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A RATE-INDEPENDENT GRADIENT SYSTEM IN DAMAGE COUPLED WITH PLASTICITY VIA STRUCTURED STRAINS *

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Abstract. This contribution deals with a class of models combining isotropic damage with plasticity. It has been inspired by a work by Freddi and Royer-Carfagni [FRC10], including the case where the inelastic part of the strain only evolves in regions where the material is damaged. The evolution both of the damage and of the plastic variable is assumed to be rate-independent. Existence of solutions is established in the abstract energetic framework elaborated by Mielke and coworkers (cf., e.g., [Mie05, Mie11b]).

Résumé. Ce papier regarde l'étude d'une classe de modèles combinant l'endommagement isotrope avec la plasticité. Ce travail a été inspiré par des résultats de Freddi et Royer Carfagni [FRC10], qui ont considéré le cas où la partie non élastique du tenseur de déformation évolue seulement dans les zones endommagées du matériel. L'évolution de l'endommagement et de la plasticité est *rate-independent*. On preuve l'existence d'une solution dans le cadre des solutions énergétiques introduisées par Mielke et collaborateurs (cf., p. ex. [Mie05, Mie11b]).

Introduction

It is well known that damage in a material can be interpreted as a degradation of its elastic properties due to the failure of its microscopic structure. Such macroscopic mechanical effects take their origin from the formation of micro-cracks and cavities at a microscopic scale. Macroscopically, these degeneracy effects may be described by the incorporation of an internal variable into the model, the damage parameter, which in particular features a decrease of stiffness with ongoing damage. However, some materials show a more complex behavior, possibly presenting different

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responses to traction and compression loading, or exhibiting some plastic-like behavior when the damage process is activated.

The study of plastic material behavior at small strains in itself has a long tradition, cf. e.g. [Hil50, Lub90], and numerous analytical and numerical results exist, cf. e.g. [Tem85, HR99, RDG08, Kne09, Kne10, JRZ13, BMR12, BR08, DMS14, DMDMM08, DMDM06]. Also isotropic damage in itself nowadays is a well-investigated phenomenon and it has been treated in the spirit of phase-field theories from the point of view of modeling, analysis, and computations, cf. e.g. [FN96, BS04, BSS05, MR06, TM10, FKS12, KRZ13, FG06, GL09, MRZ10, JZ15, Gia05, PM13, DMI13]. In this family of models, a scalar internal variable is introduced to denote the local proportion of active micro-bonds vs. the damaged ones. Nonetheless, this approach neither permits to distinguish different kinds of anisotropic behavior, nor the appearance of an unknown transformation strain, as it occurs in plasticity. Thus, it is of some interest to combine scalar and tensorial variables to describe both of these effects.

More precisely, in this contribution, we assume that a "transition strain", a *structured strain* as it is called in [FRC10], may appear and evolve during the damage process. The latter in itself decreases the stiffness of the material during the evolution. The first effect makes our model akin to a plasticity model, in which the plastic strain is activated through damage and its norm depends on the damage level; we refer to [AMV14] for an alternative model for damage coupled with plasticity, recently analyzed in [Cri15, CL16]. As a consequence, we deal with two internal variables: a scalar one χ, standardly denoting the local proportion of active bonds in the micro-structure of the material, and a tensorial one D, which stands for the transformation strain arising during the damage evolution. The behavior of these two variables is recovered by a generalization of the principle of virtual powers, in which micro-forces responsible for the formation of micro-cracks and micro-slips are included; we confine the discussion to the small-strain regime and the isothermal case, though. The momentum balance equation for the displacement $\mathbf{u}$ is eventually written in the quasi-static case, while the evolution of the internal variables χ and D is governed by an energy functional and a 1-homogeneous dissipation potential, leading to a rate-independent evolution of these variables and possibly including irreversibility constraints.

All in all, the resulting PDE system in the variable $\mathbf{q} = (\mathbf{u}, \chi, D)$ pertains to the class of abstract gradient systems of the form

$$\partial\mathcal{R}(\partial_t \mathbf{q}) + \mathrm{D}\mathcal{E}(t, \mathbf{q}(t)) \ni 0 \quad \text{in } (0, T), \tag{0.1}$$

driven by an energy functional $\mathcal{E}$ and a dissipation potential $\mathcal{R}$, positively homogeneous of degree 1 and only acting on the dissipative variables (χ, D). For the analysis of this system, we will resort to the *energetic formulation* for rate-independent systems developed by Mielke and coworkers, cf. [MT04, Mie05, MM05, Mie11b]. We will thus prove the existence of energetic solutions by applying an abstract existence result from [Mie11b].

Plan of the paper. The derivation of the model will be carried out in Section 1. The precise mathematical assumptions are collected in Section 2. The existence theorem (Thm. 3.4) is stated in Section 3 in the framework of energetic solutions. Finally, its proof is carried out in Section 4.

1. Continuum mechanical derivation of the model

Along a time-interval $[0, T]$, we study the mechanical behavior of a body, occupying a domain $\Omega \subset \mathbb{R}^d$, $1 < d \in \mathbb{N}$. The body is exposed to time-dependent external loadings, which possibly cause a degradation of the micro-structure of the material, leading to inelastic responses. In particular, we restrict ourselves to a small-strain regime and introduce the vector $\mathbf{u}$ of small displacements. Hence, as already mentioned in the introduction, we shall formulate the model in terms of the strain and in terms of two further state variables $\chi : [0, T] \times \Omega \to [0, 1]$ and $D : [0, T] \times \Omega \to \mathbb{R}^{d \times d}$, which are internal variables more specifically related to the description of damage and plastic-like behavior. Accordingly, in view of the conjugate approach, the free energy will depend on the strain and on these two internal variables, and the stress shall be derived in terms of them.

In particular, using the approach of [FRC10], we first suppose that the symmetric gradient of the displacement $\mathbf{u}$

$$e(\mathbf{u}) = (\nabla \mathbf{u} + \nabla^\top \mathbf{u})/2$$

is decomposed in two parts:

$$e(\mathbf{u}) = E_{\mathrm{el}} + \Xi \tag{1.1}$$

where $E_{\mathrm{el}} \in \mathbb{R}^{d\times d}$ represents the *elastic* part of the strain and $\Xi \in \mathbf{S} \subset \mathbb{R}^{d\times d}$ the *inelastic* one, associated with the formation of micro-cracks or micro-slips. It is indeed known that (see, e.g., [Kac90, Fré02]) for an inelastic body the strain is determined by the stress and by some additional (internal) variable, which may be interpreted within the framework of a general plasticity theory. In this spirit, the set $\mathbf{S} \subset \mathbb{R}^{d\times d}$ can, e.g., be the subspace of symmetric matrices or, as in plasticity theory, the subspace of deviatoric (i.e., trace-free) $\mathbb{R}^{d\times d}$-matrices. In order to allow for the treatment of different types of inelastic phenomena we keep $\mathbf{S} \subset \mathbb{R}^{d\times d}$ general, and refer to Remark 1.2 below for more details on specific choices of $\mathbf{S}$ and their meaning.

In the context of this damage model, we prescribe that the inelastic part of the strain depends on the state of the internal bonds acting at a microscopic level in the material. We also assume that the phenomenon of damage is *progressive*, in the sense that within the same body there may be regions where the material is completely damaged and regions where the microstructure is lost, but not yet failed. As it is common in the modeling of isotropic damage, the variable χ is therefore linked to the proportion of active or inactive bonds in a neighborhood of material-dependent size (representative volume element) centered around any material point $x \in \Omega$. Hence, χ takes values in the interval $[0, 1]$. Throughout this work we will assume that χ stands for the proportion of *active bonds* at the micro scale in the material, thus, with the value 1 in the sound regions and 0 in a failed zone. Along the footsteps of [FRC10] (cf. Remark 1.2 later on), we introduce a second internal variable $D \in \mathbf{S}$ of type "transformation" strain leading to plastic effects and developing in the regions where the material is damaged; it shall hereafter be formally referred to as *plastic strain*. Thus, following [FRC10], the inelastic part of the strain is a function of χ and D,

$$\Xi : [0, 1] \times \mathbf{S} \to \mathbf{S} \quad \text{s.t.} \quad \Xi(1, D) = 0 \text{ and } \Xi(0, D) = D \qquad \text{for every } D \in \mathbf{S}\,. \tag{1.2}$$

As a particular choice for the function Ξ one may consider

$$\Xi(\chi, D) = (1 - \chi)D\,. \tag{1.3}$$

As a general feature of Ξ, note that, in view of (1.1), for $\chi = 1$ we have $E_{\mathrm{el}} = e(\mathbf{u})$, whereas for $\chi = 0$ we have $E_{\mathrm{el}} = e(\mathbf{u}) - D$.

Following the continuum-mechanical modeling perspective of Frémond, cf. e.g. [Fré02], we shall now introduce the constitutive functionals and equations specifying the damage-plasticity model under consideration. Let us point out that this approach is mainly based on a variational principle, i.e. the (generalized) principle of virtual powers. The main idea is that forces acting at a microscopic level in the material, responsible for the formation of micro-cracks and thus activating damage, have to be included in the whole energy balance of the mechanical system. Hence, as a prerequisite we shall postulate that the powers of the interior forces P_i, the exterior forces P_e, and the acceleration forces P_a, acting on the elasto-plastic and damageable body, occupying the (reference) domain $\Omega \subset \mathbb{R}^d$, are balanced, i.e.,

$$P_i + P_e = P_a \qquad \text{and, here, } P_a = 0\,, \tag{1.4}$$

as we will confine our discussion to a quasistatic evolution.

The principle of virtual powers. The principle of virtual powers now postulates that the above balance of powers has to hold on any subdomain $\omega \subset \Omega$, thus leading to the *virtual powers* of this subdomain, which are assumed to be given in integral form by

$$P_e(\omega) = \int_\omega p_e \,\mathrm{d}x + \int_{\partial\omega} \widetilde{p}_e \,\mathrm{d}S \qquad \text{and} \qquad P_i(\omega) = \int_\omega p_i \,\mathrm{d}x.$$

Consequently, different kinds of virtual velocities are introduced: macroscopic velocities $\mathbf{v}$, microscopic scalar velocities γ, and microscopic tensorial velocities V. Under the assumption that no external forces act on the microscopic level, we can prescribe the virtual external power P_e of the subdomain $\omega \subset \Omega$ in the form

$$P_e(\omega) = \int_\omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x + \int_{\partial\omega} \mathbf{t}\cdot\mathbf{v}\,\mathrm{d}S \tag{1.5}$$

(where $\cdot$ denotes the scalar product between two vectors in $\mathbb{R}^d$), for any macroscopic virtual velocity $\mathbf{v} : \omega \to \mathbb{R}^d$ and for the given volumetric force $\mathbf{f} : \omega \to \mathbb{R}^d$ and the given surface force $\mathbf{t} : \partial\omega \to \mathbb{R}^d$ acting on $\omega \subset \Omega$. Similarly, the virtual internal power of ω is given in integral form as the product of the internal forces and virtual velocities. Due to the fact that the body is exposed to elasto-plastic defomations and damage, the internal forces consist of the macroscopic stress $\sigma : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and additional internal micro-stresses $B : \omega \to \mathbb{R}$, $\mathbf{J} : \omega \to \mathbb{R}^d$, $X : \omega \to \mathbb{R}^{d\times d}$, and $\mathbf{Y} : \omega \to \mathbb{R}^{d^3}$ related to damage and the plastic deformation. In what follows, the symbol : stands for the product both in the space of d^2- and of d^3-tensors. For any macroscopic virtual velocity $\mathbf{v}$, for all microscopic velocities $\gamma : \omega \to \mathbb{R}$, and for all microscopic tensorial velocities $V : \omega \to \mathbf{S}$, the virtual internal power of ω is thus given by

$$P_i(\omega) = \int_\omega p_i\,\mathrm{d}x = -\int_\omega \sigma : e(\mathbf{v})\,\mathrm{d}x + \int_\omega (B\gamma + \mathbf{J} : \nabla\gamma + X : V + \mathbf{Y} : \nabla V)\,\mathrm{d}x\,. \tag{1.6}$$

Let us point out that the first contribution to P_i is classical. Instead, the other terms are introduced in [Fré02] to account for the power of interior forces involving microscopic velocities, which represent the microscopic motions changing the structure of the material. Note that (1.6) reflects the fact that the power of the interior forces is zero for any (macroscopic) rigid motion. Indeed, if the body is subjected to a rigid motion, the distance between particles does not change, so that the microscopic velocities and their gradients are zero.

Now, taking into account that the relations (1.4)–(1.6) shall hold for any subdomain $\omega \subset \Omega$ and for any virtual velocity, the resulting balance equations are

$$-\mathrm{div}\,\sigma = \mathbf{f} \quad \text{in } \Omega, \qquad \sigma\mathbf{n} = \mathbf{t} \quad \text{on } \partial\Omega\,, \tag{1.7a}$$
$$B - \mathrm{div}\,\mathbf{J} = 0 \quad \text{in } \Omega, \qquad \mathbf{J}\mathbf{n} = 0 \quad \text{on } \partial\Omega\,, \tag{1.7b}$$
$$X - \mathrm{div}\,\mathbf{Y} = 0 \quad \text{in } \Omega, \qquad \mathbf{Y}\mathbf{n} = 0 \quad \text{on } \partial\Omega\,, \tag{1.7c}$$

with $\mathbf{n}$ the outward unit normal to $\partial\Omega$.

The constitutive relations. Following [Fré02, Chap.s 3, 4], we assume that the constitutive relations are comprised in two functionals, the free enery functional $\mathcal{F}$ and the pseudo-potential of dissipation $\mathcal{R}$ in integral form, with densities Ψ and Φ, respectively:

$$\mathcal{F}(\mathbf{u},\chi,D) := \int_\Omega \Psi\,\mathrm{d}x \qquad \text{and} \qquad \mathcal{R}(\partial_t\chi,\partial_t D) := \int_\Omega \Phi\,\mathrm{d}x\,. \tag{1.8}$$

Formally using the above localization arguments, and in view of (1.7), we prescribe the following constitutive relations

$$\sigma = \frac{\partial\Psi}{\partial e}, \quad B = \frac{\partial\Phi}{\partial(\partial_t\chi)} + \frac{\partial\Psi}{\partial\chi}, \quad \mathbf{J} = \frac{\partial\Psi}{\partial\nabla\chi}, \quad X = \frac{\partial\Phi}{\partial(\partial_t D)} + \frac{\partial\Psi}{\partial D}, \quad \mathbf{Y} = \frac{\partial\Psi}{\partial\nabla D}\,. \tag{1.9}$$

Choice of the constitutive functions. We choose the density of the pseudo-potential of dissipation of the form

$$\begin{aligned} &\Phi(\partial_t D,\partial_t\chi) := R_{\mathrm{inel}}(\partial_t D) + R_{\mathrm{dam}}(\partial_t\chi), \qquad \text{where} \\ &R_{\mathrm{inel}}(\partial_t D) := \mu|\partial_t D| \ \text{ and } \ R_{\mathrm{dam}}(\partial_t\chi) := \nu|\partial_t\chi| + I_{(-\infty,0]}(\partial_t\chi) \end{aligned} \tag{1.10}$$

for material parameters $\mu, \nu > 0$. Note that both R_{inel} and R_{dam} are positively 1-homogeneous, thus featuring a rate-independent evolution of the variables D and χ. In particular, for the definition of R_{dam}, observe that the indicator term $I_{(-\infty,0]}$ enforces the unidirectionality constraint $\partial_t \chi \leq 0$, i.e. that the parameter χ is a non-increasing function of time, ensuring that damage can only increase. In turn, the free energy density Ψ shall feature an indicator term acting on χ, cf. (1.13), which forces the χ-component of the solution to the evolutionary system we shall derive to take positive values. Starting from an initial datum χ_0 with $\chi_0(x) \in [0,1]$ for almost all x in Ω, we will thus obtain that $\chi(x,t) \in [0,1]$ for almost all $(x,t) \in \Omega \times [0,T]$, in accordance with the physical meaning of χ as a *proportion* of active bonds.

For the definition of the free energy density Ψ we assume, in the spirit of linear elasticity, that Ψ consists of a quadratic elastic contribution, a coupling term H, and terms J and G also featuring regularizations for the damage variable and the plastic strain, respectively. In particular, Ψ shall take the following form:

$$\begin{aligned} \Psi(e(\mathbf{u}), \chi, D, \nabla\chi, \nabla D) :=& \tfrac{1}{2}\big(e(\mathbf{u}) - \Xi(\chi, D)\big) : \mathbb{K}(\chi) : \big(e(\mathbf{u}) - \Xi(\chi, D)\big) \\ &+ H(\chi, D) + J(\chi, \nabla\chi) + G(D, \nabla D)\,. \end{aligned} \tag{1.11}$$

Here, $\Xi : \mathbb{R} \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is defined as in (1.2). Moreover, as usual in damage models we consider a χ-dependent stiffness tensor $\mathbb{K}$, such that $\mathbb{K}(\chi)$ is a symmetric $\mathbb{R}^{d^4}$-tensor for every χ, with the property that a decrease of the value of χ leads to a decrease of the quadratic contribution of the energy term. In principle, with the choice of $\mathbb{K}$ we can incorporate in the model both the case in which the stiffness degenerates when the material is completely damaged (i.e., for $\chi = 0$, the tensor $\mathbb{K}(0)$ is no longer positive definite, in particular it might happen that $\mathbb{K}(0) = 0$, cf. e.g. [BMR09, MRZ10, Mie11a, HK15]), and the case in which some residual stiffness is guaranteed even for $\chi = 0$ (i.e., $K(\chi)$ is positive definite for every χ). In fact, in what follows we shall confine our analysis to the latter case.

The coupling term H shall take into account different cohesive properties of the material and the plastic behavior. A possible choice could be

$$H(\chi, D) = w(1-\chi) + \tfrac{1}{2}|D|^2(1-\chi) \qquad \text{with } w > 0\,. \tag{1.12}$$

As usual in damage models (see, e.g, [FN96]), the function H plays the role of a cohesion energy, in that it forces the parameter χ to take the value 1, since $w > 0$. Note that, in the case $\chi = 0$, i.e. when the material is maximally "broken", the term H leads to a hardening effect for the plasticity variable D.

The function J for the damage variable shall guarantee the modeling assumption $\chi \in [0,1]$ a.e. in Ω introducing some internal constraint. A possible choice would be

$$J(\chi, \nabla\chi) := I_{[0,1]}(\chi) + \tfrac{\alpha}{2}|\nabla\chi|^2 + W_1(\chi) \tag{1.13}$$

with $\alpha > 0$ a fixed constant.

Remark 1.1. Let us point out that, setting $W_1(\chi) = \frac{1}{\alpha}(1-\chi)^2$ with small $\alpha > 0$ would rather inhibit damage. However, we may allow also for non-convex choices of W_1, e.g. in terms of a double-well potential. As analyzed in [Tho13] in the context of brittle damage, the choice $W_1(\chi) = \frac{1}{\alpha}\chi^2(1-\chi)^2$ with small $\alpha > 0$ will yield that $\chi \in \{0,1\}$ a.e. in Ω as $\alpha \to 0$, thus accounting only for the sound and the maximally damaged state of the material in the limit.

In the same manner, the term G may confine the plastic strain to a closed, convex subset K of the subspace $\mathbf{S} \subset \mathbb{R}^{d\times d}$; we refer to Remark 1.2 for different choices of K and $\mathbf{S}$. As a possible form of G we may consider

$$G(D, \nabla D) := I_K(D) + (|D|^2 - 1)^2 + \tfrac{1}{q}|\nabla D|^q \qquad \text{with } q \in (1, \infty)\,. \tag{1.14}$$

Remark 1.2 (Comparison with [FRC10] and possible choices of $\mathbf{S}$ and K)**.** In [FRC10], following [DPO93] the authors refer to Ξ as the *structured strain*, and postulate for it the form (1.3) as a function of D, which, as a function of $x \in \Omega$, in turn represents the *structured strain* that would develop in a neighbourhood of x if the material was completely disgregated. In [FRC10] it is in fact remarked that the form of the field $D = D(x)$ may depend upon the material microstructure and the local defects of the body, so that its complete characterization is an open problem. Hence, the authors propose a *mesoscopic representation* for D as a function of $e(\mathbf{u})$.

More precisely, the relation between D and $e(\mathbf{u})$ is established through the minimization of the quadratic elastic energy (i.e. the first term in (1.11)), for $\chi = 0$, over suitable classes of admissible structured strains. This reflects the fact that D physically represents the strain that a completely disgregated body may attain without energy consumption in order to accomodate the boundary data. Different choices for the class $\mathbf{S}$ of admissible strains lead to models with different types of material responses to damage and fractures.

For example, taking $\mathbf{S}$ as the space of all symmetric tensors it is possible to recover a model describing the formation of *cleavage fractures*, viz. fractures directly proportional to the macroscopic deformation. Indeed, minimizing the elastic contribution to the free energy in (1.11) for $\chi = 0$, in the case in which $\mathbb{K}(0)$ is positive definite, yields $D = e(\mathbf{u})$. Observe that, when $D = e(\mathbf{u})$ we recover the original form of the elastic part in the free energy

$$\frac{1}{2}\big(e(\mathbf{u}) - (1-\chi)D\big) : \mathbb{K}(\chi) : \big(e(\mathbf{u}) - (1-\chi)D\big) = \frac{1}{2}\chi^2|e(\mathbf{u})|^2.$$

Setting $\mathbf{S}$ as the space of symmetric tensors with null deviatoric part (i.e., *trace-free* matrices) leads to a model for the formation of less brittle fractures, like those occurring in materials like stones. In this connection, as common in plasticity models (cf. e.g. [HR99]), we might choose $K \subset \mathbf{S}$ as a closed and convex subset of the set of deviatoric matrices $\mathbf{S}$.

The final set of constitutive equations. Combining in (1.7a)-(1.7c) the constitutive relations (1.9) with (1.10) and (1.11) we obtain the set of constitutive equations, to be satisfied in $\Omega \times (0, T)$:

$$-\operatorname{div}\big(\mathbb{K}(\chi) : \big(e(\mathbf{u}) - \Xi(\chi, D)\big)\big) = \mathbf{f}\,, \tag{1.15a}$$

$$\begin{aligned} &\partial R_{\mathrm{dam}}(\partial_t \chi) + \partial_\chi J(\chi, \nabla\chi) - \operatorname{div} \tfrac{\partial J(\chi,\nabla\chi)}{\partial(\nabla\chi)} \\ &\quad \ni -\tfrac{1}{2}\big(e(\mathbf{u}) - \Xi(\chi, D)\big) : \mathbb{K}'(\chi) : \big(e(\mathbf{u}) - \Xi(\chi, D)\big) \\ &\qquad + \big(e(\mathbf{u}) - \Xi(\chi, D)\big) : \mathbb{K}(\chi) : \tfrac{\partial \Xi(\chi,D)}{\partial\chi} - \tfrac{\partial H(\chi,D)}{\partial\chi}\,, \end{aligned} \tag{1.15b}$$

$$\begin{aligned} &\partial R_{\mathrm{inel}}(\partial_t D) + \partial_D G(D, \nabla D) - \operatorname{div} \tfrac{\partial G(D,\nabla D)}{\partial(\nabla D)} \\ &\quad - \mathbb{K}(\chi)\,(e(\mathbf{u}) - \Xi(\chi, D)) : \tfrac{\partial \Xi(\chi,D)}{\partial D} + \tfrac{\partial H(\chi,D)}{\partial D} \ni 0 \end{aligned} \tag{1.15c}$$

with $\mathbf{f}$ the volume force from (1.5). We shall assume that the inelastic stress function Ξ and the material tensor $\mathbb{K}$ are suitably smooth. We will supplement the rate-independent system (1.15) with the boundary conditions

$$\begin{aligned} &\mathbf{u}(x,t) = \mathbf{u}_D(t) \text{ on } \Gamma_D, &&\mathbb{K}(\chi) : \big(e(\mathbf{u}(x,t)) - \Xi(\chi, D)\big)\mathbf{n} = \mathbf{t} \text{ on } \Gamma_N\,, \\ &\tfrac{\partial\chi}{\partial\mathbf{n}} = 0 \text{ in } \partial\Omega \times (0,T), &&\tfrac{\partial D}{\partial\mathbf{n}} = 0 \text{ in } \partial\Omega \times (0,T), \end{aligned} \tag{1.16}$$

with Γ_{D} and Γ_{N} the Dirichlet and the Neumann parts of the boundary $\partial\Omega$, respectively. In what follows, we will address the existence of solutions to the boundary-value problem (1.15)–(1.16) in a suitably weak sense. We will discuss a suitable solution concept in Section 3 ahead.

Remark 1.3. Let us point out that system (1.15) is related, for special choices of the involved functionals, to well-known models in plasticity and phase transitions processes. Indeed, taking, for example,

$$\mathbb{K}(\chi) = \chi\mathrm{Id} \in \mathbb{R}^{d^4}, \quad \Xi(\chi, D) = (1-\chi)D, \quad \text{and} \quad H(\chi, D) = w(1-\chi) + \frac{1}{2}|D|^2(1-\chi),$$

we get the following PDE system in $\Omega \times (0,T)$

$$-\operatorname{div}\left(e(\mathbf{u}) - (1-\chi)D\right) = \mathbf{f}\,, \tag{1.17a}$$

$$\begin{aligned} &\partial R_{\mathrm{dam}}(\partial_t\chi) + \partial_\chi J(\chi, \nabla\chi) - \operatorname{div} \tfrac{\partial J(\chi,\nabla\chi)}{\partial(\nabla\chi)} \\ &\quad \ni -\tfrac{1}{2}\left|e(\mathbf{u}) - (1-\chi)D\right|^2 - (e(\mathbf{u}) - (1-\chi)D) \colon \chi\mathrm{Id} \colon D + \frac{1}{2}|D|^2 + w, \end{aligned} \tag{1.17b}$$

$$\begin{aligned} &\partial R_{\mathrm{inel}}(\partial_t D) + \partial_D G(D, \nabla D) - \operatorname{div} \tfrac{\partial G(D,\nabla D)}{\partial(\nabla D)} \\ &\quad - \left(e(\mathbf{u}) - (1-\chi)D\right)(1-\chi) + (1-\chi)D \ni 0\,. \end{aligned} \tag{1.17c}$$

In particular, without terms as J and G in the free energy, the resulting equations correspond to a rate independent evolution for the parameter χ, governed by a quadratic source of damage (including strain and structured/plastcity strain), and a plasticity equation with hardening contribution (obtained by the third equation in the case $\chi = 0$).

2. Assumptions and notation

In the following, given a Banach space $\mathbf{B}$, we shall denote by $\mathbf{B}^*$ its dual space, by $\|\cdot\|_{\mathbf{B}}$ its norm, and by $\langle\cdot,\cdot\rangle_{\mathbf{B}}$ the duality pairing between $\mathbf{B}^*$ and $\mathbf{B}$. We set $R_\infty := \mathbb{R} \cup \{+\infty\}$.

In the next lines, we specify the mathematical assumptions for the quantities introduced so far.
Assumptions on the domain: We assume that

$$\begin{aligned} &\Omega \subset \mathbb{R}^d\,,\ d \in \mathbb{N}\,,\ \text{is a bounded domain with Lipschitz-boundary } \partial\Omega \text{ such that} \\ &\Gamma_{\mathrm{D}} \subset \partial\Omega \text{ is nonempty and relatively open and } \Gamma_{\mathrm{N}} := \partial\Omega\backslash\Gamma_{\mathrm{D}}\,. \end{aligned} \tag{2.1}$$

Function spaces: We fix the function spaces as follows

$$\mathbf{U} := \{u \in H^1(\Omega, \mathbb{R}^d),\ u = 0 \text{ on } \Gamma_D\}\,, \tag{2.2a}$$

$$\mathbf{Z} := L^1(\Omega)\,, \tag{2.2b}$$

$$\mathbf{M} := \{z \in \mathbf{Z},\ z \in [0,1] \text{ a.e. in } \Omega\}\,, \tag{2.2c}$$

$$\mathbf{X} := \{z \in W^{1,r}(\Omega)\}\,, \qquad r > 1, \tag{2.2d}$$

$$\mathbf{V} := L^1(\Omega; \mathbb{R}^{d\times d})\,, \tag{2.2e}$$

$$\mathbf{H} := W^{1,q}(\Omega; \mathbf{S})\,, \qquad q \geq 2d/(d+2), \tag{2.2f}$$

$$\mathbf{H}_1 := L^{q_1}(\Omega; \mathbf{S})\,, \qquad q_1 \geq \max\{q, 2\}, \tag{2.2g}$$

and we recall that $\mathbf{S}$ is a subspace of $\mathbb{R}^{d\times d}$. With (2.7c) ahead we shall further specify the conditions on the indices q and q_1.
Assumptions on the given data: Given $\mathbf{U}$ from (2.2a), we shall assume that the volume forces $\mathbf{f}$ and the surface forces $\mathbf{t}$ from (1.5) are comprised in a time-dependent functional $F : [0,T] \to \mathbf{U}^*$. Moreover, for all $t \in [0,T]$, we suppose that the Dirichlet datum $\mathbf{u}_{\mathrm{D}}(t)$ has an extension from Γ_{D} into the domain Ω, also denoted by $\mathbf{u}_{\mathrm{D}}(t)$. In particular, we make the following regularity assumptions:

$$\begin{aligned} &F \in C^1([0,T]; \mathbf{U}^*) \text{ comprises both volume forces and Neumann data,} \\ &\text{such that } \|F\|_{C^1([0,T];\mathbf{U}^*)} \leq C_F, \end{aligned} \tag{2.3a}$$

$$\begin{aligned} &\mathbf{u}_{\mathrm{D}} \in C^1([0,T]; \mathbf{U}) \text{ is an extension of the Dirichlet datum,} \\ &\text{such that } \|\mathbf{u}_{\mathrm{D}}\|_{C^1([0,T];\mathbf{U})} \leq C_{\mathrm{D}} \text{ and } e_{\mathrm{D}} := e(\mathbf{u}_{\mathrm{D}})\,. \end{aligned} \tag{2.3b}$$

Furthermore, for the elastic tensor $\mathbb{K} : [0,1] \to \mathbb{R}^{d^4}$ we assume symmetry and positive definiteness, i.e.,

$$\begin{aligned} &\forall \chi \in [0,1] : \ \mathbb{K}(\chi) \text{ is symmetric}, \\ &\exists K_1, K_2 > 0 \, \forall e \in \mathbb{R}^{d\times d} \, \forall \chi \in [0,1] : \ |e|^2 K_1 \le e : \mathbb{K}(\chi) : e \le K_2 |e|^2 . \end{aligned} \tag{2.3c}$$

Recall that the positive definiteness of $\mathbb{K}(\chi)$ for any $\chi \in [0,1]$ ensures some residual stiffness of the material even in the case of maximal damage $\chi = 0$.
Assumptions on the inelastic strain Ξ: For the inelastic strain $\Xi : [0,1] \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ introduced in (1.2) we make the following assumptions:

$$\Xi \in C^0([0,1] \times \mathbf{S}; \mathbf{S}) \quad \text{s.t.} \tag{2.4a}$$

$$\forall \chi_1 < \chi_2 \in [0,1],\ D \in \mathbf{S} : \ |\Xi(\chi_2, D)| \le |\Xi(\chi_1, D)| \,.\ \Xi(1,D) = 0 \text{ and } \Xi(0,D) = D \,. \tag{2.4b}$$

For later use we remark that (2.4b) in particular implies that

$$|\Xi(\chi, D)| \le |D| \qquad \text{for all } (\chi, D) \in [0,1] \times \mathbf{S} \,. \tag{2.5}$$

Assumptions on the damage regularization: In view of (1.11), given $\mathbf{Z}$ from (2.2b), we define the damage regularization functional in terms of

$$\begin{aligned} &\mathcal{J} : \mathbf{Z} \to \mathbb{R}_\infty,\ \mathcal{J}(\chi) := \int_\Omega J(\chi, \nabla\chi)\,\mathrm{d}x \quad \text{with} \\ &J(\chi, \nabla\chi) := I_{[0,1]}(\chi) + \tilde{J}(\chi, \nabla\chi) \end{aligned} \tag{2.6a}$$

and we assume that $\tilde{J}$ has the following properties:

$$\text{Continuity: } \tilde{J} \in C^0(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}) \,, \tag{2.6b}$$

$$\begin{aligned} \text{Growth: } &\exists\, c_J, \tilde{c}_J, C_J > 0,\ \exists\, r \in (1,\infty),\ \forall \chi \in [0,1],\ A \in \mathbb{R}^d : \\ &c_J(|A|^r - \tilde{c}_J) \le \tilde{J}(\chi, A) \le C_J(|A|^r + 1) \,, \end{aligned} \tag{2.6c}$$

$$\text{Convexity: } \forall \chi \in [0,1] : \ \tilde{J}(\chi, \cdot) \text{ is convex on } \mathbb{R}^{d\times d} . \tag{2.6d}$$

Under these assumptions $\mathcal{J}$ may e.g. be a pure (convex) gradient regularization (i.e. $\tilde{J}$ does not depend on χ), but it may also incorporate terms like $(1-\chi)^2$ enforcing χ to stay close to 1, hence inhibiting damage. Also nonconvex terms of lower order, e.g. double well potentials, may contribute to $\tilde{J}$, provided the leading term is convex. In particular, note that the density J considered in (1.13) is comprised in this set of assumptions.
Assumptions on the plastic regularization: In view of (1.11), given $\mathbf{V}$ from (2.2e) and the subspace $\mathbf{S} \subset \mathbb{R}^{d\times d}$, we introduce the plastic regularization functional as follows

$$\begin{aligned} &\mathcal{G} : \mathbf{V} \to \mathbb{R}_\infty,\ \mathcal{G}(D) := \begin{cases} \int_\Omega G(D, \nabla D)\,\mathrm{d}x & \text{if } G(D, \nabla D) \in L^1(\Omega), \\ \infty & \text{otw.} \end{cases} \quad \text{with} \\ &G(D, \nabla D) := I_K(D) + \tilde{G}(D, \nabla D) \,, \end{aligned} \tag{2.7a}$$

where K is a closed, convex subset of $\mathbf{S}$, and we suppose that $\tilde{G}$ has the following properties:

$$\text{Continuity: } \tilde{G} \in C^0(\mathbb{R}^{d\times d} \times \mathbb{R}^{d^4}; \mathbb{R}) \,, \tag{2.7b}$$

$$\begin{aligned} \text{Growth: } &\exists\, c_G, \tilde{c}_G, C_G > 0,\ \exists\, q,\ q_1 \text{ with } \tilde{q} := \tfrac{2d}{d+2} \le q < q_1 \in [2,\infty),\ \forall D \in \mathbb{R}^{d\times d},\ A \in \mathbb{R}^{d^4} : \\ &c_G(|A|^q + |D|^{q_1} - \tilde{c}_G) \le \tilde{G}(D, A) \le C_G(|A|^q + |D|^{q_1} + 1) \,, \end{aligned} \tag{2.7c}$$

$$\text{Convexity: } \forall D \in \mathbb{R}^{d\times d} : \ \tilde{G}(D, \cdot) \text{ is convex on } \mathbb{R}^{d^4} . \tag{2.7d}$$

The assumption on the exponents $q < q_1$ (which are the ones associated with the spaces $\mathbf{H}$ and $\mathbf{H}_1$, cf. (2.2f) and (2.2g)), implies the coercivity of the integral functional $\mathcal{G}$ wrt. the space $W^{1,q}(\Omega;\mathbb{R}^{d\times d})$. Moreover, $q_1 \geq 2$ yields that $D \in L^2(\Omega;\mathbb{R}^{d\times d})$ on energy sublevels, whereas the lower bound $\frac{2d}{d+2} \leq q$ ensures that

$$W^{1,q}(\Omega;\mathbb{R}^{d\times d}) \Subset L^2(\Omega;\mathbb{R}^{d\times d}) \text{ compactly, with embedding constant } C_{\mathbf{H}\to L^2}\,. \tag{2.8}$$

Note that, for instance, G from (1.14) complies with the above growth assumptions if $q \in [\tilde{q}, 4]$.
Assumptions on the coupling term: In view of (1.11), (2.6a), and (2.7a) we introduce the coupling term as follows

$$\mathcal{H} : \mathbf{Z}\times\mathbf{V} \to \mathbb{R} \quad \mathcal{H}(\chi, D) := \begin{cases} \int_\Omega H(\chi, D)\,\mathrm{d}x & \text{if } (\chi, D) \in \mathbf{M}\times(\mathbf{H}\cap\mathbf{H}_1), \\ \infty & \text{otw.,} \end{cases} \tag{2.9a}$$

and for the density H we assume

$$\text{Continuity: } H \in C^0([0,1]\times\mathbb{R}^{d\times d};\mathbb{R})\,, \tag{2.9b}$$

$$\text{Growth: } \exists\, C_H > 0,\ \exists\, q_2 \in [1, q_*)\forall\,(\chi, D) \in [0,1]\times\mathbb{R}^{d\times d}:\ 0 \leq H(\chi, D) \leq C_H(|D|^{q_2}+1)\,, \tag{2.9c}$$

where $q_* = dq/(d-q)$ if $q < d$ and $q_* = \infty$ if $q \geq d$. Note that $q_2 \in [1, q_1]$ would be sufficient to ensure the integrability of H. But for the continuity of $\mathcal{H}$ it is required that $q_2 \in [1, q_*)$. Also note that the special choice (1.12) of H complies with the above assumptions with $q_2 \geq 2$.

3. Energetic solutions for the rate-independent system with damage and plasticity

In view of the positively 1-homogeneous character of the pseudo-potential of dissipation $\mathcal{R}$ from (1.10), system (1.15) is rate-independent. Therefore, for the analysis of the associated initial-boundary value problem we will resort to a weak solvability concept for rate-independent systems, namely the notion of energetic solution, cf. [MT04, Mie05]. In order to give it in the context of the present system with damage and plasticity we now introduce the energy functional, depending on $t \in [0, T]$ and on the state variables $(\mathbf{u}, \chi, D)$, and the dissipation potential associated with (1.15). In accordance with Sec. 2 we set

$$\mathcal{R} = \mathcal{R}_{\mathrm{inel}} + \mathcal{R}_{\mathrm{dam}} : \mathbf{V}\times\mathbf{Z} \to \mathbb{R}_\infty, \quad \text{where,} \tag{3.1a}$$

$$\mathcal{R}_{\mathrm{inel}} : \mathbf{V} \to [0,\infty), \quad \mathcal{R}_{\mathrm{inel}}(A) := \int_\Omega R_{\mathrm{inel}}(A)\,\mathrm{d}x \quad \text{with } R_{\mathrm{inel}}(A) := \mu|A|\,, \tag{3.1b}$$

$$\mathcal{R}_{\mathrm{dam}} : \mathbf{Z} \to [0,\infty], \quad \mathcal{R}_{\mathrm{dam}}(z) := \int_\Omega R_{\mathrm{dam}}(z)\,\mathrm{d}x \quad \text{with } R_{\mathrm{dam}}(z) := \nu|z| + I_{(-\infty,0]}(z)\,, \tag{3.1c}$$

$$\begin{aligned} \mathcal{E} : [0,T]\times\mathbf{U}\times\mathbf{Z}\times\mathbf{V} \to \mathbb{R}_\infty\,,\ \mathcal{E}(t, \mathbf{u}, \chi, D) :&= \mathcal{F}(\mathbf{u}, \chi, D) - \langle F(t), \mathbf{u}\rangle_{\mathbf{U}} \\ &= \mathcal{W}(t, \mathbf{u}, \chi, D) + \mathcal{J}(\chi) + \mathcal{G}(D) + \mathcal{H}(\chi, D)\,, \end{aligned} \tag{3.1d}$$

$$\mathcal{W} : [0,T]\times\mathbf{U}\times\mathbf{M}\times L^2(\Omega;\mathbb{R}^{d\times d}) \to \mathbb{R}\,, \tag{3.1e}$$

$$\begin{aligned} \mathcal{W}(t, \mathbf{u}, \chi, D) :=& \frac{1}{2}\int_\Omega \big(e(\mathbf{u}) + e_{\mathrm{D}}(t) - \Xi(\chi, D)\big) : \mathbb{K}(\chi) : \big(e(\mathbf{u}) + e_{\mathrm{D}}(t) - \Xi(\chi, D)\big)\,\mathrm{d}x \\ & - \langle F(t), \mathbf{u}\rangle_{\mathbf{U}}\,. \end{aligned}$$

In order to formulate the concept of energetic solution, we shall use the shorthand notation $\mathbf{q} = (\mathrm{u}, \chi, D)$ and set $\mathcal{Q} := \mathbf{U}\times\mathbf{Z}\times\mathbf{V}$, the state space where $\mathbf{q}$ varies, whereas $\mathcal{Z} := \mathbf{Z}\times\mathbf{V}$ stands for the space of the *dissipative* variables $z := (\chi, D)$. In this way, we shall now state the definition of energetic solutions in an abstract form.

Definition 3.1 (Energetic formulation of rate-independent processes)**.** *For the initial datum* $\mathbf{q}_0 \in \mathcal{Q}$ *find* $\mathbf{q}:[0,T]\to\mathcal{Q}$ *such that for all* $t \in [0,T]$ *the global stability* (3.2a) *and the global energy balance* (3.2b) *hold*

$$\text{Stability :}\quad \textit{for all } \tilde{\mathbf{q}} \in \mathcal{Q}: \quad \mathcal{E}(t,\mathbf{q}(t)) \le \mathcal{E}(t,\tilde{\mathbf{q}}) + \mathcal{R}(\tilde{\mathbf{q}}-\mathbf{q}(t)), \tag{3.2a}$$

$$\text{Energy balance :}\quad \mathcal{E}(t,\mathbf{q}(t)) + \mathrm{Diss}_{\mathcal{R}}(\mathbf{q},[s,t]) = \mathcal{E}(0,\mathbf{q}(0)) + \int_s^t \partial_t\mathcal{E}(\xi,\mathbf{q}(\xi))\,\mathrm{d}\xi \tag{3.2b}$$

with $\mathrm{Diss}_{\mathcal{R}}(\mathbf{q},[0,t]) := \sup\left\{\sum_{j=1}^N \mathcal{R}(\mathbf{q}(\xi_j)-\mathbf{q}(\xi_{j-1}))\,|\, s=\xi_0<\ldots<\xi_N=t,\, N\in\mathbb{N}\right\}$*, where* $\mathcal{R}(\mathbf{q}_1-\mathbf{q}_2)$ *has to be understood as* $\mathcal{R}(z_1-z_2)$ *with* $\mathbf{q}_i=(\mathbf{u}_i,z_i)$ *for* $i=1,2$.

The claim that (3.2) has to hold for all $t\in[0,T]$ entails that the energetic formulation is only solvable for initial data $\mathbf{q}_0$ which satisfy (3.2a) for $t=0$, which is equivalent to requiring that $\mathbf{q}_0$ solves the minimum problem $\min_{\tilde{\mathbf{q}}\in\mathcal{Q}}\{\mathcal{E}(0,\tilde{\mathbf{q}})+\mathcal{R}(\tilde{\mathbf{q}}-\mathbf{q}_0)\}$. For later convenience we introduce the set of stable states at time $t\in[0,T]$

$$\mathcal{S}(t) := \{\mathbf{q}\in\mathcal{Q},\ \mathbf{q} \text{ satisfies (3.2a) wrt. } \mathcal{E}(t,\cdot) \text{ and } \mathcal{R}\}\,. \tag{3.3}$$

A solution in terms of the energetic formulation is called an energetic solution to the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{R})$.

In what follows we will investigate the existence of energetic solutions (3.2) for the rate-independent system with damage and plasticity defined by the functionals $\mathcal{E}$ and $\mathcal{R}$ from (3.1) by verifying the assumptions of an abstract existence theorem given in [Mie11b], cf. also [MT04, Mie05, MRS08]. We now shortly recap this result, highlighting the role of a series of conditions on the driving functionals $\mathcal{E}$ and $\mathcal{R}$ enucleated below, cf. (3.4)–(3.9).

This result is proved by passing to the limit in a time-discretization scheme, where discrete energetic solutions are constructed via time-incremental minimization of a functional involving the sum of the dissipation potential $\mathcal{R}$ and the energy $\mathcal{E}$. The existence of minimizers follows from the *direct method*, provided that the energy functional $\mathcal{E}$ complies with a standard coercivity requirement, cf. (3.4a) ahead. It is shown that the discrete solutions fulfill the stability condition and a discrete energy *inequality*. The proof of the discrete stability relies on the fact that the dissipation distance $\mathcal{D}$ induced by $\mathcal{R}$ complies with the triangle inequality, see (3.8a) later on. From the discrete energy inequality all a priori estimates are derived. For this, a crucial role is played by a condition ensuring that the power of the external forces $\partial_t\mathcal{E}$ is controlled by the energy $\mathcal{E}$ itself, cf. (3.4b) below, so that the last integral term on the right-hand side of (3.2b) is estimated in terms of the energy, and Gronwall's lemma can be applied. All in all, $\mathcal{E}$ has to satisfy the following properties:

Compactness of energy sublevels: $\forall\, t\in[0,T]\ \forall\, E\in\mathbb{R}$:
$L_E(t) := \{\mathbf{q}\in\mathcal{Q}\,|\,\mathcal{E}(t,\mathbf{q})\le E\}$ is weakly seq. compact. (3.4a)

Uniform control of the power:
$\exists\, c_0\in\mathbb{R}\ \exists\, c_1>0\ \forall\,(t,\mathbf{q})\in[0,T]\times\mathcal{Q}$ with $\mathcal{E}(t,\mathbf{q})<\infty$:
$\mathcal{E}(\cdot,\mathbf{q})\in C^1([0,T])$ and $|\partial_t\mathcal{E}(t,\mathbf{q})|\le c_1(c_0+\mathcal{E}(t,\mathbf{q}))$. (3.4b)

Remark 3.2. Observe that condition (3.4b) in fact guarantees a Lipschitz estimate for $\mathcal{E}$ with respect to time via Gronwall's lemma, namely

$$|\mathcal{E}(t,\mathbf{q})-\mathcal{E}(s,\mathbf{q})| \le \left(\mathrm{e}^{c_1|t-s|}-1\right)(\mathcal{E}(t,\mathbf{q})+c_0) \le \mathrm{e}^{c_1 T}(\mathcal{E}(t,\mathbf{q})+c_0)|t-s|\,. \tag{3.5}$$

Hence, if $\mathcal{E}(t,\mathbf{q})<E$ for $E\in\mathbb{R}$, then, for $c_E := \mathrm{e}^{c_1T}(E+c_0)$, estimate (3.5) implies

$$|\mathcal{E}(t,\mathbf{q})-\mathcal{E}(s,\mathbf{q})| \le c_E|t-s|\,. \tag{3.6}$$

As for the dissipation potential $\mathcal{R}$, we require that the induced dissipation distance

$$\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty] \quad \text{given by} \quad \mathcal{D}(z, \tilde{z}) := \mathcal{R}(\tilde{z} - z) \quad \text{for all } z, \tilde{z} \in \mathcal{Z}, \tag{3.7}$$

fulfills

$$\textit{Quasi-distance:} \quad \forall\, z_1, z_2, z_3 \in \mathcal{Z} : \quad \begin{array}{l} \mathcal{D}(z_1, z_2) = 0 \ \Leftrightarrow z_1 = z_2 \ \text{ and} \\ \mathcal{D}(z_1, z_3) \leq \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3); \end{array} \tag{3.8a}$$

$$\begin{array}{l} \textit{Semi-continuity:} \\ \qquad \mathcal{D} : \ \mathcal{Z} \times \mathcal{Z} \to [0, \infty] \text{ is weakly sequentially lower semi-continuous.} \end{array} \tag{3.8b}$$

The abstract existence proof then consists in passing to the limit in the discrete energy inequality by lower semicontinuity arguments, leading to an upper energy estimate, and in the discrete stability condition, leading to (3.2a). The lower energy estimate which ultimately yields the energy balance (3.2b) then follows from a by now classical procedure, based on the combination of the previously proved (3.2a) with a Riemann-sum argument. For the limit passage in the discrete energy inequality and in the discrete stability, the following compatibility conditions are required: *For every stable sequence* $(t_k, \mathbf{q}_k)_{k\in\mathbb{N}}$ *with* $t_k \to t$, $\mathbf{q}_k \rightharpoonup \mathbf{q}$ *in* $[0, T] \times \mathcal{Q}$ *we have*

$$\text{Convergence of the power of the energy:} \quad \partial_t \mathcal{E}(t_k, \mathbf{q}_k) \to \partial_t \mathcal{E}(t, \mathbf{q}), \tag{3.9a}$$

$$\text{Closedness of sets of stable states:} \quad \mathbf{q} \in \mathcal{S}(t). \tag{3.9b}$$

With these prerequisites at hand the abstract existence result reads as follows:

Theorem 3.3 (Abstract main existence theorem [Mie11b])**.** *Let the rate-independent system* $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ *satisfy conditions* (3.4) *and* (3.8)*. Moreover, let the compatibility conditions* (3.9) *hold.*
Then, for each $\mathbf{q}_0 \in \mathcal{S}(0)$ *there exists an energetic solution* $\mathbf{q} : [0, T] \to \mathcal{Q}$ *for* $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ *satisfying* $\mathbf{q}(0) = \mathbf{q}_0$.

From Thm. 3.3 we will derive our own existence result for the rate-independent system with damage and plasticity.

Theorem 3.4 (Existence of energetic solutions for the rate-independent system from (3.1))**.** *Let the assumptions* (2.1) *and* (2.3)–(2.9) *stated in Sec. 2 be satisfied. Then the rate-independent system for damage and plasticity* $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ *given by* (3.1) *satisfies the properties* (3.4)*,* (3.8) *&* (3.9)*, and hence, for each* $\mathbf{q}_0 \in \mathcal{Q}$ *with* $\mathbf{q}_0 \in \mathcal{S}(0)$ *it admits an energetic solution in the sense of Def. 3.1.*

4. Proof of Theorem 3.4

In the following lines, we shall denote by the symbols c, $\tilde{c}$, C, $\tilde{C}$ various positive constants depending only on known quantities.

It is immediate to observe that the dissipation distance generated by the potential $\mathcal{R}$ from (3.1c) via formula (3.7) satisfies the abstract condition (3.8). Thus, it remains to verify that the energy functional $\mathcal{E}$ from (3.1d) satisfies the basic properties (3.4). In addition, the compatibility conditions (3.9) have to be deduced.

To this aim, we start with verifying the following regularity property for the inelastic strain.

Lemma 4.1. *Let* (2.4) *and* (2.8) *hold true, let* $\alpha \in [1, \infty)$*. Then,* $\Xi : \mathbf{M} \times L^2(\Omega; \mathbf{S}) \to L^2(\Omega; \mathbf{S})$ *is continuous wrt. the* $L^\alpha(\Omega) \times L^2(\Omega; \mathbf{S})$*-topology.*

Proof. Consider $(\chi_k, D_k)_k \subset \mathbf{M} \times L^2(\Omega; \mathbf{S})$ such that $(\chi_k, D_k)_k \to (\chi, D)$ in $L^\alpha(\Omega) \times L^2(\Omega; \mathbf{S})$. Hence, up to a subsequence we find that $(\chi_k, D_k)_k \to (\chi, D)$ pointwise a.e. in Ω. Thanks to (2.4a) we find that $|\Xi(\chi_k, D_k)|^2 \to |\Xi(\chi, D)|^2$ pointwise a.e. in Ω. Moreover, (2.4b) ensures $|\Xi(\chi_k, D_k)|^2 \leq |\Xi(0, D_k)|^2 = |D_k|^2$ for all $k \in \mathbb{N}$, which thus serves as a convergent majorant. Hence, $\Xi(\chi_k, D_k) \to \Xi(\chi, D)$ in $L^2(\Omega; \mathbb{R}^{d\times d})$ by the dominated convergence theorem. ∎

The above continuity property is an important ingredient for the verification of the following properties of the functional $\mathcal{W}$:

Lemma 4.2 (Properties of $\mathcal{W}$)**.** *The functional* $\mathcal{W} : [0,T] \times \mathbf{U} \times \mathbf{M} \times L^2(\Omega;\mathbf{S}) \to \mathbb{R}$ *from* (3.1e) *has the following properties:*

$$\begin{aligned} &\textit{bound from below:} && \exists\, c_W, \tilde{c}_W, C_W > 0,\ \forall (t,\mathbf{u},\chi,D) \in [0,T] \times \mathbf{U} \times (\mathbf{X} \cap \mathbf{M}) \times \mathbf{H} : \\ & && \mathcal{W}(t,\mathbf{u},\chi,D) \geq c_W \|\mathbf{u}\|^2_{\mathbf{U}} - \tilde{c}_W \|D\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} - C_W\,, \qquad (4.1) \\ &\textit{lower semicontinuity:} && \mathcal{W}(t,\cdot,\cdot,\cdot) \textit{ is lower semicontinuous wrt. the weak convergence in } \mathbf{U} \\ & && \textit{and strong convergence in } L^\alpha(\Omega) \times L^2(\Omega;\mathbf{S}) \textit{ for all } \alpha \in [1,\infty) \qquad (4.2) \end{aligned}$$

and for every $t \in [0,T]$*. Moreover, for all* $(t,\mathbf{u},\chi,D) \in [0,T] \times \mathbf{U} \times \mathbf{X} \cap \mathbf{M} \times (\mathbf{H} \cap \mathbf{H}_1)$ *the partial time-derivative* $\partial_t \mathcal{W}$ *is given by*

$$\partial_t \mathcal{W}(t,\mathbf{u},\chi,D) := \int_\Omega \big(e(\mathbf{u}) + e_{\mathrm{D}}(t) - \Xi(\chi,D)\big) : \mathbb{K}(\chi) : \partial_t e_{\mathrm{D}}(t)\,\mathrm{d}x - \langle \partial_t F(t), \mathbf{u}\rangle_{\mathbf{U}} \qquad (4.3)$$

and $\mathcal{W} + \mathcal{G}$ *satisfies relation* (3.4b).

Proof. We split the proof in several steps.

Bound from below (4.1)**:** Thanks to the positive definiteness of $\mathbb{K}$, the bounds on the given data F, g, cf. (2.3), and the properties of Ξ from (2.4), also using Young's and Korn's inequality as well as estimate (2.5), we find for all $(t,\mathbf{u},\chi,D) \in [0,T] \times \mathbf{U} \times (\mathbf{X} \cap \mathbf{M}) \times (\mathbf{H} \cap \mathbf{H}_1)$

$$\begin{aligned} \mathcal{W}(t,\mathbf{u},\chi,D) &\geq K_1 \|e(\mathbf{u}) + e_{\mathrm{D}} - \Xi(\chi,D)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} - C_F \|\mathbf{u}\|_{\mathbf{U}} \\ &\geq K_1 \big(\|e(\mathbf{u})\|_{L^2} - (C_{\mathrm{D}} + \|\Xi(\chi,D)\|_{L^2})\big)^2 - C_F \|\mathbf{u}\|_{\mathbf{U}} \\ &\overset{(2.5)}{\geq} \tfrac{K_1}{2} \|e(\mathbf{u})\|^2_{L^2} - 2(C_{\mathrm{D}} + \|D\|_{L^2})^2 - C_F \|\mathbf{u}\|_{\mathbf{U}} \\ &\geq \tfrac{K_1 C_{\mathrm{K}}^2}{2} \|\mathbf{u}\|^2_{\mathbf{U}} - 4(C_{\mathrm{D}}^2 + \|D\|^2_{L^2}) - \tfrac{K_1 C_{\mathrm{K}}^2}{4} \|\mathbf{u}\|^2_{\mathbf{U}} - \tfrac{C_F^2}{K_1 C_{\mathrm{K}}^2}, \end{aligned} \qquad (4.4)$$

using the short-hand $\|\cdot\|_{L^2}$ for $\|\cdot\|_{L^2(\Omega;\mathbb{R}^{d\times d})}$, and with C_{K} the constant in Korn's inequality. This proves (4.1).

Lower semicontinuity (4.2)**:** For every $t \in [0,T]$ we observe that $\mathcal{W}(t,\cdot,\cdot,\cdot)$ is continuous wrt. the strong convergence in $\mathbf{U} \times L^\alpha(\Omega) \times L^2(\Omega;\mathbf{S})$, also due to estimate (2.5). Moreover, for every (t,χ,D) in $[0,T] \times (L^\alpha(\Omega) \cap \mathbf{M}) \times L^2(\Omega;\mathbf{S})$ the functional $\mathcal{W}(t,\cdot,\chi,D) : \mathbf{U} \to \mathbb{R}$ is convex and we have $\mathcal{W}(t,\mathbf{u},\chi,D) \geq -K_1 \int_\Omega |D||e(\mathbf{u}) + e_{\mathrm{D}}(t)|\mathrm{d}x - \langle F(t), \mathbf{u}(t)\rangle_{\mathbf{U}}$. Hence, [FL07, p. 492, Thm. 7.5] guarantees the lower semicontinuity statement (4.2).

(4.3) **& relation** (3.4b)**:** Formula (4.3) ensues from a direct calculation, taking into account that $\mathcal{W}(t,\cdot,\chi,D)$ is Fréchet-differentiable in $\mathbf{U}$, as well as the regularity properties of F and $\mathbf{u}_{\mathrm{D}}$, cf. (2.3). Note now that $\partial_t(\mathcal{W} + \mathcal{G}) = \partial_t \mathcal{W}$. In order to find the bound (3.4b) on $|\partial_t \mathcal{W}(t,\mathbf{u},\chi,D)|$ we make use of the growth properties of $\mathbb{K}$, cf. (2.3), Hölder's and Young's inequality, and exploit the already deduced bound (4.1). This yields

$$\begin{aligned} \big|\partial_t \mathcal{W}(t,\mathbf{u},\chi,D)\big| &\leq K_2 C_{\mathrm{D}} \big(\|e(\mathbf{u})\|_{L^2} + \|e_{\mathrm{D}}\|_{L^2} + \|\Xi(\chi,D)\|_{L^2}\big) + \tfrac{1}{2}(C_F^2 + \|\mathbf{u}\|^2_{\mathbf{U}}) \\ &\leq C\|\mathbf{u}\|^2_{\mathbf{U}} + c\|D\|^2_{L^2} + C_3 \\ &\leq \tilde{C}\mathcal{W}(t,\mathbf{u},\chi,D) + \tilde{c}\|D\|^{q_1}_{L^{q_1}} + C_4 \\ &\leq C_5\big(\mathcal{W}(t,\mathbf{u},\chi,D) + \mathcal{G}(D)\big) + C_6\,, \end{aligned}$$

where the last-but-one estimate follows (4.4), and the last one from the coercivity properties of $\mathcal{G}$, cf. (4.8b). This finishes the proof of (3.4b). ∎

Lemma 4.3 (Weak lower semicontinuity)**.** *Let* $\mathbf{B}_1 \subset \mathbf{B}_2$ *with a continuous embedding be separable Banach spaces and* $\mathbf{B}_1$ *reflexive. Assume that the functional* $\mathcal{E}_1 : \mathbf{B}_1 \to \mathbb{R}$ *is weakly sequentially* *lower semicontinuous and coercive. Then the extended functional $\mathcal{E}_2 : \mathbf{B}_2 \to \mathbb{R}_\infty$ is also weakly sequentially lower semicontinuous, where*

$$\mathcal{E}_2(v) := \begin{cases} \mathcal{E}_1(v) & \text{if } v \in \mathbf{B}_1, \\ \infty & \text{if } v \in \mathbf{B}_2 \backslash \mathbf{B}_1. \end{cases} \tag{4.5}$$

Proof. Consider a sequence $v_k \rightharpoonup v$ in $\mathbf{B}_2$. If $v_k \in \mathbf{B}_2\backslash\mathbf{B}_1$ for $k \in \mathbb{N}$ except of a finite number of indices, then $\mathcal{E}_2(v) \le \liminf_{k\to\infty} \mathcal{E}_2(v_k) = \infty$. Also, if $\|v_k\|_{\mathbf{B}_1} \to \infty$ for any subsequence, then $\mathcal{E}_2(v) \le \liminf_{k\to\infty} \mathcal{E}_2(v_k) = \infty$. Hence, in these two cases there is nothing to prove. Thus, assume that $\|v_k\|_{\mathbf{B}_1} \le C$ for a not relabelled subsequence $(v_k)_k$ and a constant $C > 0$. From the reflexivity of $\mathbf{B}_1$ we now conclude that there is a further, not relabelled subsequence and an element $\tilde{v} \in \mathbf{B}_1$ such that $v_k \rightharpoonup \tilde{v}$ in $\mathbf{B}_1$. By the uniqueness of the limit in $\mathbf{B}_2 \supset \mathbf{B}_1$ we have that $\tilde{v} = v$. Now the weak lower semicontinuity of $\mathcal{E}_1 : \mathbf{B}_1 \to \mathbb{R}$ implies that $\mathcal{E}_2(v) = \mathcal{E}_1(v) \le \liminf_{k\to\infty} \mathcal{E}_1(v_k) \le \liminf_{k\to\infty} \mathcal{E}_2(v_k)$, which proves the assertion. ∎

Lemma 4.4 (Properties of $\mathcal{H}$, $\mathcal{J}$, and $\mathcal{G}$)**.** *Let the coupling term $\mathcal{H}$ be defined as in* (2.9) *and let $\alpha \in [1, \infty)$. Then $\mathcal{H}$ has the following properties:*

$$\begin{aligned} &\textit{proper domain:} && \operatorname{dom}\mathcal{H} = \mathbf{M} \times (\mathbf{H}\cap\mathbf{H}_1)\,, && (4.6a)\\ &\textit{bound from below:} && \forall(\chi, D) \in \mathbf{Z} \times \mathbf{V} :\ \mathcal{H}(\chi, D) \ge 0\,, && (4.6b)\\ &\textit{lower semicontinuity:} && \mathcal{H} : \mathbf{Z} \times \mathbf{V} \to [0, \infty] \textit{ is lower semicontinuous wrt. the} && (4.6c)\\ & && \textit{strong convergence in } L^\alpha(\Omega) \times L^{q_2}(\Omega, \mathbb{R}^{d\times d}) \textit{ for every } \alpha \in [1, \infty)\,. \end{aligned}$$

Let the damage regularization $\mathcal{J}$ be given by (2.6a)*. Then, the following properties hold true:*

$$\begin{aligned} &\textit{proper domain:} && \operatorname{dom}\mathcal{J} = (\mathbf{X} \cap \mathbf{M})\,, && (4.7a)\\ &\textit{bound from below:} && \exists\, c_\mathcal{J}, C_\mathcal{J} > 0,\ \forall\chi \in (\mathbf{X} \cap \mathbf{M}) :\ \mathcal{J}(\chi) \ge c_\mathcal{J}\|\chi\|^r_\mathbf{X} - C_\mathcal{J}\,, && (4.7b)\\ &\textit{lower semicontinuity:} && \mathcal{J} \textit{ is lower semicontinuous wrt. the weak convergence in } \mathbf{X}\,. && (4.7c) \end{aligned}$$

Let the plastic regularization $\mathcal{G}$ be given as in (2.7a)*. Then, the following properties are satisfied:*

$$\begin{aligned} &\textit{proper domain:} && \operatorname{dom}\mathcal{G} \subset (\mathbf{H}\cap\mathbf{H}_1) \textit{ is a closed, convex subset}\,, && (4.8a)\\ &\textit{bound from below:} && \exists\, c_\mathcal{G}, C_\mathcal{G} > 0,\ \forall D \in \operatorname{dom}\mathcal{G} :\ \mathcal{G}(D) \ge c_\mathcal{G}\|D\|^q_\mathbf{H} - C_\mathcal{G}\,, && (4.8b)\\ & && \mathcal{G}(D) \ge c_\mathcal{G}\|D\|^{q_1}_{\mathbf{H}_1} - C_\mathcal{G}\,,\\ &\textit{lower semicontinuity:} && \mathcal{G} \textit{ is lower semicontinuous wrt. weak convergence in } (\mathbf{H} \cap \mathbf{H}_1)\,. && (4.8c) \end{aligned}$$

Hence, the functionals $\mathcal{J} : \mathbf{Z} \to \mathbb{R}_\infty$ and $\mathcal{G} : \mathbf{V} \to \mathbb{R}_\infty$ are weakly sequentially lower semicontinuous.

Proof. We split the proof in several steps.

(4.6)**:** The domain property (4.6a) and the boundedness from below are a direct consequence of definition (2.9a) and (2.9c). The lower semicontinuity can be concluded from the continuity (2.9b) and the growth property (2.9c) as follows. Given a sequence $(\chi_k, D_k)_k \subset (\mathbf{Z}\backslash\mathbf{M}) \times \mathbf{V}$, with $(\chi_k, D_k) \to (\chi, D)$ in $L^\alpha(\Omega)\times L^{q_2}(\Omega, \mathbb{R}^{d\times d})$, we immediately find that $\infty = \liminf_{k\to\infty} \mathcal{H}(\chi_k, D_k) \ge \mathcal{H}(\chi, D)$, while no matter occurs if $(\chi, D) \in \mathbf{M} \times \mathbf{V}$. Hence assume that there is a (not relabelled) subsequence $(\chi_k, D_k)_k \subset \mathbf{M} \times \mathbf{V}$ such that $(\chi_k, D_k) \to (\chi, D)$ in $L^\alpha(\Omega) \times L^{q_2}(\Omega, \mathbb{R}^{d\times d})$. Upon extraction of a further subsequence that converges pointwise a.e. in Ω we find that the limit $(\chi, D) \in \mathbf{M} \times \mathbf{V}$. Moreover, thanks to the continuity (2.9b) we have that $H(\chi_k, D_k) \to H(\chi, D)$ a.e. in Ω along this subsequence. In addition, for each $k \in \mathbb{N}$, the growth property (2.9c) guarantees that $C_H(|D|^{q_2} + 1)$ is a convergent majorant of $H(\chi_k, D_k)$, so that the convergence of the respective integral terms is implied by the dominated convergence theorem. Thus, altogether, we have verified the lower semicontinuity property stated in (4.6c).

(4.7) **&** (4.8)**:** Properties (4.7a) and (4.8a) are implied by (2.6a)& (2.6c), and by (2.7a)&(2.7c), respectively. The bounds (4.7b) and (4.8b) immediately follow from the growth properties (2.6c) and (2.7c). Invoking [FL07, p. 492, Thm. 7.5], the latter, together with the continuity (2.6b), resp. (2.7b) and the convexity property (2.6d), resp. (2.7d), also ensure the weak sequential lower semicontinuity.

The last statement of the lemma follows from (4.7c) and (4.8c) as a direct consequence of Lemma 4.3. ∎

We are now in the position to conclude properties (3.4) for $\mathcal{E}$ in consequence of Lemmata 4.1–4.4.

Corollary 4.5 (Properties (3.4) of $\mathcal{E}$)**.** *Let the functional* $\mathcal{E} : [0,T]\times\mathbf{U}\times\mathbf{Z}\times\mathbf{V}\to\mathbb{R}_\infty$ *be defined as in* (3.1d)*. Let the assumptions of Lemmata 4.1–4.4 hold true. Then, the functional* $\mathcal{E}$ *satisfies properties* (3.4)*.*

Proof. We split the proof in several steps.

Compactness of the sublevels (3.4a)**:** Comparing (4.1) with (4.8b), using Hölder's and Young's inequality, we first deduce for D:

$$-\bar c_W\|D\|^2_{L^2(\Omega,\mathbb{R}^{d\times d})}\ge-\|D\|^2_{\mathbf{H}_1}\bar c_W\mathcal{L}^d(\Omega)^{(q_1-2)/q_1}$$
$$\ge-\tfrac{2c_\mathcal{G}}{q_1}\|D\|^{q_1}_{\mathbf{H}_1}-\tfrac{q_1-2}{q_1}\big(c_\mathcal{G}^{-2/q_1}\bar c_W\mathcal{L}^d(\Omega)^{(q_1-2)/q_1}\big)^{(q_1-2)/q_1},$$

where $2/q_1<1$ according to (2.7c). Thus, combining bounds (4.1), (4.6b), (4.7b), and (4.8b) yields that $\mathcal{E}$ has bounded sublevels in $\mathbf{U}\times\mathbf{X}\times(\mathbf{H}\cap\mathbf{H}_1)$. Since this space is reflexive, the sublevels are then sequentially weakly compact, and so they are in $\mathcal{Q}=\mathbf{U}\times\mathbf{Z}\times\mathbf{V}$.

Uniform control of the power (3.4b)**:** Since $\partial_t\mathcal{E}(t,\mathbf{u},\chi,D)=\partial_t\mathcal{W}(t,\mathbf{u},\chi,D)$, given by (4.3), the last statement of Lemma 4.2 ensures (3.4b) for $\mathcal{E}$ upon adding $\mathcal{H}(\chi,D)>0$ and $\mathcal{J}(\chi)+C_\mathcal{J}>0$. ∎

Finally, we verify that the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{R})$ for plasticity and damage satisfies the compatibility conditions (3.9).

Proposition 4.6 (Compatibility conditions (3.9))**.** *Let the assumptions of Theorem 3.4 hold true. Then the rate-independent system* $(\mathcal{Q},\mathcal{E},\mathcal{R})$ *for plasticity and damage satisfies the compatibility conditions* (3.9)*.*

Proof. In view of (3.6) we infer for any stable sequences $(t_k,\mathbf{u}_k,\chi_k,D_k)_k\subset\mathcal{S}(t)$ that there is a constant $E>0$ such that this sequence belongs to same the energy sublevel $L_E(t)$, which is bounded in $\mathbf{U}\times\mathbf{X}\times(\mathbf{H}\cap\mathbf{H}_1)$ as guaranteed by Cor. 4.5. Hence, we deduce the following convergence properties along a (not relabelled) subsequence:

$$\mathbf{u}_k\rightharpoonup\mathbf{u}\quad\text{in }\mathbf{U},\tag{4.9a}$$
$$D_k\rightharpoonup D\quad\text{in }\mathbf{H},\tag{4.9b}$$
$$D_k\to D\quad\text{in }L^{q_1}(\Omega,\mathbb{R}^{d\times d})\cap L^{q_2}(\Omega,\mathbb{R}^{d\times d}),\tag{4.9c}$$
$$\chi_k\rightharpoonup\chi\quad\text{in }\mathbf{X},\tag{4.9d}$$
$$\chi_k\to\chi\quad\text{in }L^\alpha(\Omega)\text{ for any }\alpha\in[1,\infty).\tag{4.9e}$$

(3.9a) **convergence of the power** $\partial_t\mathcal{E}(t_k,q_k)$**:** In view of the above convergences, property (3.9a) can be concluded from weak-strong convergence arguments using that $\partial_te_\mathrm{D}(t_k)\to\partial_te_\mathrm{D}(t)$ strongly in $\mathbf{U}$ and $\partial_tF(t_k)\to\partial_tF(t)$ strongly in $\mathbf{U}^*$ thanks to the regularity assumptions (2.3).

Closedness of sets of stable states (3.9b)**:** In order to deduce (3.9b), we make use of the so-called mutual recovery condition, i.e. *for every sequence* $(\mathbf{u}_k,\chi_k,D_k)_k\subset\mathcal{S}(t)$ *converging to a limit* $(\mathbf{u},\chi,D)$ *in the sense of* (4.9)*, and any competitor* $(\hat{\mathbf{u}},\hat\chi,\hat D)$*, it must be possible to construct a mutual recovery sequence* $(\hat{\mathbf{u}}_k,\hat\chi_k,\hat D_k)_k$ *such that*

$$\limsup_{k\to\infty}\big(\mathcal{E}(t,\hat q_k)-\mathcal{E}(t,q_k)+\mathcal{R}(\hat q_k-q_k)\big)\le\mathcal{E}(t,\hat q)-\mathcal{E}(t,q)+\mathcal{R}(\hat q-q),\tag{4.10}$$

where we again abbreviated $\hat q_k=(\hat{\mathbf{u}}_k,\hat\chi_k,\hat D_k)$*, etc..*

Let $\hat{q} = (\hat{\mathbf{u}}, \hat{\chi}, \hat{D})$ such that $\mathcal{E}(t, \hat{q}) < \infty$. Then, a suitable recovery sequence is defined by

$$\hat{\mathbf{u}}_k := \hat{\mathbf{u}}\,, \tag{4.11a}$$
$$\hat{\chi}_k := \min\{\chi_k, \max\{0, \hat{\chi} - \delta_k\}\}\,, \tag{4.11b}$$
$$\hat{D}_k := \hat{D}\,, \tag{4.11c}$$

where δ_k in (4.11b) is suitably chosen in dependence of $\|\chi_k - \chi\|_{\mathbf{X}}$ such that $\delta_k \to 0$ as $k \to \infty$, see [TM10] for the details. We refer to [LRTT14, Thm. 4.5] for the proof of the following convergence property:

$$\hat{\chi}_k \rightharpoonup \hat{\chi} \text{ in } \mathbf{X} \text{ as well as } \limsup_{k\to\infty} \big(\mathcal{J}(\hat{\chi}_k) - \mathcal{J}(\chi_k)\big) \leq \mathcal{J}(\hat{\chi}) - \mathcal{J}(\chi)\,. \tag{4.12}$$

The convergence stated in (4.12) together with (4.11c) yields that $\mathcal{R}(\hat{\chi}_k - \chi_k, \hat{D}_k - D_k) \to \mathcal{R}(\hat{\chi} - \chi, \hat{D} - D)$. Moreover, upon choosing a further subsequence $(\hat{\mathbf{u}}_k, \hat{\chi}_k, \hat{D}_k)_k$, which converges pointwise a.e. in Ω, and by making use of the bounds (2.3), we may conclude via the dominated convergence theorem, also taking into account the growth properties (2.9c) of $\mathcal{H}$, that

$$\mathcal{W}(t, \hat{\mathbf{u}}_k, \hat{\chi}_k, \hat{D}_k) \to \mathcal{W}(t, \hat{\mathbf{u}}, \hat{\chi}, \hat{D}) \text{ and } \mathcal{H}(\hat{\chi}_k, \hat{D}_k) \to \mathcal{H}(\hat{\chi}, \hat{D}),$$

whereas we clearly have $\mathcal{G}(\hat{D}_k) \to \mathcal{G}(\hat{D})$. The respective expressions for $(\mathbf{u}_k, \chi_k, D_k)$ can be handled by weak lower semicontinuity. Ultimately, we conclude (4.10), which finishes the proof. ∎

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