_a(Z_a,t) F_b(Z_b,t). \tag{45}$$

The correlation function of the two Klimontovich densities is split into two contributions of different orders. First, for $a = b$ and $Z_a = Z_b$ one obtains the self correlation term equal to the one particle distribution function. The contribution for different species, $a \neq b$, or for different particles of the same species, $a = b$ and $Z_a \neq Z_b$, is by definition the two-particle distribution function that can be expanded as $F_a(Z_a,t)F_b(Z_b,t) + g_{a,b}(Z_a, Z_b, t)$. One thus obtains:

$$C_{a,b}(Z_a, Z_b, t) = \delta(a,b)\delta(Z_a - Z_b) F_a(Z_a,t) + g_{a,b}(Z_a, Z_b, t). \tag{46}$$

In Eq.(44), the self-correlation term does not contribute since it leads to self-Coulomb interaction which is set to zero. The lowest order contribution of the correlation function then yields the Vlasov equation. One must therefore determine the higher order contribution, $g_{a,b}(Z_a, Z_b, t)$ to obtain a non-vanishing collision operator.

$$\frac{\partial F_a(Z_a,t)}{\partial t} - \Big[H_a(Z_a,t), F_a(Z_a,t)\Big] - \frac{\boldsymbol{\partial}}{\boldsymbol{\partial P}_a} \cdot \sum_b \int dZ_b \frac{\boldsymbol{\partial}}{\boldsymbol{\partial X}_a} \left(\frac{\alpha\ q_a\ q_b}{|\boldsymbol{X}_a - \boldsymbol{X}_b|} \right) g_{a,b}(Z_a, Z_b, t) = 0. \tag{47}$$

3.1.1. *Formal solution for the fluctuations of the Klimontovich density*

Given the definition of the correlation function $C_{a,b}(Z_a, Z_b, t)$, one can either solve Eq.(38) to determine the evolution of the fluctuating densities f_a and f_b or address the evolution of the correlation function. Formally, Eq.(38), can be written as:

$$\partial_t f_a(Z_a, t) + L_a(t) f_a(Z_a, t) = K_a(t) F_a(Z_a, t). \tag{48}$$

The evolution of f_a then depends on two contributions, a source term via the operator K_a that explicitly depends on the fluctuating part of the Hamiltonian acting on the one-particle distribution function, and a contribution that only depends on the fluctuating part of the Klimontovich density. The solution of Eq.(48) can be written as:

$$f_a(Z_a, t) - \Lambda(t, t_0) f_a(Z_a, t_0) = \int_{t_0}^{t} dt' \partial_{t'} \Big(\Lambda_a(t, t') f_a(Z_a, t') \Big) = \int_{t_0}^{t} dt' \Lambda_a(t, t') K_a(t') F_a(Z_a, t'), \tag{49}$$

where the propagator $\Lambda_a(t, t')$ is defined by:

$$\partial_t \Lambda_a(t, t') = -L_a(t) \Lambda_a(t, t'). \tag{50}$$

The fluctuating particle density f_a then depends on two terms, the propagation with Λ_a of the fluctuations at time t_0 and the generation and propagation of fluctuations due to the source term, therefore:

$$f_a(Z_a, t) = \Lambda_a(t, t_0) f_a(Z_a, t_0) + \int_{t_0}^{t} dt' \Lambda_a(t, t') K_a(t') F_a(Z_a, t'). \tag{51}$$

3.1.2. *Formal solution for the correlation function of Klimontovich density fluctuations*

For convenience we note this solution as $f_a = p_a + s_a$, the propagation on initial fluctuations p_a and the generation part due to the source s_a. With this notation the correlation function for different species, or different particles of the same species, $C_{a,b}(Z_a, Z_b, t) = g_{a,b}(Z_a, Z_b, t)$ takes the form:

$$g_{a,b}(Z_a, Z_b, t) = \Big\langle p_a p_b + (p_a s_b + p_b s_a) + s_a s_b \Big\rangle_P. \tag{52}$$

In order to maintain the explicit symmetry between species a and b in the calculation, we use the operator $\mathbb{P}_{a,b}$, which permutes species a and b, hence:

$$g_{a,b}(Z_a, Z_b, t) = \left\langle \frac{\mathbb{I} + \mathbb{P}_{a,b}}{2} \Big(p_a p_b + s_a s_b \Big) + \Big(\mathbb{I} + \mathbb{P}_{a,b} \Big) s_a p_b \right\rangle_P. \tag{53}$$

Using the two forms for the source term Eq.(49) one then obtains:

$$\frac{\mathbb{I} + \mathbb{P}_{a,b}}{2}\ s_a s_b = \frac{\mathbb{I} + \mathbb{P}_{a,b}}{2}\ \int_{t_0}^{t} dt' \Lambda_a(t, t') K_a(t') F_a(Z_a, t') \int_{t_0}^{t} dt'' \partial_{t''} \Big(\Lambda_b(t, t'') f_b(Z_b, t'') \Big). \tag{54}$$

Given the symmetry that is enforced by the operator $\mathbb{I} + \mathbb{P}_{a,b}$ between the two integrals, one can order the time integrals so that:

$$\frac{\mathbb{I} + \mathbb{P}_{a,b}}{2}\ s_a s_b = \Big(\mathbb{I} + \mathbb{P}_{a,b} \Big) \int_{t_0}^{t} dt' \Lambda_a(t, t') K_a(t') F_a(Z_a, t') \int_{t_0}^{t'} dt'' \partial_{t''} \Big(\Lambda_b(t, t'') f_b(Z_b, t'') \Big), \tag{55}$$

which then yields:

$$\begin{aligned}\frac{\mathbb{I}+\mathbb{P}_{a,b}}{2}\ s_a s_b &= \Big(\mathbb{I}+\mathbb{P}_{a,b}\Big)\int_{t_0}^{t} dt'\Lambda_a(t,t')K_a(t')F_a(Z_a,t')\Big(\Lambda_b(t,t')f_b(Z_b,t')-\Lambda_b(t,t_0)f_b(Z_b,t_0)\Big)\\ &= \Big(\mathbb{I}+\mathbb{P}_{a,b}\Big)\int_{t_0}^{t} dt'\Lambda_a(t,t')\Lambda_b(t,t')K_a(t')f_b(Z_b,t')F_a(Z_a,t')-\Big(\mathbb{I}+\mathbb{P}_{a,b}\Big)\ s_a p_b. \qquad (56)\end{aligned}$$

Similarly to the fluctuating particle density Eq.(51), one obtains the correlation function as the sum of two terms: one determined by the evolution of initial fluctuations and the other related to the evolution of fluctuations generated by the collisions that act as a source term.

$$\begin{aligned}g_{a,b}(Z_a,Z_b,t) &= \frac{\mathbb{I}+\mathbb{P}_{a,b}}{2}\Big\langle\Lambda_a(t,t_0)f_a(Z_a,t_0)\Lambda_b(t,t_0)f_b(Z_b,t_0)\Big\rangle_P\\ &+\Big(\mathbb{I}+\mathbb{P}_{a,b}\Big)\int_{t_0}^{t} dt'\Big\langle\Lambda_a(t,t')\Lambda_b(t,t')K_a(t')f_b(Z_b,t')\Big\rangle_P F_a(Z_a,t'). \qquad (57)\end{aligned}$$

It is to be noted that the evolution operator Λ also depends on the fluctuations and therefore does not commute with the averaging procedure.

3.1.3. *Self-similarity of the correlation function*

Let us now consider the two operators $K_a(t')$ and $\Lambda(t,t')$ defined by Eq.(48) and Eq.(38) together with Eq.(50). One then obtains:

$$K_a(t)F_a(Z_a,t) = \Big[h,F_a(Z_a,t)\Big] = \sum_c\int dZ_c\frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}_a}\frac{\alpha\ q_a\ q_c}{|\boldsymbol{X}_a-\boldsymbol{X}_c|}f_c(Z_c,t)\cdot\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}F_a(Z,t), \qquad (58)$$

$$L_a f_a(Z,t) = \Big[H,f_a(Z_a,t)\Big]+\Big[h,f_a(Z_a,t)\Big]-\Big\langle\Big[h,f_a(Z_a,t)\Big]\Big\rangle_P. \qquad (59)$$

Both operators depend on the fluctuations. The operator K_a directly describes the binary interaction between the particles of the Klimontovich density. This is the source of the fluctuations in the present description. Conversely the operator L_a depends on the fluctuations only via a second order impact on the particle trajectories and therefore on the evolution of the Klimontovich density. We concentrate here on the lowest order effect, namely the source of the fluctuations via the binary interactions. We thus consider the evolution operator L_a^0 and the associated propagator $U_a(t,t_0)$ such that:

$$\begin{aligned}L_a^{(0)}f_a(Z,t) &= -\Big[H,f_a(Z,t)\Big], \qquad (60)\\ \partial_t U_a(t,t') &= -L_a^{(0)}\ U_a(t,t'). \qquad (61)\end{aligned}$$

With this approximation one then obtains:

$$\begin{aligned}g_{a,b}(Z_a,Z_b,t) &= \Big(\frac{\mathbb{I}+\mathbb{P}_{a,b}}{2}\Big)U_a(t,t_0)U_b(t,t_0)\Big\langle f_a(Z_a,t_0)f_b(Z_b,t_0)\Big\rangle_P\\ &+\Big(\mathbb{I}+\mathbb{P}_{a,b}\Big)\int_{t_0}^{t} dt' U_a(t,t')U_b(t,t')\\ &\sum_c\int dZ_c\frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}}\frac{\alpha\ q_a\ q_c}{|\boldsymbol{X}_a-\boldsymbol{X}_c|}\cdot\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}F_a(Z_a,t')\Big\langle f_c(Z_c,t)f_b(Z_b,t')\Big\rangle_P, \qquad (62)\end{aligned}$$

and therefore:

$$g_{a,b}(Z_a, Z_b, t) \;=\; \left(\frac{\mathbb{I}+\mathbb{P}_{a,b}}{2}\right) U_a(t,t_0)U_b(t,t_0)\; C_{a,b}(Z_a, Z_b, t_0)$$
$$+\;\Big(\mathbb{I}+\mathbb{P}_{a,b}\Big)\int_{t_0}^{t} dt' U_a(t,t')U_b(t,t')\sum_c \int dZ_c\; q_a q_c R_a(Z_a, Z_c, t')F_a(Z_a, t')\; C_{c,b}(Z_c, Z_b, t'). \quad (63)$$

Where we have now introduced the operator $R_a(Z_a, Z_c, t)$ such that:

$$R_a(Z_a, Z_c, t)F_a(Z_a, t) = \frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}_a}\left(\frac{\alpha}{|\boldsymbol{X}_a - \boldsymbol{X}_c|}\right)\cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} F_a(Z_a, t). \quad (64)$$

We thus find that the correlation function depends on the correlation function at previous times in a self-similar fashion with two contributions, the free evolution of the initial correlations at $t = t_0$, first term on the right hand side of Eq.(63) and the creation via binary interaction and subsequent free evolution of correlations, second term on the right hand side of Eq.(63).

Following Bogoliubov, we consider a scale separation in time, assuming that the characteristic time of the correlations is much shorter than the characteristic evolution time of the one particle distribution function, typically $t - t_0$ in Eq.(57). We can then neglect the evolution of the initial fluctuations that will have died away long before the reference time t.

$$g_{a,b}(Z_a, Z_b, t) \;=\; \Big(\mathbb{I}+\mathbb{P}_{a,b}\Big)\int_{t_0}^{t} dt' U_a(t,t')U_b(t,t')\sum_c \int dZ_c\; q_a q_c R_a(Z_a, Z_c, t')F_a(Z_a, t')\; C_{c,b}(Z_c, Z_b, t'). \quad (65)$$

In this expression one can notice the development of a collective behaviour via the summation on the species c as well as the dependence on previous times t'.

3.2. Landau collision term

Given the self-similar form of the correlation function Eq.(65) and the dependence of the correlation function on the one and two particle distribution functions Eq.(46), one obtains:

$$g_{a,b}(Z_a, Z_b, t) \;=\; q_a\; q_b\Big(\mathbb{I}+\mathbb{P}_{a,b}\Big)\int_{t_0}^{t} dt' U_a(t,t')U_b(t,t')R_a(Z_a, Z_b, t')U_a(t',t)U_b(t',t)F_a(Z_a, t)F_b(Z_b, t)$$
$$+\Big(\mathbb{I}+\mathbb{P}_{a,b}\Big)\int_{t_0}^{t} dt' U_a(t,t')U_b(t,t')\sum_c \int dZ_c\; q_a q_c R_a(Z_a, Z_c, t')F_a(Z_a, t')g_{c,b}(Z_c, Z_b, t'). \quad (66)$$

In this expression we have introduced the inverse operators $U_a(t,t')^{-1} = U_a(t',t)$ and $U_b(t,t')^{-1} = U_b(t',t)$ in the lowest order contribution to the right and side. In order to address the structure of the kernel of this contribution let us use the expression of the operator R_a Eq.(64) in terms of its Fourier transform in space.

$$R_a(Z_a, Z_b, t')F_a(Z_a, t') = \int d\boldsymbol{K} e^{i\boldsymbol{K}\cdot(\boldsymbol{X}_a - \boldsymbol{X}_b)}\; q_a\; q_b\; V(K)\; i\; \boldsymbol{K}\cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} F_a(Z_a, t'). \quad (67)$$

The kernel is then given by:

$$U_a(t,t')U_b(t,t')R_a(Z_a, Z_b, t')U_a(t',t)U_b(t',t)$$
$$= \int d\boldsymbol{K}\; U_a(t,t')U_b(t,t')e^{i\boldsymbol{K}\cdot(\boldsymbol{X}_a - \boldsymbol{X}_b)}U_a(t',t)U_b(t',t)\; q_a\; q_b\; V(K)\; i\; \boldsymbol{K}\cdot\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}. \quad (68)$$

When considering the free streaming approximation for the propagator U_a, one has:

$$U_a(t,t') = \exp\Big(-(t-t')\boldsymbol{v}_a \cdot \boldsymbol{\nabla}_a \Big), \tag{69}$$

depending on the velocity $\boldsymbol{v} = \boldsymbol{P}/m$ and where $\boldsymbol{\nabla}_a$ is the compact form of the derivative with respect to the position is space $\boldsymbol{X}_a$:

$$\boldsymbol{\nabla}_a = \frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}_a}. \tag{70}$$

The calculation of the kernel of the operator is then straightforward and yields:

$$\begin{aligned} &U_a(t,t')U_b(t,t')R_a(Z_a,Z_b,t')U_a(t',t)U_b(t',t) \\ &\quad = \int d\boldsymbol{K}\ \exp\Big(i\boldsymbol{K}\cdot(\boldsymbol{X}_a - \boldsymbol{X}_b - (t-t')\boldsymbol{v}_{a,b})\Big)\ q_a\ q_b\ V(K)\ i\ \boldsymbol{K}\cdot\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}, \end{aligned} \tag{71}$$

where $\boldsymbol{v}_{ab} = \boldsymbol{v}_a - \boldsymbol{v}_b$ is the relative velocity.

The lowest non-vanishing collision operator, the Landau collision operator, is then determined by the correlation function:

$$\begin{aligned} g^{(1)}_{a,b}(Z_a,Z_b,t)\ =\ & \Big(\mathbb{I} + \mathbb{P}_{a,b}\Big)\int_{t_0}^{t} dt' \int d\boldsymbol{K}\ \exp\Big(i\boldsymbol{K}\cdot(\boldsymbol{X}_a - \boldsymbol{X}_b - (t-t')\boldsymbol{v}_{ab})\Big) \\ & q_a\ q_b\ V(K)\ F_b(Z_b,t)\ i\ \boldsymbol{K}\cdot\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} F_a(Z_a,t), \end{aligned} \tag{72}$$

which then yields:

$$\begin{aligned} \mathcal{C}_a\Big(F_a(Z_a,t)\Big)\ =\ & -\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\cdot\sum_b \int dZ_b \int d\boldsymbol{K}'\ \boldsymbol{K}' \exp\Big(i\boldsymbol{K}'\cdot(\boldsymbol{X}_a - \boldsymbol{X}_b)\Big) q_a\ q_b\ V(K') \\ & \Big(\mathbb{I} + \mathbb{P}_{a,b}\Big)\int_{t_0}^{t} dt' \int d\boldsymbol{K}\ \exp\Big(i\boldsymbol{K}\cdot(\boldsymbol{X}_a - \boldsymbol{X}_b - (t-t')\boldsymbol{v}_{ab})\Big) \\ & q_a\ q_b\ V(K)\ F_b(Z_b,t)\ \boldsymbol{K}\cdot\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} F_a(Z_a,t). \end{aligned} \tag{73}$$

To recover the standard form of the collision operator, we introduce another scale separation such that the scale of variation of the one particle distribution function is much larger than the relative distance between the particle during the binary interaction $\boldsymbol{X}_{ab} = \boldsymbol{X}_a - \boldsymbol{X}_b$. The integration over $d\boldsymbol{X}_b$ then leads to $\boldsymbol{K} = -\boldsymbol{K}'$ with the ordering operator $\mathbb{I}$ and to $\boldsymbol{K} = \boldsymbol{K}'$ with the ordering operator $\mathbb{P}_{a,b}$ so that:

$$\begin{aligned} \mathcal{C}_a\Big(F_a(Z_a,t)\Big)\ =\ & -\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\cdot\sum_b \int d\boldsymbol{P}_b \int d\boldsymbol{K} \int_0^{t-t_0} d\tau'\ e^{-i\tau \boldsymbol{K}\boldsymbol{v}_{ab}} \\ & (2\pi)^3\Big(\ q_a\ q_b\ V(K)\Big)^2\ \boldsymbol{K}\otimes\boldsymbol{K}\cdot\ \Big(\mathbb{I} - \mathbb{P}_{a,b}\Big)F_b(Z_b,t)\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}F_a(Z_a,t). \end{aligned} \tag{74}$$

Within the Bogoliubov point of view $t - t_0 \to +\infty$ so that the integration over τ yields a Dirac distribution and:

$$\mathcal{C}_a\Big(F_a\Big) = \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} \cdot \sum_b \int d\boldsymbol{P}_b \int_{Kmin}^{Kmax} dK \frac{1}{K}\ \pi^2\ (2\pi)^3\ \left(\frac{q_a\ q_b\ V(K)}{K^2}\right)^2 \frac{v_{ab}^2 \mathbb{I} - \boldsymbol{v}_{ab} \otimes \boldsymbol{v}_{ab}}{v_{ab}^3} \cdot \Big(\mathbb{I} - \mathbb{P}_{a,b}\Big) F_b(Z_b,t) \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} F_a(Z_a,t). \quad (75)$$

3.3. **Further expanding the correlation function**

3.3.1. *Evolution equation of the two-particle distribution function*

The self-similar structure of the correlation function Eq.(66) can drive an expansion procedure. We follow a slightly different path by first deriving an evolution equation for the correlation function. Indeed, given Eq.(48), one can determine the evolution of $C_{a,b}(Z,Z',t) =< f_a(Z,t) f_b(Z',t) >_{P_L}$ by combining $f_b(Z',t)\partial_t f_a(Z,t)$ and $f_a(Z,t)\partial_t f_b(Z',t)$ so that:

$$\partial_t\Big(f_a(Z_a,t) f_b(Z_b,t)\Big) + \Big(L_a(t) + L_b(t)\Big) f_a(Z_a,t) f_b(Z_b,t) = f_b(Z_b,t) K_a(t) F_a({}_aZ,t) + f_a(Z_a,t) K_b(t) F_b(Z_b,t). \quad (76)$$

This equation can also be written as:

$$\begin{aligned}
&\partial_t\Big(f_a(Z_a,t) f_b(Z_b,t)\Big) + \Big(L_a(t) + L_b(t)\Big)\Big(f_a(Z_a,t) f_b(Z_b,t)\Big) \\
&= f_b(Z_b,t) \sum_c \int dZ_c \frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}_a} \frac{\alpha\ q_a\ q_c}{|\boldsymbol{X_a} - \boldsymbol{X}_c|} \cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} F_a(Za,t) f_c(Z_c,t) \\
&\quad + f_a(Z_a,t) \sum_c \int dZ_c \frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}_b} \frac{\alpha\ q_b\ q_c}{|\boldsymbol{X_b} - \boldsymbol{X}_c|} f_c(Z_c,t) \cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_b} F_b(Z_b,t). \qquad (77)
\end{aligned}$$

For two distinct particles, $Z \neq Z_b$ or $a \neq b$, and with the approximation $L(Z,t) \approx \bar{L}(Z,t)$ and $\big\langle L \big\rangle_{P_L} = \bar{L}$, one can average Eq.(77) and obtain the equation for $g_{a,b}(Z_a,Z_b,t)$:

$$\begin{aligned}
&\partial_t g_{a,b}(Z_a,Z_b,t) + \Big(\bar{L}_a(t) + \bar{L}_b(t)\Big) g_{a,b}(Z_a,Z_b,t) \\
&= \sum_c \int dZ_c\ q_a\ q_c R_a(Z_a,Z_c,t) F_a(Z_a,t) \Big\langle f_c(Z_c,t) f_b(Z_b,t) \Big\rangle_{P_L} \\
&\quad + \sum_c \int dZ_c\ q_b\ q_c R_b(Z_b,Z_c,t) F_b(Z_b,t) \Big\langle f_c(Z_c,t) f_a(Z_a,t) \Big\rangle_{P_L}. \qquad (78)
\end{aligned}$$

Following Eq.(46), the two last terms yield two contributions: a lower order contribution due to the self-correlations and that due to the two particle function g.

$$\begin{aligned}
&\partial_t g_{a,b}(Z_a, Z_b, t) + \Big(\bar{L}_a(t) + \bar{L}_b(t)\Big) g_{a,b}(Z_a, Z_b, t) \\
&\quad = \sum_c \int dZ_c q_a\ q_c R_a(Z_a, Z_c) F_a(Z_a, t) \Big(\delta_{b,c}\delta_{Z_b,Z_c} F_b(Z_b, t) + g_{b,c}(Z_b, Z_c, t)\Big) \\
&\qquad + \sum_c \int dZ_c q_b\ q_c R_b(Z_b, Z_c, t) \Big(\delta_{a,c}\delta_{Z_a,Z_c} F_a(Z_a, t) + g_{a,c}(Z_a, Z_c, t)\Big) \\
&\quad = S_{a,b}(Z_a, Z_b, t) + \sum_c \int dZ_c q_a\ q_c R_a(Z_a, Z_c, t) F_a(Z_a, t) g_{b,c}(Z_b, Z_c, t) \\
&\qquad + \sum_c \int dZ_c q_b\ q_c R_b(Z_b, Z_c, t) F_b(Z_b, t)\ g_{a,c}(Z_a, Z_c, t).
\end{aligned} \tag{79}$$

The right hand side of the latter equation is split into the source term $S_{a,b} = (\mathbb{I} + \mathbb{P}_{a,b})S^a_{a,b}$ and two operators acting on the global sum for species c, the global sum being the sum over all species and over the whole phase space for each species.

$$\begin{aligned}
S_{a,b}(Z_a, Z_b, t) &= \big(\mathbb{I} + \mathbb{P}_{a,b}\big) \left(\frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}_a} \frac{\alpha\ q_a\ q_b}{|\boldsymbol{X}_a - \boldsymbol{X}_b|} \cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\right) \Big(F_a(Z_a, t) F_b(Z_b, t)\Big) \\
&= \frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}_a} \frac{\alpha\ q_a\ q_b}{|\boldsymbol{X}_a - \boldsymbol{X}_b|} \big(\mathbb{I} - \mathbb{P}_{a,b}\big) \cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} \Big(F_a(Z_a, t) F_b(Z_b, t)\Big) \\
&= (\mathbb{I} + \mathbb{P}_{a,b}) S^a_{a,b} = (\mathbb{I} + \mathbb{P}_{a,b})\ q_a\ q_b\ R_a(Z_a, Z_b, t) \Big(F_b(Z_b, t) F_a(Z_a, t)\Big).
\end{aligned} \tag{80}$$

The source like the other terms of Eq.(79) is thus symmetric by permutation of the indexes a and b. However, in the collision term, this symmetry is lost since the two particle distribution $g_{a,b}$ is integrated over species and phase space for the index b Eq.(47).

3.3.2. *Low order propagator for the evolution of the two-particle distribution function*

Although the structure given by equation Eq.(79) seems appropriate to generate a solution by iteration, we have found more convenient to select the interaction pattern leading to the collective screening by determining the evolution operator for the reduced system :

$$\begin{aligned}
\partial_t G^a_{a,b}(Z_a, Z_b, t) \quad + \quad \bar{L}_a(t) G^a_{a,b}(Z_a, Z_b, t) \ - \sum_c \int dZ_c\ q_a\ q_c R_a(Z_a, Z_c, t) F_a(Z_a, t) G^c_{c,b}(Z_c, Z_b, t) \\
= S^a_{a,b}(Z_a, Z_b, t).
\end{aligned} \tag{81}$$

In this calculation F_b is fixed so that $G^a_{a,b}$ has a linear dependence on F_a. It is then possible to find the solution of Eq.(81) by expanding $G^a_{a,b}(Z_a, Z_b, t)$ following the ordering of small binary collisions compared to the particle free motion. The term $\bar{L}_a G^a_{a,b}$ and the source term are considered to be of the same order while the interaction mediated by other particles, typically $R_{a,c} F_a(Z_a, t) G_{c,b}$, is a higher order in powers of the collision term. We therefore expand $G^a_{a,b}$

$$G^a_{a,b}(Z_a, Z_b, t) = \sum_{j=0}^{+\infty} G^{a,(j)}_{a,b}(Z_a, Z_b, t), \tag{82}$$

so that the different terms in this expansion are solution the following set of equations that yield a procedure to address the problem order after order.

$$\partial_t G^{a,(j)}_{a,b}(Z_a, Z_b, t) \quad + \quad \bar{L}_a(t) G^{a,(j)}_{a,b}(Z_a, Z_b, t) = \delta(j,0) S^a_{a,b}(Z_a, Z_b, t)$$
$$+\Big(1 - \delta(j,0)\Big) = \sum_c \int dZ_c \; q_a q_c R_a(Z_a, Z_c, t) F_a(Z_a, t) G^{c,(j-1)}_{c,b}(Z_c, Z_b, t). \qquad (83)$$

Given the propagator $U_c(t,t')$ Eq.(61) with the assumption $\bar{L} \approx L^{(0)}$, and neglecting the propagation of the initial conditions, one then obtains the solution by iteration:

$$\text{for j} = 0: \quad G^{a,(0)}_{a,b}(Z_a, Z_b, t) = \int_{t_0}^{t} dt' U_a(t,t') \; S^a_{a,b}(Z_a, Z'_b, t'), \qquad (84)$$
$$\text{for j} \neq 0: \quad G^{c_1,(j)}_{c_1,b}(Z_b, Z_{c_2}, t) = \; = \sum_c \int dZ_c \int_{t_0}^{t} dt_{c_1} \; q_c q_{c_1} \; U_{c_1}(t, t_{c_1}) R_{c_1}(Z_{c_1}, Z_c, t_{c_1}) U_{c_1}(t_{c_1}, t)$$
$$F_{c_1}(Z_{c_1}, t)) \; G^{c,(j-1)}_{c,b}(Z_b, Z_c, t). \qquad (85)$$

Despite the iteration structure, one cannot readily recognise a pattern since the global sum over species c involves species c both in iterations j and $j-1$. One must therefore reorder the terms in the chain rule to sort out the parts that depend on a single species. For that purpose we combine two successive steps of the iteration process.

Let us first express $G^{(j)}$ according to its j iterations Eq.(83):

$$G^{a,(j)}_{a,b}(Z_a, Z_b, t) = \int_{t_0}^{t} dt_a \; U_a(t, t_a) \sum_{c_1} \int dZ_{c_1} \; q_a \; q_{c_1} \; R_a(Z_a, Z_{c_1}, t_a) U_a(t_a, t)$$
$$\prod_{k=2}^{k=j} \int_{t_0}^{t} dt_{k-1} \; U_{c_{k-1}}(t, t_{k-1}) \sum_{c_k} \int dZ_{c_k} \; q_{c_{k-1}} \; q_{c_k} R_{c_{k-1}}(Z_{c_{k-1}}, Z_{c_k}, t_{k-1}) U_{c_{k-1}}(t_{k-1}, t) F_{c_{k-1}}(Z_{c_{k-1}}, t)$$
$$\int_{t_0}^{t} dt_j \; U_{c_j}(t, t_j) S^{c_j}_{c_j,b}(Z_{c_j}, Z_b, t_j). \qquad (86)$$

In order to identify a relevant pattern we select iterations $k-1$ and k, hence:

$$
\begin{aligned}
\mathcal{G}(k-1,k+1) &= \int_{t_0}^{t} dt_{k-1}\; U_{c_{k-1}}(t,t_{k-1}) \sum_{c_k} \int dZ_{c_k}\; q_{c_{k-1}}\; q_{c_k}\; R_{c_{k-1}}(Z_{c_{k-1}}, Z_{c_k}, t_{k-1}) \\
&\qquad \int_{t_0}^{t} dt_k\; U_{c_k}(t,t_k) \sum_{c_{k+1}} \int dZ_{c_{k+1}}\; q_{c_k}\; q_{c_{k+1}}\; R_{c_k}(Z_{c_k}, Z_{c_{k+1}}, t_k) \\
&= \frac{1}{q_{c_{k-1}}} \int_{t_0}^{t} dt_{k-1}\; U_{c_{k-1}}(t,t_{k-1}) \\
&\qquad \int d\boldsymbol{X}_{c_k}\; q_{c_{k-1}}^2\; \frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}_{c_{k-1}}} \left(\frac{\alpha}{|\boldsymbol{X}_{c_{k-1}} - \boldsymbol{X}_{c_k}|} \right) \cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_{c_{k-1}}} F_{c_{k-1}}(Z_{c_{k-1}}, t_{k-1}) \\
&\qquad\quad \sum_{c_k} \int d\boldsymbol{P}_{c_k} \int_{t_0}^{t} dt_k\; U_{c_k}(t,t_k) \\
&\qquad \int d\boldsymbol{X}_{c_{k+1}}\; q_{c_k}^2 \frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}_{c_k}} \left(\frac{\alpha}{|\boldsymbol{X}_{c_k} - \boldsymbol{X}_{c_{k+1}}|} \right) \cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_{c_k}} F_{c_k}(Z_{c_k}, t_k) \\
&\qquad\quad \sum_{c_{k+1}} \int d\boldsymbol{P}_{c_{k+1}}\; q_{c_{k+1}}.
\end{aligned} \tag{87}
$$

One finds therefore that the distance that governs the Coulomb interaction is the only term that depends on two successive iterations. To split this term, it is convenient to introduce the Fourier transform Eq.(67).

$$
\begin{aligned}
\mathcal{G}(k-1,k+1) &= \frac{1}{q_{c_{k-1}}} \int_{t_0}^{t} dt_{k-1}\; U_{c_{k-1}}(t,t_{k-1}) \\
&\int d\boldsymbol{K}_{k-1} \int d\boldsymbol{X}_{c_k}\; q_{c_{k-1}}^2\; V(K_{k-1})\; e^{i\boldsymbol{K}_{k-1}\cdot(\boldsymbol{X}_{c_{k-1}} - \boldsymbol{X}_{c_k})} \left(i\boldsymbol{K}_{k-1} \cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_{c_{k-1}}} \right) F_{c_{k-1}}(Z_{c_{k-1}}, t_{k-1}) \\
&\sum_{c_k} \int d\boldsymbol{P}_{c_k} \int_{t_0}^{t} dt_k\; U_{c_k}(t,t_k) \\
&\quad \int d\boldsymbol{K}_k \int d\boldsymbol{X}_{c_{k+1}}\; q_{c_k}^2\; V(K_k)\; e^{i\boldsymbol{K}_k\cdot(\boldsymbol{X}_{c_k} - \boldsymbol{X}_{c_{k+1}})} \left(i\boldsymbol{K}_k \cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_{c_k}} \right) F_{c_k}(Z_{c_k}, t_k) \\
&\qquad \sum_{c_{k+1}} \int d\boldsymbol{P}_{c_{k+1}}\; q_{c_{k+1}}.
\end{aligned} \tag{88}
$$

The next logical step is to put together the terms associated to a given location. However, one must take into account the commutation with the propagators. Using $U(t',t)$ as the inverse propagator of $U(t,t')$ and shifting the reference time of the one particle distribution functions to time t with the inverse propagator, one finds therefore:

$$
\begin{aligned}
\mathcal{G}(k-1,k+1) \quad = \quad & \frac{1}{q_{c_{k-1}}}\int_{t_0}^{t} dt_{k-1}\int d\boldsymbol{K}_{k-1}\ U_{c_{k-1}}(t,t_{k-1})e^{i\boldsymbol{K}_{k-1}\cdot\boldsymbol{X}_{c_{k-1}}}\ U_{c_{k-1}}(t_{k-1},t)\\
& \int d\boldsymbol{X}_{c_k}\ q^2_{c_{k-1}}\ V(K_{k-1})\ e^{-i\boldsymbol{K}_{k-1}\cdot\boldsymbol{X}_{c_k}}\left(i\boldsymbol{K}_{k-1}\cdot\frac{\boldsymbol{\partial}}{\partial\boldsymbol{P}_{c_{k-1}}}\right)F_{c_{k-1}}(Z_{c_{k-1}},t)\\
& \sum_{c_k}\int d\boldsymbol{P}_{c_k}\int_{t_0}^{t} dt_k\int d\boldsymbol{K}_k\ U_{c_k}(t,t_k)\ e^{i\boldsymbol{K}_k\cdot\boldsymbol{X}_{c_k}}\ U_{c_k}(t_k,t)\\
& \quad\int d\boldsymbol{X}_{c_{k+1}}\ q^2_{c_k}\ V(K_k)\ e^{-i\boldsymbol{K}_k\cdot\boldsymbol{X}_{c_{k+1}}}\ \left(i\boldsymbol{K}_k\cdot\frac{\boldsymbol{\partial}}{\partial\boldsymbol{P}_{c_k}}\right)F_{c_k}(Z_{c_k},t)\\
& \qquad\sum_{c_{k+1}}\int d\boldsymbol{P}_{c_{k+1}}\ q_{c_{k+1}}.
\end{aligned}
\tag{89}
$$

Following Eq.(69) and Eq.(70) one then readily obtains:

$$
\begin{aligned}
\mathcal{G}(k-1,k+1) \quad = \quad & \frac{1}{q_{c_{k-1}}}\int_{t_0}^{t} dt_{k-1}\int d\boldsymbol{K}_{k-1}e^{i\boldsymbol{K}_{k-1}\cdot\boldsymbol{X}_{c_{k-1}}}\quad e^{-i\boldsymbol{K}_{k-1}\cdot\boldsymbol{v}_{c_{k-1}}(t-t_{k-1})}\\
& \int d\boldsymbol{X}_{c_k}\ q^2_{c_{k-1}}\ V(K_{k-1})\ e^{-i\boldsymbol{K}_{k-1}\cdot\boldsymbol{X}_{c_k}}\left(i\boldsymbol{K}_{k-1}\cdot\frac{\boldsymbol{\partial}}{\partial\boldsymbol{P}_{c_{k-1}}}\right)F_{c_{k-1}}(Z_{c_{k-1}},t)\\
& \sum_{c_k}\int d\boldsymbol{P}_{c_k}\int_{t_0}^{t} dt_k\int d\boldsymbol{K}_k\ e^{i\boldsymbol{K}_k\cdot\boldsymbol{X}_{c_k}}\quad e^{-i\boldsymbol{K}_k\cdot\boldsymbol{v}_{c_k}(t-t_k)}\\
& \quad\int d\boldsymbol{X}_{c_{k+1}}\ q^2_{c_k}\ V(K_k)\ e^{-i\boldsymbol{K}_k\cdot\boldsymbol{X}_{c_{k+1}}}\ \left(i\boldsymbol{K}_k\cdot\frac{\boldsymbol{\partial}}{\partial\boldsymbol{P}_{c_k}}\right)F_{c_k}(Z_{c_k},t)\\
& \qquad\sum_{c_{k+1}}\int d\boldsymbol{P}_{c_{k+1}}\ q_{c_{k+1}}.
\end{aligned}
\tag{90}
$$

It is then possible to associate the terms with the same spacial location. The calculation is performed with a few changes in the ordering of the integrals and introducing an operator acting on $\mathcal{G}$ for symmetry purposes.

$$
\begin{aligned}
& \int d\boldsymbol{X}_{c_{k-1}}e^{-i\boldsymbol{K}\cdot\boldsymbol{X}_{c_{k-1}}}\mathcal{G}(k-1,k+1) = \frac{1}{q_{c_{k-1}}}\int_{t_0}^{t} dt_{k-1}\int d\boldsymbol{K}_{k-1}\int d\boldsymbol{X}_{c_{k-1}}e^{i(\boldsymbol{K}_{k-1}-\boldsymbol{K})\cdot\boldsymbol{X}_{c_{k-1}}}\\
& e^{-i\boldsymbol{K}_{k-1}\cdot\boldsymbol{v}_{c_{k-1}}(t-t_{k-1})}\\
& \int d\boldsymbol{K}_k\int d\boldsymbol{X}_{c_k}\ q^2_{c_{k-1}}\ V(K_{k-1})\ e^{i(\boldsymbol{K}_k-\boldsymbol{K}_{k-1})\cdot\boldsymbol{X}_{c_k}}\left(i\boldsymbol{K}_{k-1}\cdot\frac{\boldsymbol{\partial}}{\partial\boldsymbol{P}_{c_{k-1}}}\right)F_{c_{k-1}}(Z_{c_{k-1}},t)\\
& \sum_{c_k}\int d\boldsymbol{P}_{c_k}\int_{t_0}^{t} dt_k\ e^{-i\boldsymbol{K}_k\cdot\boldsymbol{v}_{c_k}(t-t_k)}\ q^2_{c_k}\ V(K_k)\quad\left(i\boldsymbol{K}_k\cdot\frac{\boldsymbol{\partial}}{\partial\boldsymbol{P}_{c_k}}\right)F_{c_k}(Z_{c_k},t)\\
& \qquad\sum_{c_{k+1}}\int d\boldsymbol{P}_{c_{k+1}}\ q_{c_{k+1}}\int d\boldsymbol{X}_{c_{k+1}}\ e^{-i\boldsymbol{K}_k\cdot\boldsymbol{X}_{c_{k+1}}}\quad .
\end{aligned}
\tag{91}
$$

One can then notice that the terms at a given location combine a phase shift depending on the difference between two subsequent wave vectors in the expansion as well as the one particle distribution function. We shall assume that one can neglect the variation in space of the distribution functions on the short interaction distance. Integrating in space then yields a delta function for the difference between the subsequent wave-vectors.

Consequently, this leads the result in terms of a single wave vector.

$$\left(\int d\boldsymbol{X}_{c_{k-1}} e^{-i\boldsymbol{K}\cdot\boldsymbol{X}_{c_{k-1}}}\right)\mathcal{G}(k-1,k+1) = \frac{1}{q_{c_{k-1}}}\int_{t_0}^{t} dt_{k-1}\ \ e^{-i\boldsymbol{K}\cdot\boldsymbol{v}_{c_{k-1}}(t-t_{k-1})}$$
$$q_{c_{k-1}}^2 (2\pi)^3\ V(K)\ \left(i\boldsymbol{K}\cdot\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_{c_{k-1}}}\right) F_{c_{k-1}}(\boldsymbol{P}_{c_{k-1}},t)$$
$$\sum_{c_k}\int d\boldsymbol{P}_{c_k}\int_{t_0}^{t} dt_k\ e^{-i\boldsymbol{K}\cdot\boldsymbol{v}_{c_k}(t-t_k)}\ q_{c_k}^2(2\pi)^3\ V(K)\ \ \left(i\boldsymbol{K}\cdot\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_{c_k}}\right) F_{c_k}(\boldsymbol{P}_{c_k},t)$$
$$\sum_{c_{k+1}}\int d\boldsymbol{P}_{c_{k+1}}\ q_{c_{k+1}}\int d\boldsymbol{X}_{c_{k+1}}\ e^{-i\boldsymbol{K}\cdot\boldsymbol{X}_{c_{k+1}}}. \tag{92}$$

This expression allows one to identify the following pattern:

$$\begin{aligned}\mathcal{G}(k-1,k+1) &= \mathcal{P}(k-1,k)\ \mathcal{P}(k,k+1),\\ \mathcal{P}(k,k+1) &= \left(\sum_{c_k}\int dZ_{c_k}\ q_k\ e^{-i\boldsymbol{K}\cdot\boldsymbol{X}_{c_k}}\right)^{-1} E(\boldsymbol{K})\ \left(\sum_{c_{k+1}}\int dZ_{c_{k+1}}\ q_{c_{k+1}} e^{-i\boldsymbol{K}\cdot\boldsymbol{X}_{c_{k+1}}}\right).\end{aligned} \tag{93}$$

The pattern $\mathcal{P}$ thus depends on the rank k via the two operators that dress the propagator $E(\boldsymbol{K})$ that is species independent since it contains a sum over all species,

$$E(\boldsymbol{K}) = \sum_c \int d\boldsymbol{P}_c \left(i\int_{t_0}^{t} dt'\ e^{-i\boldsymbol{K}\cdot\boldsymbol{v}_c(t-t')}\right)\ q_c^2(2\pi)^3\ V(K)\ \ \boldsymbol{K}\cdot\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_c} F_c(\boldsymbol{P}_c,t). \tag{94}$$

With this expression one can readily determine the function $G_{a,b}^{(j)}$ Eq.(86):

$$\begin{aligned}G_{a,b}^{a,(j)}(Z_a,Z_b,t) &= \int_{t_0}^{t} dt_a\ U_a(t,t_a)\sum_{c_1}\int dZ_{c_1}\ q_a\ q_{c_1}\ R_a(Z_a,Z_{c_1},t_a)\\ &\left(\sum_{c_1}\int dZ_{c_1}\ q_1\ e^{-i\boldsymbol{K}\cdot\boldsymbol{X}_{c_1}}\right)^{-1} E(\boldsymbol{K})^{j-1}\ \left(\sum_{c_j}\int dZ_{c_j}\ q_{c_j} e^{-i\boldsymbol{K}\cdot\boldsymbol{X}_{c_j}}\right)\\ &\int_{t_0}^{t} dt_j\ U_{c_j}(t,t_j) S_{c_j,b}^{c_j}(Z_{c_j},Z_b,t_j).\end{aligned} \tag{95}$$

Given the definition of R_a, Eq.(64) and Eq.(80) one can then write:

$$\begin{aligned}G_{a,b}^{a,(j)}(Z_a,Z_b,t) &= \int_{t_0}^{t} dt_a U_a(t,t_a)\int d\boldsymbol{K}\ E(\boldsymbol{K})^j\ q_a\ q_b\ e^{i\boldsymbol{K}\cdot(\boldsymbol{X}_a-\boldsymbol{X}_b)}\ V(K)\\ &i\boldsymbol{K}\cdot\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} F_a(\boldsymbol{P}_a,t_a) F_b(\boldsymbol{P}_b).\end{aligned} \tag{96}$$

In this expression one recognises the Fourier transform of the source term $S_{a,b}^a$ combined to $E(K)^j$ with F_b constant as assumed in the definition of $G_{a,b}^a$. It is then possible to sum the solution Eq.(96) for the values of

j according to Eq.(82) so that one has:

$$G^a_{a,b}(Z_a, Z_b, t) = \int_{t_0}^{t} dt_a U_a(t, t_a) \int d\boldsymbol{K}\ (1 - E(\boldsymbol{K}))^{-1}\ q_a\ q_b\ e^{i\boldsymbol{K}\cdot(\boldsymbol{X}_a - \boldsymbol{X}_b)}\ V(K)$$
$$i\boldsymbol{K} \cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} F_a(\boldsymbol{P}_a, t_a) F_b(\boldsymbol{P}_b). \tag{97}$$

When the source term is written in Fourier space, one finds an explicit expression of the propagator $M_a(t, t_a)$ that yields the evolution of $G^a_{a,b}$, namely:

$$\hat{M}_a(t, t') = \frac{\hat{U}_a(t, t')}{\mathcal{E}\big(\boldsymbol{K}\big)}, \tag{98}$$

where $\hat{U}_a(t, t') = exp(-\boldsymbol{K} \cdot \boldsymbol{v}_a(t - t'))$ is the free streaming propagator in Fourier space, and where $\mathcal{E}$ is the plasma permittivity:

$$\mathcal{E}\big(\boldsymbol{K}\big) = \Big(1 - E\big(\boldsymbol{K}\big)\Big)^{-1}. \tag{99}$$

Let us consider the evolution equation for g such that:

$$\frac{\partial g}{\partial t} = \Big(\mathcal{L}_a + \mathcal{L}_b\Big)\ g + S. \tag{100}$$

Let $\mathcal{M}_a$ be the propagator associated to $\mathcal{L}_a$, hence:

$$\frac{\partial}{\partial t}\mathcal{M}_a(t, t') = \mathcal{L}_a\ \mathcal{M}_a(t, t'). \tag{101}$$

We then define the propagator $\mathcal{M}$ for $\mathcal{L} = \mathcal{L}_a + \mathcal{L}_b$. We look for a solution of the form $\mathcal{M}(t, t') = \mathcal{M}_a(t, t')\mathcal{M}_b(t, t')$ and thus find that the propagator $\mathcal{M}_b(t, t')$ is defined as the solution of:

$$\frac{\partial}{\partial t}\mathcal{M}_b(t, t')) = \Big(\mathcal{M}_a(t, t')\mathcal{L}_b\ \mathcal{M}_a(t, t')^{-1}\Big)\mathcal{M}_b(t, t'). \tag{102}$$

At lowest order, for the free streaming operator with the permittivity correction, the operators $\mathcal{M}_a(t, t')$ and $\mathcal{L}_b$ commute so that $\mathcal{M}_b(t, t') = \mathbb{P}_{a,b}\mathcal{M}_a(t, t')$ if $\mathcal{L}_b = \mathbb{P}_{a,b}\mathcal{L}_a$. Given the propagator Eq.(98), one can then determine the two particle distribution function $g_{a,b}(\boldsymbol{P}_a, \boldsymbol{P}_b, t)$ at low but non-trivial order:

$$g_{a,b}(Z_a, Z_b, t) = q_b\ q_a \int d\boldsymbol{K} \int_{t_0}^{t} dt' \left(\frac{\hat{U}_a(t, t')}{\mathcal{E}\big(\boldsymbol{K}\big)}\right)\left(\frac{\hat{U}_b(t, t')}{\mathcal{E}\big(-\boldsymbol{K}\big)}\right) e^{i\boldsymbol{K}\cdot(\boldsymbol{X}_a - \boldsymbol{X}_b)} V(K)$$
$$\Big(\mathbb{I} - \mathbb{P}_{a,b}\Big)\Big(i\boldsymbol{K}\Big) \cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} F_a(\boldsymbol{P}_a, t') F_b(\boldsymbol{P}_b, t'). \tag{103}$$

3.3.3. *Lenard-Balescu collision operator*

This solution for $g_{a,b}$ is quite similar to Eq.(72) giving the two particle distribution function as lowest non-trivial order. The only difference is introduced by the permittivity $|\mathcal{E}(\boldsymbol{K})|^2 = \mathcal{E}(\boldsymbol{K})\mathcal{E}(-\boldsymbol{K})$. As a consequence, the collision operator including the permittivity, and therefore the cut-off at the Debye scale, the so-called

Lenard-Balescu collision operator, will have exactly the same structure as the Landau operator Eq.(75).

$$\mathcal{C}_a\Big(F_a\Big) = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \boldsymbol{P}_a} \cdot \sum_b \int d\boldsymbol{P}_b \int_0^{Kmax} dK \frac{1}{K}\, 2\pi^2\, (2\pi)^3 \left(\frac{q_a\, q_b\, V(K)}{K^2 |\mathcal{E}(\boldsymbol{K})|^2}\right)^2 \frac{v_{ab}^2 \mathbb{I} - \boldsymbol{v}_{ab} \otimes \boldsymbol{v}_{ab}}{v_{ab}^3} \cdot \Big(\mathbb{I} - \mathbb{P}_{a,b}\Big) F_b(Z_b,t) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \boldsymbol{P}_a} F_a(Z_a,t). \tag{104}$$

4. Beyond the Landau collision operator

4.1. Gradients at the scale of the Debye length

In order to address the effect of small scale gradients let us rewrite the Landau collision term as:

$$\mathcal{C}_a\Big(Z_a,t\Big) = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \boldsymbol{P}_a} \cdot \sum_b \int dZ_b \int_0^{+\infty} d\tau \quad \boldsymbol{\nabla}_a V\left(|\boldsymbol{X}_{ab}|\right) \otimes \left(U_a(t,t-\tau) U_b(t,t-\tau) \boldsymbol{\nabla}_a V\left(|\boldsymbol{X}_{ab}|\right) \cdot (\mathbb{I} - \mathbb{P}_{a,b}) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \boldsymbol{P}_a} U_a(t-\tau,t) U_b(t-\tau,t)\right) F_a(Z_a,t) F_b(Z_b,t). \tag{105}$$

The time τ is the duration of the correlation, which is assumed short compared to the relevant mesoscopic scale of interest. The upper bound in the τ integral is then set at $+\infty$ wo that the integration will lead to distribution functions without enlargement effects due to the slow time scale changes. Gradients appear in the formulation due to two different effects. On the one hand the distribution of species b is not localised at the point where the collision operator. This characterises the non-local feature of the Coulomb interaction. Taylor expanding the distribution function F_b it is then possible to localise the interaction at the cost of the gradient expansion, characterised by the Debye scale, that modifies the collision operator compared to the homogeneous case. Similarly, the derivative with respect to $\boldsymbol{P}$ does not commute with the propagator U thus leading to gradients acting on both distribution functions. On the other hand, for magnetised plasmas, the change of coordinates to guiding centre coordinates introduces a velocity dependence in the particle position and therefore, via the $\boldsymbol{P}$ derivative, a gradient relative to the guiding centre position. The characteristic scale associated to the latter effect being the Larmor radius. On the basis of these remarks one obtains the following expression of the collision operator:

$$\mathcal{C}_a\Big(Z_a,t\Big) = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \boldsymbol{P}_a} \cdot \sum_b \int d\boldsymbol{P}_b \int d\boldsymbol{X}\ \delta(\boldsymbol{X} - \boldsymbol{X}_a) \int d\boldsymbol{X}_{ab} \int_0^{+\infty} d\tau \quad \boldsymbol{\nabla}_{ab} V\left(|\boldsymbol{X}_{ab}|\right) \otimes \left(U_a(t,t-\tau) U_b(t,t-\tau) \boldsymbol{\nabla}_{ab} V\left(|\boldsymbol{X}_{ab}|\right) U_a(t-\tau,t) U_b(t-\tau,t)\right) \cdot (\mathbb{I} - \mathbb{P}_{a,b}) \left(U_a(t,t-\tau) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \boldsymbol{P}_a} U_a(t-\tau,t)\right) e^{\boldsymbol{X}_{ab} \cdot \boldsymbol{\nabla}} \quad F_a(Z_a,t) F_b(\boldsymbol{X}, \boldsymbol{P}_b, t). \tag{106}$$

The propagator for magnetised trajectories at lowest order is that in cylindrical geometry with homogeneous magnetic field and thus governs a rotation of the velocity transverse to the magnetic field and the time-integration of this rotating velocity to obtain the displacement. Following the same procedure as in Section 3.3.2, Eqs.(100, 101 , 102), one then finds the propagator for particles of species a:

$$U_a(t,t-\tau) = e^{\Omega_a \tau \partial_{\varphi_a}} \quad e^{-\boldsymbol{v} \mathbb{G}_a(\tau) \cdot \boldsymbol{\nabla}} = e^{\Omega_a \tau \partial_{\varphi_a}} \quad e^{-\left(\mathbb{G}_a^t(\tau) \boldsymbol{v}\right) \cdot \boldsymbol{\nabla}}, \tag{107}$$

where φ_a is the cyclotron angle of the velocity and Ω_a the cyclotron frequency. The velocity variation is then characterised by a rotation with rotation operator $\mathbb{R}(\Omega_a \tau)$ where $\Omega_a \tau$ is the rotation angle and the axis is defined by the magnetic field $\boldsymbol{B}$. The operator $\mathbb{G}_a(\tau)$, transposed operator $\mathbb{G}_a^t(\tau)$, is the integration of the rotation operator:

$$\mathbb{G}_a(\tau) = \int_0^\tau dt\ \mathbb{R}(\Omega_a t). \tag{108}$$

One readily recovers the non-magnetised propagator with the limit $\Omega_a\tau \to 0$ so that the propagator acting on the cyclotron angle tends to the identity as well as the rotation operator $\mathbb{R}(\Omega_a\tau) \to \mathbb{I}$ so that $\mathbb{G}_a(\tau) \to \tau\ \mathbb{I}$. Given Eq.(108) one then finds:

$$U_a(t, t-\tau)\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}U_a(t-\tau, t) = \mathbb{R}(\Omega_a\tau)\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} - \frac{1}{m_a}\mathbb{G}_a(\tau)\boldsymbol{\nabla}_a, \tag{109}$$

as well as the displacement due to the motion of the particle labelled by a:

$$U_a(t, t-\tau)\ \boldsymbol{X}\ U_a(t-\tau, t) = \boldsymbol{X} - \boldsymbol{d}_a(\tau) \qquad ; \qquad \boldsymbol{d}_a(\tau) = \mathbb{G}_a^t(\tau)\boldsymbol{v}. \tag{110}$$

One then readily obtains:

$$U_a(t, t-\tau)U_b(t, t-\tau)\ V\left(|\boldsymbol{X}_a - \boldsymbol{X}_b|\right) = V\left(|\boldsymbol{X}_{a,b} - \boldsymbol{d}_{a,b}(\tau)|\right) U_a(t, t-\tau)U_b(t, t-\tau). \tag{111}$$

Given the two commutation relations Eq.(111) and Eq.(109), the expression of the collision operator is the following:

$$\begin{aligned} \mathcal{C}_a\Big(Z_a, t\Big) \quad = \quad & \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\cdot\sum_b \int d\boldsymbol{P}_b \int d\boldsymbol{X}\ \delta(\boldsymbol{X} - \boldsymbol{X}_a)\int d\boldsymbol{X}_{ab} \\ & \int_0^{+\infty} d\tau \quad \boldsymbol{\nabla}_{ab}V\left(|\boldsymbol{X}_{ab}|\right) \otimes \boldsymbol{\nabla}_{ab}V\left(|\boldsymbol{X}_{ab} - \boldsymbol{d}_{ab}|\right) \\ & \cdot\left(\mathbb{I} - \mathbb{P}_{a,b}\right)\left(\mathbb{R}(\Omega_a\tau)\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} - \frac{1}{m_a}\mathbb{G}_a(\tau)\boldsymbol{\nabla}_a\right) e^{\boldsymbol{X}_{ab}\cdot\boldsymbol{\nabla}} \quad F_a(Z_a, t)F_b(\boldsymbol{X}, \boldsymbol{P}_b, t). \end{aligned} \tag{112}$$

In a further step, one must commute the displacement operator $\exp(\boldsymbol{X}_{ab}\cdot\boldsymbol{\nabla})$ with the gradient operators $\boldsymbol{\nabla}_a$ and $\boldsymbol{\nabla}_b$, hence:

$$\begin{aligned} \mathcal{C}_a\Big(Z_a, t\Big) \quad = \quad & \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\cdot\sum_b \int d\boldsymbol{P}_b \int d\boldsymbol{X}_b\ \delta(\boldsymbol{X}_b - \boldsymbol{X}_a)\int_0^{+\infty} d\tau \\ & \int d\boldsymbol{X}_{ab}\ \boldsymbol{\nabla}_{ab}V\left(|\boldsymbol{X}_{ab}|\right) \otimes \boldsymbol{\nabla}_{ab}V\left(|\boldsymbol{X}_{ab} - \boldsymbol{d}_{ab}|\right) e^{\boldsymbol{X}_{ab}\cdot\boldsymbol{\nabla}_b} \\ & \cdot\left(\left(\mathbb{I} - \mathbb{P}_{a,b}\right)\mathbb{R}(\Omega_a\tau)\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} - \frac{1}{m_a}\mathbb{G}_a(\tau)\boldsymbol{\nabla}_a + \frac{1}{\mu_{ab}}\mathbb{G}_{ab}(\tau)\boldsymbol{\nabla}_b\right) F_a(Z_a, t)F_b(\boldsymbol{X}_b, \boldsymbol{P}_b, t), \end{aligned} \tag{113}$$

where $1/\mu_{ab} = 1/m_a + 1/m_b$ is the reduced mass and:

$$\frac{1}{\mu_{ab}}\mathbb{G}_{ab}(\tau) = \frac{1}{m_b}\mathbb{G}_b(\tau) + \frac{1}{m_a}\mathbb{G}_a(\tau). \tag{114}$$

It is to be noted that a loss in symmetry is thus introduced between species a and b because the collision operator is located at $\boldsymbol{X}_a$, and consequently, the relative motion of particle b with respect to a includes the magnetised trajectory of particle a. The operator $\mathbb{G}_{ab}$ is thus introduced to take into account the complex time dependence of the relative motion of two particles following helical trajectories with different axis, cyclotron frequency, and consequently radius, as well as different parallel velocity. Eq.(113) corrects the results in [2] where this complex dependence is missing. Let us now define the kernel of the collision operator $\mathbb{K}_{ab}$ such that:

$$\begin{aligned}\mathcal{C}_a\Big(Z_a,t\Big) &= \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\cdot\sum_b\int d\boldsymbol{P}_b\int d\boldsymbol{X}_b\;\delta(\boldsymbol{X}_b-\boldsymbol{X}_a)\int_0^{+\infty}d\tau\;\mathbb{K}_{ab}\\ &\cdot\left((\mathbb{I}-\mathbb{P}_{a,b})\,\mathbb{R}(\Omega_a\tau)\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}-\frac{1}{m_a}\mathbb{G}_a(\tau)\boldsymbol{\nabla}_a+\frac{1}{\mu_{ab}}\mathbb{G}_{ab}(\tau)\boldsymbol{\nabla}_b\right)F_a(Z_a,t)F_b(\boldsymbol{X}_b,\boldsymbol{P}_b,t). \end{aligned} \tag{115}$$

The operator $\mathbb{K}_{ab}$ can readily be computed in Fourier space leading to:

$$\mathbb{K}_{ab}=(2\pi)^3\int d\boldsymbol{K}\;V\Big(|\boldsymbol{K}|\Big)\;\;V\Big(|\boldsymbol{K}+i\boldsymbol{\nabla}_b|\Big)\Big(\boldsymbol{K}+i\boldsymbol{\nabla}_b\Big)\otimes\boldsymbol{K}e^{i\boldsymbol{K}\cdot\boldsymbol{d}_{ab}}. \tag{116}$$

In this expression, the relative displacement $\boldsymbol{d}_{ab}$ does not depend on space so that commutation with operator $\boldsymbol{\nabla}_b$ is straightforward. Regarding the latter, one can obtain a more convenient expression with a Taylor expansion of $V\big(|\boldsymbol{K}+i\boldsymbol{\nabla}_b|\big)$, hence:

$$\begin{aligned}\mathbb{K}_{ab} &= (2\pi)^3\int d\boldsymbol{K}\;V(K)\left(\sum_{j=0}^{+\infty}\frac{1}{j!}\left(i\frac{\boldsymbol{K}\cdot\boldsymbol{\nabla}_b}{k}\frac{d}{dk}\right)^j V(K)\right)\boldsymbol{K}\otimes\boldsymbol{K}e^{i\boldsymbol{K}\cdot\boldsymbol{d}_{ab}}\\ &+ i\;(2\pi)^3\int d\boldsymbol{K}\;V(K)\left(\sum_{j=0}^{+\infty}\frac{1}{j!}\left(i\frac{\boldsymbol{K}\cdot\boldsymbol{\nabla}_b}{k}\frac{d}{dk}\right)^j V(K)\right)\boldsymbol{\nabla}_b\otimes\boldsymbol{K}e^{i\boldsymbol{K}\cdot\boldsymbol{d}_{ab}}. \end{aligned} \tag{117}$$

The non-local feature of collisions within the Debye sphere is thus equivalent to an infinite expansion in terms of powers of $\boldsymbol{\nabla}_b$. The two contributions have opposite parity with respect to $\boldsymbol{K}$ as also readily noticeable with the difference in powers of i. As a consequence, the associated distribution functions stemming from the integral on τ will be either Dirac distributions as in the case of the homogeneous limit, Eq.(75), or principal parts.

The calculation of the collisional contribution for the terms depending linearly on the gradients, relative to the Landau collision term, yields terms scaling like $\lambda_D/\Big(L\;Log(\Lambda)\Big)$ where L is the characteristic gradient length. One thus finds that for $L\gg\lambda_D$ these terms are small contributions compared to the standard Landau collision operator. Conversely for $L\ll\lambda_D$ these collisional transport terms will be larger and will smooth out the gradients on these scales, which justifies a posteriori that homogeneous one particle distribution functions be considered when deriving the Lenard-Balescu collision operator.

4.2. **Magnetised collisions**

Magnetised plasmas correspond in practise to two different limits. The first limit depends the characteristic time of interest. Most mechanisms relevant to fusion plasmas exhibit a characteristic time $1/\omega\gg 1/\Omega$ so that the free trajectories, neglecting the collisions, can be averaged over the fast cyclotron motion. This leads to the so-called guiding centre coordinates where the velocity is reduced to $2D$: the parallel velocity $v_\parallel$ and the magnetic moment μ, namely the motion adiabatic invariant associated to the fast cyclotron phase φ_c that is averaged out. This limit is addressed in the gyrokinetic framework [5]. The second limit is reached when $\lambda_D\gg\rho_a$, a situation that is marginally met for the electrons.

In the gyrokinetic framework, such that $\lambda_D\ll\rho_a$, the free trajectories are toroidal, and close to cylindrical at small scales. However, the collisions are still characterised by a spherical symmetry that corresponds to the usual Landau collision operator. In particular the displacement does not depend on the magnetic field $\boldsymbol{d}_{ab}(\tau)=\tau\boldsymbol{v}_{ab}$.

Consequently, the collision operator defined by Eq.(115) and Eq.(117) is to be evaluated in the limit $\Omega\tau \to 0$. The difference, however, is that the location of the particle considered in the distribution functions is the guiding centre position, typically located at the barycentre of the cyclotron motion, therefore displaced by the Larmor radius from the particle position, hence $\boldsymbol{X}_G = \boldsymbol{X} - \boldsymbol{\rho}$, see Eq.(3). In order to address the effect on collisions, hence on short scales and fast times, we approximate the particle motion to its cylindrical limit, hence with homogeneous magnetic field. In that case the averaged distribution function is such that $F_G(\boldsymbol{X}_G, V_G)$ depends on $\boldsymbol{X_G}$, defined above, and $V_{CG} = (v_\parallel, v_\perp)$. As a consequence, one finds that:

$$\begin{aligned} \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}} F_G\Big(\boldsymbol{X}_G, v_\parallel, v_\perp\Big) &= \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}} (\boldsymbol{X} - \frac{m}{q\ B} \boldsymbol{b} \times \boldsymbol{v}) \cdot \boldsymbol{\nabla} F_G + \frac{1}{m} \frac{\partial F_G}{\partial v_\perp} \hat{\boldsymbol{v}}_\perp + \frac{1}{m} \frac{\partial F_G}{\partial v_\parallel} \boldsymbol{b} \\ &= \frac{1}{m} \left(\frac{1}{\Omega_L} \boldsymbol{b} \times \boldsymbol{\nabla} F_G + \frac{\partial F_G}{\partial v_\perp} \hat{\boldsymbol{v}}_\perp + \frac{\partial F_G}{\partial v_\parallel} \boldsymbol{b} \right). \end{aligned} \tag{118}$$

One thus finds that changing coordinates to that more appropriate to the actual properties of the particle trajectories reintroduces a gradient operator that acts at the Larmor radius scale. One also changes the velocity derivatives that only depends on two components: one in the $\hat{\boldsymbol{v}}_\perp$ direction that exhibits a high gyration frequency and the other in the direction parallel to the magnetic field. The first term in Eq.(118) is thus governed by a drift motion transverse to the magnetic field and distribution function gradient. This is a non local effect induced by the fact that the collisional pitch angle scattering leads in practise to a displacement of the guiding centre. The second term governs the change in magnetic moment generated by the collisions. Such a high frequency term will not be averaged out provided it is combined to the similar contribution stemming from the other velocity derivative of the collision operator, thus yielding a quadratic magnetic moment exchange term. Finally the third term, ruling the parallel velocity component is not modified by the magnetic field and is thus comparable to the case of a non magnetised plasma. This analysis of the effect of the velocity derivative of the gyrokinetic distribution function only provides a qualitative description of the modifications of the collision operator. The complete collision operator with this dual symmetry is far more difficult to handle and ongoing mathematical developments address possible linearised forms of the collision operator [10] suitable for gyrokinetic codes such as GYSELA [6].

In the second limit, one addresses magnetised collisions, such that the displacement $\boldsymbol{d}_{ab}(\tau)$ depends on the gyrating motion. Within the collision term, this is only valid when the impact parameter is such that $\rho \leq b \leq \lambda_D$. The collisions with a smaller impact parameter remaining non-magnetised. One then finds that the displacement is bounded in the transverse direction and that the Larmor rotation tends to average out the pitch angle scattering effect. As a consequence, it is shown that the correction is mainly that of the Coulomb logarithm since a good approximation in many cases is obtained by replacing λ_D by the Larmor radius ρ [2]. Physically this corresponds to an ineffective deflection in the directions transverse to the magnetic field. For standard conditions of plasmas confined by a magnetic field, such a correction remains negligible, typically a reduced collision efficiency of the order of $Log(\lambda_D/\rho_L)/Log(\Lambda) < 0.05$, see fig.(3, 4). However, it can also lead to an increased exchange due to the parallel relative velocity contribution. Indeed, unlike the non-magnetised case, the weight of this contribution is not vanishing since it is restricted to $1D$.

The case of a large amplitude magnetic field thus appears to have a two fold effect. Regarding magnetically confined plasmas, the main issue is to develop a collision operator that is consistent with the gyrokinetic framework while remaining local, and such that linear approximations can be found that satisfy all the conservation constraints [10]. In this first case, the collision operator does not depend on the magnetic field and its symmetry remains spherical. At larger magnetic fields, when the Larmor radius of the particle trajectories becomes significantly smaller than the Debye length, the collision contribution encompasses two classes of particles, that with small impact parameter that remain non-magnetised regarding the collisions, and that at larger impact parameter such that the helical structure of the trajectory must be taken into account during the collision. One finds, Ref([2]), that this second class of collisions is described by very complicated expressions, in particular due to the expansion in terms of Bessel functions, with possible resonances of the particle cyclotron motions. However, the bottom line of these complex formulas is that collisions in this interaction regime are rather ineffective so that their main impact is to reduce the upper bound of the Coulomb logarithm from the Debye length to the Larmor radius.

5. Conservation properties of the collision operator

5.1. Conservation laws for the full distribution function

Conservation properties are embedded in that of the initial equation (29) which is written for an isolated system without creation or destruction of particles, hence the Liouville equation $d\mathcal{F}/dt = 0$. Given this structure one readily finds that $\int dZ\ G(\mathcal{F})$ is conserved. Here $G' = dG/d\mathcal{F}$ is used for convenience.

$$\begin{aligned}
&\int dZ\ G'(\mathcal{F})\left(\frac{\partial \mathcal{F}(Z,t)}{\partial t} - \Big[\mathcal{H}, \mathcal{F}(Z,t)\Big]\right) \\
&\qquad = \int dZ\ \left(G'(\mathcal{F})\partial_t \mathcal{F}(Z,t) - \int dZ\ G'(\mathcal{F})\Big[\mathcal{H}, \mathcal{F}(Z,t)\Big]\right) \\
&\qquad = \int dZ \left(\partial_t\ G(\mathcal{F}) - \ \Big[\mathcal{H}, G(\mathcal{F}(Z,t))\Big]\right) = \frac{\partial}{\partial t}\left(\int dZ\ G(\mathcal{F})\right) = 0.
\end{aligned} \tag{119}$$

One thus uses the properties of the Poisson brackets to include the function G' in the Poisson bracket that stands for the divergence of the phase space flux. One can then use the key property, namely that the phase space flux integrated over the phase space is zero which stems from the assumption of a closed system, hence with no outflux in physical space, including outflux to smaller scales, and no outflux in momentum usually attached to the assumption that $\mathcal{F}$ vanishes at large value of the momentum faster than any function of the momentum. The conservation law Eq.(119) thus stems directly from our assumptions.

For the particular choice of $G'(\mathcal{F}) = -\Big(1 + Log(\mathcal{F})\Big)$, hence $G(\mathcal{F}) = -\mathcal{F}\ Log(\mathcal{F}))$, one immediately obtains the entropy conservation law for the entropy $\mathcal{S} = \int dZ\ S(\mathcal{F})$ where the entropy density $S(\mathcal{F})$ is defined as $S(\mathcal{F}) = -\mathcal{F}\ Log(\mathcal{F})$.

$$\frac{\partial}{\partial t}\left(\int dZ\ S(\mathcal{F})\right) = \frac{\partial \mathcal{S}}{\partial t} = 0. \tag{120}$$

Similarly, taking benefit of the other dependence in the Poisson bracket, one can also address conservation equations for functions of $\mathcal{H}$, hence $G(\mathcal{H})$ leading to the conservation of $\mathcal{F}G(\mathcal{H})$.

From these properties one readily finds the particle conservation law $G(\mathcal{F}) = 1$ for the whole phase space or locally in physical space.

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\int dZ\ \mathcal{F}\right) &= 0, \\
\frac{\partial}{\partial t}\left(\int d\boldsymbol{P}\ \mathcal{F}\right) + \frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}}\left(\int d\boldsymbol{P}\ \frac{\boldsymbol{\partial}\mathcal{H}}{\partial \boldsymbol{P}}\ \ \mathcal{F}\right) &= 0.
\end{aligned} \tag{121}$$

Similarly for $\boldsymbol{P} = \boldsymbol{\partial_P}\mathcal{H}$ one recovers the momentum conservation law:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\int dZ\ \boldsymbol{P}\ \mathcal{F}\right) &= \boldsymbol{0}, \\ \frac{\partial}{\partial t}\left(\int d\boldsymbol{P}\ \boldsymbol{P}\ \mathcal{F}\right) + \frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}}\left(\int d\boldsymbol{P}\ \frac{\boldsymbol{\partial}\mathcal{H}}{\partial \boldsymbol{P}} \otimes\ \ \boldsymbol{P}\ \mathcal{F}\right) &= 0. \end{aligned} \tag{122}$$

Finally, for $G(\mathcal{H}) = \mathcal{H}$, the energy conservation law is then given by:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\int dZ\ \mathcal{H}\ \mathcal{F}\right) &= \boldsymbol{0}, \\ \frac{\partial}{\partial t}\left(\int d\boldsymbol{P}\ \mathcal{H}\ \mathcal{F}\right) + \frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}}\left(\int d\boldsymbol{P}\ \frac{\boldsymbol{\partial}\mathcal{H}}{\partial \boldsymbol{P}}\ \ \boldsymbol{H}\ \mathcal{F}\right) &= 0. \end{aligned} \tag{123}$$

In all these expressions, one can conveniently use the definition $\boldsymbol{\partial_P}\mathcal{H} = \boldsymbol{v}$ to replace $\boldsymbol{\partial_P}\mathcal{H}$ by $\boldsymbol{v}$.

All these conservation laws are thus straightforward consequences of the initial assumptions on the system that is considered. However, when addressing the reduced system Eq.(37), closed by the expression of the collision operator Eq.(105), it is important to check that the conservation properties are recovered and to determine which key property is involved in the extension of these conservation laws to the reduced system. Since the actual change in the system is due to the collision operator with its specific structure one can concentrate on the conservation properties of the latter.

5.2. Particle balance

Particle balance can be discussed in terms of the local property for a given species. In then takes the form:

$$\begin{aligned} \int d\boldsymbol{P}_a\ \mathcal{C}_a\Big(Z,t\Big) &= \int d\boldsymbol{P}_a\ \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\cdot\sum_b\int dZ_b\int_0^{+\infty} d\tau\quad \boldsymbol{\nabla}_a V\left(|\boldsymbol{X}_a - \boldsymbol{X}_b|\right)\otimes \\ &\quad \left(U_a(t,t-\tau)U_b(t,t-\tau)\boldsymbol{\nabla}_a V\left(|\boldsymbol{X}_a - \boldsymbol{X}_b|\right)\cdot\left(\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} - \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_b}\right)U_a(t-\tau,t)U_b(t-\tau,t)\right) \\ &\quad F_a(Z_a,t)F_b(Z_b,t). \end{aligned} \tag{124}$$

which, upon immediate integration over $\boldsymbol{P}_a$, is thus determined by the boundary conditions on $F_a(Z_a,t)$ that ensure that the integral is null. The particle conservation law for the collision operator, $\int d\boldsymbol{P}_a\ \mathcal{C}_a\Big(Z,t\Big)$, is thus recovered for the reduced system. One can readily trace back this property to the Liouville conservation equation and to the term of constructive interference of fluctuations that leads to the collision term, Eq.(37) and Eq.(43), namely:

$$\left\langle\Big[h, f_a(Z,t)\Big]\right\rangle_P = \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\cdot\sum_b\left\langle\int dZ_b\frac{\boldsymbol{\partial}}{\partial \boldsymbol{X}_a}\left(\frac{\alpha\ q_a\ q_b}{|\boldsymbol{X}_a - \boldsymbol{X}_b|}\right) f_b(Z_b,t)f_a(Z_a,t)\right\rangle_P = 0. \tag{125}$$

The structure of the collision operator that allows one to perform the integration with $\boldsymbol{P}_a$ is therefore intrinsic to the term from which is derived the collision operator. It can simply be stated as the fact that all forces acting on the plasma particles, including the fluctuating force term, are assumed not to create or destroy particles. This is readily expected for the weak binary collisions and would require modifications in the strong collision case that yields the fusion reactions.

5.3. Momentum balance

The momentum balance via Coulomb interaction stems from the action-reaction principle and the symmetry by particle permutation of the Coulomb force. In order to use this property we consider the calculation of the moment of the function $G(\boldsymbol{P}_a)$ with the collision operator, hence:

$$\begin{aligned}\int d\boldsymbol{P}_a\ G(\boldsymbol{P}_a)\ \mathcal{C}_a\Big(Z,t\Big) &= \sum_b \int d\boldsymbol{X}_b \int d\boldsymbol{P}_a \int d\boldsymbol{P}_b\ G(\boldsymbol{P}_a)\ \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\cdot \boldsymbol{D}_{a,b}\ \ F_a(Z_a,t)F_b(Z_b,t) \\ &= \sum_b \int d\boldsymbol{X}_b \int d\boldsymbol{P}_a \int d\boldsymbol{P}_b\ \left(\boldsymbol{D}_{a,b}\cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\ G(\boldsymbol{P}_a)\right) F_a(Z_a,t)F_b(Z_b,t), \quad (126)\end{aligned}$$

where, according to the action-reaction principle, the vector $\boldsymbol{D}_{a,b}$ is antisymmetric with respect to particle permutation, hence $\boldsymbol{D}_{a,b} = -\boldsymbol{D}_{b,a}$. One obtains therefore:

$$\Big(\mathbb{I}+\mathbb{P}_{a,b}\Big)\left(\boldsymbol{D}_{a,b}\cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\ G(\boldsymbol{P}_a)\right) = \left(\boldsymbol{D}_{a,b}\cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a}\ G(\boldsymbol{P}_a) - \boldsymbol{D}_{a,b}\cdot \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_b}\ G(\boldsymbol{P}_b)\right). \quad (127)$$

In the case of momentum balance, the function $G(\boldsymbol{P}_a) = \boldsymbol{P}_a$ so that Eq.(127) yields $\boldsymbol{D}_{a,b} - \boldsymbol{D}_{a,b}$ which is null, hence the momentum balance by combining the effect of the collision operator for pairs of particles.

$$\sum_a \int d\boldsymbol{P}_a\ \boldsymbol{P}_a\ \mathcal{C}_a\Big(Z,t\Big) = \frac{1}{2}\sum_{a,b}\int d\boldsymbol{X}_b \int d\boldsymbol{P}_a \int d\boldsymbol{P}_b \left((\mathbb{I}+\mathbb{P}_{a,b})\boldsymbol{D}_{a,b}\right) F_a(Z_a,t)F_b(Z_b,t) = 0. \quad (128)$$

It thus requires a sum over all species a but can still be expressed as a local property. Considering Eq.(125) one readily finds that the antisymmetry property is generic. It is thus found in the constructive interference term which is the starting point of the calculation, and preserved by the reduction process yielding the collision operators, both for the Landau and Lenard-Balescu expressions.

5.4. Energy balance

The calculation of the energy balance follows the same lines as that of the momentum, Eq.(126), but with $G(\boldsymbol{P}_a) = \mathcal{K}_a = \frac{1}{2}\boldsymbol{P}_a^2/m_a$ so that $\boldsymbol{\partial}_{\boldsymbol{P}_a} G(\boldsymbol{P}_a) = \boldsymbol{v}_a$. Following Eq.(128), one then obtains:

$$\begin{aligned}\sum_a \int d\boldsymbol{P}_a\ \mathcal{K}_a\ \mathcal{C}_a\Big(Z,t\Big) &= \frac{1}{2}\sum_{a,b}\int d\boldsymbol{X}_b \int d\boldsymbol{P}_a \int d\boldsymbol{P}_b \left(\Big(\mathbb{I}+\mathbb{P}_{a,b}\Big)\boldsymbol{D}_{a,b}\cdot \boldsymbol{v}_a\right) F_a(Z_a,t)F_b(Z_b,t) \\ &= \frac{1}{2}\sum_{a,b}\int d\boldsymbol{X}_b \int d\boldsymbol{P}_a \int d\boldsymbol{P}_b \Big(\boldsymbol{D}_{a,b}\cdot \boldsymbol{v}_{a,b}\Big) F_a(Z_a,t)F_b(Z_b,t). \quad (129)\end{aligned}$$

In Fourier space,see Eq.(74), the integrand of this relation is of the form:

$$(\boldsymbol{K}\cdot \boldsymbol{v}_{a,b}) \int_0^{+\infty} d\tau\ e^{i\boldsymbol{K}\cdot \boldsymbol{v}_{a,b}\tau} \left(\boldsymbol{K}\cdot \boldsymbol{\partial}_{\boldsymbol{v}_{a,b}}\right) = (\boldsymbol{K}\cdot \boldsymbol{v}_{a,b})\,\pi\delta\Big(\boldsymbol{K}\cdot \boldsymbol{v}_{a,b}\Big)\left(\boldsymbol{K}\cdot \boldsymbol{\partial}_{\boldsymbol{v}_{a,b}}\right). \quad (130)$$

The integration on $\boldsymbol{K}$ then yields zero because of the term of the form $x\delta(x)$. Regarding the physics, one thus finds that the kinetic energy of the two particles is redistributed between the two colliding particles, but that it is conserved as a whole since the Coulomb collisions are elastic.

5.5. **Entropy production**

For the entropy production, one is lead to consider the one particle distribution function for all species within the plasma, $F = \prod_a F_a$. The entropy is then defined by (setting $k_B = 1$ for convenience):

$$\begin{aligned} S &= \left(\prod_c \int dZ_c \Big(- F\ Log(F) \Big)\right) = - \prod_c \int dZ_c \Big(1 + \sum_a\ Log(F_a) \Big) F \\ &= - \prod_c \int dZ_c\ F_c + \prod_{c \neq a} \int dZ_c\ F_c\ \sum_a \int dZ_a\ \Big(- F_a Log(F_a) \Big). \end{aligned} \tag{131}$$

The entropy production is then determined by:

$$\begin{aligned} \frac{d}{dt} S &= \prod_{c \neq a} \int dZ_c\ F_c \frac{d}{dt} \left(\sum_a \int dZ_a\ \Big(- F_a Log(F_a) \Big) \right) \\ &= \prod_{c \neq a} \int dZ_c\ F_c \left(- \sum_a \int dZ_a\ \Big(1 + Log(F_a) \Big) \frac{\partial F_a}{\partial t} \right). \end{aligned} \tag{132}$$

Without loss of generality, one can set for convenience $\int dZ_c F_c = 1$ to simplify this expression. Let us then examine the collisional contribution to the entropy production and define $\dot{S}_{Coll}$ such that:

$$\dot{S}_{Coll} = - \sum_a \int dZ_a\ \Big(1 + Log(F_a) \Big) C_a(Z_a, t). \tag{133}$$

Considering Eq.(124), but neglecting the spatial inhomogeneity at scales smaller than the Debye length, the entropy production by the collisions is then determined by:

$$\begin{aligned} \dot{S}_{Coll} = \sum_a \int d\boldsymbol{X}_a\ \int d\boldsymbol{P}_a\ \frac{\boldsymbol{\partial} Log(F_a)}{\partial \boldsymbol{P}_a} \cdot \sum_b \int dZ_b\ F_a(Z_a, t) F_b(Z_b, t) \\ \int_0^{+\infty} d\tau\quad \boldsymbol{\nabla}_a V\left(X_{a,b}\right) \otimes \boldsymbol{\nabla}_a V\left(|\boldsymbol{X}_{a,b} - \boldsymbol{d}_{a,b}(\tau)|\right) \cdot \left(\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} - \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_b} \right)\ Log\Big(F_a\ F_b\Big). \end{aligned} \tag{134}$$

Following the two previous calculations we now address the symmetry properties of this expression.

$$\begin{aligned} \dot{S}_{Coll} &= \frac{1}{2} \sum_{a,b} \Big(\mathbb{I} + \mathbb{P}_{a,b} \Big) \int dZ_a\ \int dZ_b\ \frac{\boldsymbol{\partial} Log(F_a)}{\partial \boldsymbol{P}_a} \cdot\ F_a(Z_a, t) F_b(Z_b, t) \\ &\int_0^{+\infty} d\tau\quad \boldsymbol{\nabla}_a V\left(X_{a,b}\right) \otimes \boldsymbol{\nabla}_a V\left(|\boldsymbol{X}_{a,b} - \boldsymbol{d}_{a,b}(\tau)|\right) \cdot \left(\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} - \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_b} \right)\ Log\Big(F_a\ F_b\Big) \\ &= \frac{1}{2} \sum_{a,b} \int dZ_a\ \int dZ_b\ \left(\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} - \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_b} \right)\ Log\Big(F_a\ F_b\Big) \cdot\ F_a(Z_a, t) F_b(Z_b, t) \\ &\int_0^{+\infty} d\tau\quad \boldsymbol{\nabla}_a V\left(X_{a,b}\right) \otimes \boldsymbol{\nabla}_a V\left(|\boldsymbol{X}_{a,b} - \boldsymbol{d}_{a,b}(\tau)|\right) \cdot \left(\frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_a} - \frac{\boldsymbol{\partial}}{\partial \boldsymbol{P}_b} \right)\ Log\Big(F_a\ F_b\Big). \end{aligned} \tag{135}$$

Switching to Fourier space, one readily computes the following result:

$$\dot{S}_{Coll} = \frac{1}{2}\sum_{a,b}\int dZ_a \int d\boldsymbol{K} \int d\boldsymbol{P}_b \int_0^{+\infty} d\tau \; e^{-i\boldsymbol{K}\cdot\boldsymbol{d}_{a,b}(\tau)} \; F_a(Z_a,t)F_b(Z_b,t)$$
$$\left((2\pi)^{3/2} \; V(K)\Big(\mathbb{I}-\mathbb{P}_{a,b}\Big)\boldsymbol{K}\cdot\frac{\boldsymbol{\partial}}{\partial\boldsymbol{P}_a}Log(F_a)\right)^2. \tag{136}$$

By construction this expression is positive so that one recovers the expected entropy production due to the collision operator. Unlike the conservation laws, this is a marked difference with the Liouville or Vlasov equations that are isentropic.

Conclusion

We propose in this paper a complete survey of collisions and the collision operators useful for magnetised plasmas. Our focus is a general and systematic approach using propagators to address several aspects of the collision operators with a single and efficient tool. Novel aspects of this procedure are the dependence of the collision operator on small scale inhomogeneities, generating large collisional transport to smooth out the sub-Debye scales as well as some aspects of the Larmor gyration motion in the collision operator. Given the need for mathematical development to derive linearised collision operators, suitable for the gyrokinetic framework and tractable in the large gyrokinetic code, we believe that such a survey of collisions will be useful. It is also to be underlined that the present work addresses multi-species collisions with no assumption regarding charge or mass. These are the basis for the collisional exchange between main species and impurities, an important issue in understanding and hopefully controlling present experiments with tungsten walls [11, 12].

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