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INVARIANCE PROPERTIES IN THE DYNAMIC GAUSSIAN COPULA MODEL *

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Abstract. Based on Gaussian tail distribution estimates of independent interest, we study the mathematical properties of the default times (or any of their minima) in the dynamic Gaussian copula model. In particular, depending on the value of the correlation parameter ϱ in the model, the so-called invariance property of Crépey and Song (2017) may be satisfied or not. This gives together an example of a model where the invariance property is satisfied but immersion does not hold, for small ϱ, and, for larger ϱ, an example of a model where the invariance property may not be satisfied.

Keywords: counterparty credit risk, wrong-way risk, Gaussian copula, dynamic copula, immersion property, invariance time, CDS.

Mathematics Subject Classification: 91G40, 60G07.

1. Introduction

Crépey, Jeanblanc, and Wu (2013) (in book form Crépey, Bielecki, and Brigo (2014, Chapter 7)) proposed a dynamic version (DGC) of the Gaussian copula model. Related models include the one-period Merton model of Fermanian and Vigneron (2015, Section 6) or other variants used in credit and counterparty risk softwares.

As demonstrated in Crépey et al. (2014, Section 7.3.3), this model yields a dynamic meaning to the ad hoc bump sensitivities that were used by traders for hedging CDO tranches by CDS contracts before the subprime crisis.

From a more topical perspective, the DGC model can be used for dealing with counterparty risk on credit derivatives (notably, portfolios of CDS contracts) traded between a bank and its counterparty, respectively labeled as -1 and 0, and referencing credit names 1 to n, for some positive integer n. Accordingly, we introduce

$$N = \{-1, 0, 1, \ldots, n\} \text{ and } N^{\star} = \{1, \ldots, n\}$$

and we focus on $\tau = \tau_{-1} \wedge \tau_0$ in the paper. However, analog properties hold for any minimum of the τ_i and, in particular, for the τ_i themselves.

In particular, the default intensities of the surviving names and therefore the value of credit protection spike at default times in the DGC model, as observed in practice. This nonimmersive feature of the model is in line with the wrong-way risk feature of counterparty risk embedded in credit derivatives, which is the adverse dependence between the default risk of a counterparty and an underlying credit derivative exposure.

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To illustrate this numerically, Figure 1 shows the valuation adjustment accounting for counterparty and funding risks (total valuation adjustment TVA) embedded in one CDS between a bank and its counterparty on a third reference firm (hence for $n = 1$).

The left graph displays the TVA computed as a function of the correlation parameter ϱ in a DGC model of the three credit names (hence $n = 1$): the bank, its counterparty and the reference credit name of the CDS. The different curves correspond to different levels of credit spread $\bar{\lambda}$ of bank: the higher $\bar{\lambda}$, the higher the funding costs for the bank, resulting in higher TVAs. All the TVA numbers are computed by a Monte Carlo scheme dubbed "FT scheme of order 3" in Crépey and Nguyen (2016, Section 6.1). FT refers to Fujii and Takahashi (2012a,b). The numerical parameters are set as in Crépey and Nguyen (2016, Section 6.1), to which we refer the reader for a complete description of the CDS contract, of the FT numerical scheme and of other numerical experiments involving CDS portfolios (as opposed to a single contract here).

The right panel of Figure 1 shows the analog of the left graph, but in a fake DGC model, where we deliberately ignore the impact of the default of the counterparty in the valuation of the CDS at time $\tau_{-1} \wedge \tau_0$ (technically, in the notation of Crépey and Song (2016, Equation (6.7)), we replace $(\widetilde{P}^e_t + \widetilde{\Delta}^e_t)$ by P_{t-} in the coefficient $\widehat{f}$), in order to kill the wrong-way risk feature of the DGC model. We can see from the figure that, for large ϱ, the corresponding fake TVA numbers are five to ten times smaller than the "true" TVA levels that can be seen in the left panel. In addition of being much smaller for large ϱ, the fake DGC TVA numbers in the right panel are mostly decreasing with ϱ. This shows that the wrong-way risk feature of the DGC model is indeed responsible for the "systemic" increasing pattern observed in the left panel.

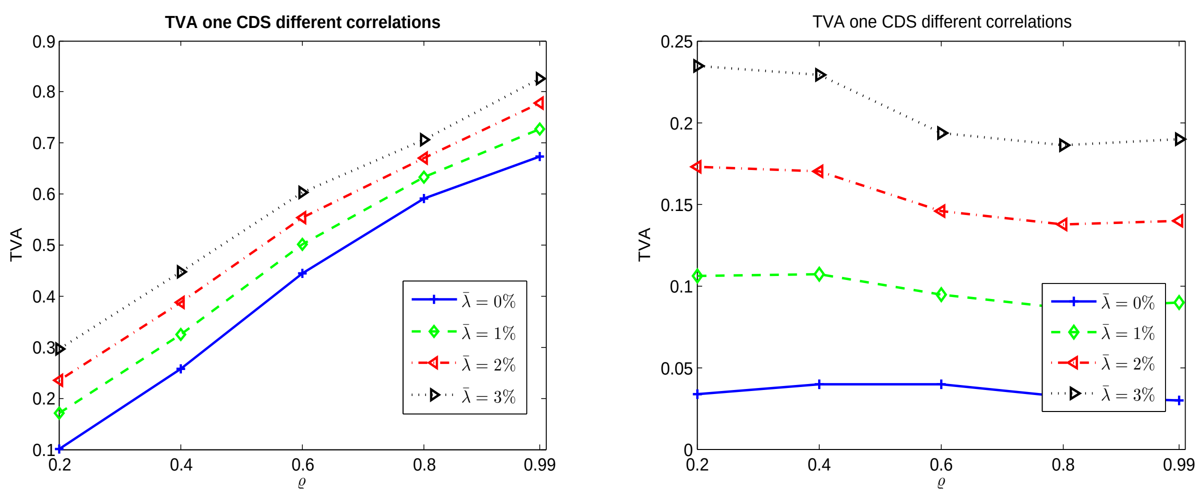

FIGURE 1. *Left*: TVA on one CDS as a function of the correlation parameter ϱ and for different bank credit spreads $\bar{\lambda}$, in a DGC model with three names: the bank, its counterparty and the reference credit name of the CDS. *Right*: Analog results in a fake DGC model without wrong-way-risk.

The dynamic Gaussian copula model has been assessed from an engineering perspective in previous work. The present paper deals with the corresponding mathematics, including explicit computations for the main model primitives and a study of the so-called invariance property as per Crépey and Song (2017). A major technical difficulty in this last regard stems from the explosion of the auxiliary functions h_l that are required for defining the default times in the model. This is addressed by means of Gaussian tail moment estimates of independent interest. We conclude that, depending on the value of the correlation parameter ϱ in the model, the so-called invariance property as per Crépey and Song (2017) may be satisfied or not: This gives together an example of a model where the invariance property is satisfied but immersion does not hold, for small ϱ, and, for larger ϱ, an example of a model where the invariance property may not be satisfied.

1.1. **Invariance Times and Probability Measures**

We work on a filtered probability space $(\Omega, \mathbb{G}, \mathcal{A}, \mathbb{Q})$. Given a $\mathbb{G}$ stopping time τ and a subfiltration $\mathbb{F}$ of $\mathbb{G}$, $\mathbb{F}$ and $\mathbb{G}$ satisfying the usual conditions, let J and S denote the survival indicator process of τ and its optional projection known as the Azéma supermartingale of τ, i.e.

$$J_t = \mathbb{1}_{\{\tau>t\}},\ \ S_t = {}^oJ_t = \mathbb{Q}(\tau > t\,|\,\mathcal{F}_t),\ \ t \geq 0.$$

The following conditions are studied in Song (2014a) and Crépey and Song (2015, 2017).

Condition (B). Any $\mathbb{G}$ predictable process U admits an $\mathbb{F}$ predictable reduction, i.e. an $\mathbb{F}$ predictable process, denoted by U', that coincides with U on $]\!]0, \tau]\!]$.

For any left-limited process Y, we denote by $Y^{\tau-} = JY + (1 - J)Y_{\tau-}$ the process Y stopped before τ.

Condition (A). Given a constant time horizon $T > 0$, there exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that $(\mathbb{F}, \mathbb{P})$ local martingales stopped before τ are $(\mathbb{G}, \mathbb{Q})$ local martingales on $[0, T]$.

If the conditions (B) and (A) are satisfied, then we say that τ is an invariance time and $\mathbb{P}$ is an invariance probability measure. If, in addition, $S_T > 0$ almost surely, then $\mathbb{F}$ predictable reductions are uniquely defined on $(0, T]$ and any inequality between two $\mathbb{G}$ predictable processes on $(0, \tau]$ implies the same inequality between their $\mathbb{F}$ predictable reductions on $(0, T]$ (see Song (2014a, Lemma 6.1)); invariance probability measures are uniquely defined on $\mathcal{F}_T$, so that one can talk of the invariance probability measure $\mathbb{P}$ (as the specification of an invariance probability measure outside $\mathcal{F}_T$ is immaterial anyway).

2. Dynamic Gaussian Copula Model

In the DGC model the default intensities of the surviving names and therefore the value of credit protection spike at default times, as observed in practice. This nonimmersive feature makes the DGC model appropriate for dealing with counterparty risk on credit derivatives (notably, portfolios of CDS contracts) traded between a bank and its counterparty, respectively labeled as -1 and 0, and referencing credit names 1 to n, for some positive integer n. Accordingly, we introduce

$$N = \{-1, 0, 1, \ldots, n\} \text{ and } N^{\star} = \{1, \ldots, n\}$$

and we focus on $\tau = \tau_{-1} \wedge \tau_0$ in the paper. However, analog properties hold for any minimum of the τ_i and, in particular, for the τ_i themselves.

2.1. **The Model**

We consider a family of independent standard linear Brownian motions Z and $Z^i, i \in N$. For $\varrho \in [0, 1)$, we define

$$B^i_t = \sqrt{\varrho} Z_t + \sqrt{1-\varrho} Z^i_t. \tag{2.1}$$

Let ς be a continuous function on $\mathbb{R}_+$ with $\int_{\mathbb{R}_+} \varsigma^2(s)ds = 1$ and $\alpha^2(t) = \int_t^{+\infty} \varsigma^2(s)ds > 0$ for all $t \in \mathbb{R}_+$. For any $i \in N$, let h_i be a continuously differentiable strictly increasing function from $\mathbb{R}^*_+$ to $\mathbb{R}$, with derivative denoted by $\dot{h}_i$, such that $\lim_{s\downarrow 0} h_i(s) = -\infty$ and $\lim_{s\uparrow +\infty} h_i(s) = +\infty$. We define

$$\tau_i = h_i^{-1}\big(\int_0^{+\infty} \varsigma(u)dB_u^i\big) = h_i^{-1}\big(\sqrt{\varrho}\int_0^{+\infty} \varsigma(u)dZ_u + \sqrt{1-\varrho}\int_0^{+\infty} \varsigma(u)dZ_u^i\big), \tag{2.2}$$

for $i \in N$. The random times $(\tau_i)_{i\in N}$ follow the standard one-factor Gaussian copula model of Li (2000) (a DGC model in abbreviation), with correlation parameter ϱ and with marginal survival function $\Phi \circ h_i$ of τ_i, where

$$\Phi(t) = \int_t^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx,\ t \in \mathbb{R}$$

is the standard normal survival function. Note that, if $\varrho < 1$, the τ_i avoid each other:

$$\mathbb{Q}(\tau_i = \tau_j) = 0,\ \text{for any } i \neq j \text{ in } N.$$

2.2. Density Property

By multivariate density default model, we mean a model with an $\mathbb{F}$ conditional density of the default times (see e.g. the condition (DH) in Pham (2010, page 1800)), given some reference subfiltration $\mathbb{F}$ of $\mathbb{G}$. This is the multivariate extension of the notion of a density time, first introduced in an initial enlargement setup in Jacod (1987) and revisited in a progressive enlargement setup in Jeanblanc and Le Cam (2009) (under the name of initial time) and El Karoui, Jeanblanc, and Jiao (2010, 2015a,b).

First we prove that the DGC model is a multivariate density model with respect to the natural filtration $\mathbb{B} = (\mathcal{B}_t)_{t\geq 0}$ of the Brownian motions Z and $Z^i, i \in N$. We introduce the following processes.

$$m_t^i = \int_0^t \varsigma(u)dB_u^i \ \text{ and } \ \bar{m}_t^i = \int_t^\infty \varsigma(u)dB_u^i = h(\tau_i) - m_t^i,\ i \in N.$$

The standard normal density function is denoted by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},\ x \in \mathbb{R},$$

and the associated survival function by $\Phi(y) = \int_y^\infty \phi(x)dx,\ y \in \mathbb{R}$.

Theorem 2.1. *The dynamic Gaussian copula model is a multivariate density model of default times (with respect to the filtration $\mathbb{B}$), with conditional Lebesgue density*

$$p_t(t_i, i \in N) = \partial_{t_{-1}} \ldots \partial_{t_n} \mathbb{Q}(\tau_i < t_i,\ i \in N \,|\, \mathcal{B}_t)$$

of the τ_i, $i \in N$, given, for any nonnegative t_i, $i \in N$, and $t \in \mathbb{R}_+$, by

$$p_t(t_i,\ i \in N) = \int_{\mathbb{R}} \phi(y) \prod_{i\in N} \phi\left(\frac{h_i(t_i) - m_t^i + \alpha(t)\sqrt{\varrho}y}{\alpha(t)\sqrt{1-\varrho}}\right) \frac{\dot{h}_i(t_i)}{\alpha(t)\sqrt{1-\varrho}} dy. \tag{2.3}$$

Proof. The conditional density function p given $\mathcal{B}_t$ can be computed thanks to the independence of increments of the processes $Z, Z^i, i \in N$. Actually, for any $t \geq 0$, we can write

$$\tau_i = h_i^{-1}(m_t^i + \sqrt{\varrho}\xi^t + \sqrt{1-\varrho}\xi_i^t),\ i \in N,$$

where ξ^t is a real normal random variable with variance α_t^2, $(\xi_j^t)_{j\in N}$ is a centered Gaussian vector independent of ξ^t with homogeneous marginal variances α_t^2 and zero pairwise correlations, and where the family $\xi^t, \xi_i^t, i \in N$, is independent of $\mathcal{B}_t$. See Crépey et al. (2014, page 172)[1]. ∎

[1]Or Crépey et al. (2013, page 3) in the journal version.

2.3. Computation of the Intensity Processes

Note that the τ_i are $\mathcal{B}_\infty$ measurable, but they are not $\mathbb{B}$ stopping times. In the DGC model, the full model filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is taken as the progressive enlargement of the Brownian filtration $\mathbb{B}$ by the $\tau_i, i \in N$, augmented so as to satisfy the usual conditions, i.e.

$$\mathcal{G}_t = \cap_{s>t}(\mathcal{B}_s \vee \bigvee_{i\in N} \sigma(\tau_i \wedge s)), \quad t \geq 0. \tag{2.4}$$

In this section we prove that the τ_i are totally inaccessible $\mathbb{G}$ stopping times with intensities that we compute explicitly.

For $t \geq 0, u > 0$, $I \subseteq N$ and $j \in N$, we define:

$$\rho^I = \frac{\varrho}{|I|\varrho + 1}, \quad (\sigma^I)^2 = (1-\varrho)\frac{|I|\varrho + 1}{|I|\varrho + 1 - \varrho}, \quad \lambda^I = \frac{\varrho}{(|I|-1)\varrho + 1},$$
$$\boldsymbol{\tau}_I = (\tau_i)_{i\in I}, \ \ Z_t^{j,I}(u) = \frac{h_j(u) - m_t^j}{\alpha(t)} - \lambda^I \sum_{i\in I} \frac{\bar{m}_t^i}{\alpha(t)}.$$

For $t \geq 0$, let

$$\mathcal{I}_t = \{i \in N : \tau_i \leq t\}$$

(representing the set of obligors in N that are in default at time t) and let

$$\rho_t = \rho^{\mathcal{I}_t}, \ \sigma_t = \sigma^{\mathcal{I}_t}, \ \mathcal{J}_t = N \setminus \mathcal{I}_t.$$

For $\sigma > 0$, $\rho \in [0,1]$ and $J \subseteq N$, we define the functions

$$\Phi_{J,\rho,\sigma}(\mathbf{z}_J) = \mathbb{Q}(\xi_j > z_j, j \in J), \ \ \psi^j_{J,\rho,\sigma}(\mathbf{z}_J) = -\frac{\partial_{z_j}\Phi_{J,\rho,\sigma}}{\Phi_{J,\rho,\sigma}}(\mathbf{z}_J), \ j \in J, \tag{2.5}$$

where $\mathbf{z}_J = (z_j)_{j\in J}$ is a real vector and $(\xi_j)_{j\in N}$ is a centered Gaussian vector with homogeneous marginal variances σ^2 and pairwise correlations ρ. Note the following:

Lemma 2.1. *For $I = N \setminus J$, the family of random variables*

$$\left(\xi_j - \frac{\rho}{(|I|-1)\rho + 1}\sum_{i\in I}\xi_i\right)_{j\in J}$$

defines a centered Gaussian vector independent of $\sigma(\xi_i, i \in I)$, with homogeneous marginal variances and pairwise correlations, respectively given as

$$(\sigma^I)^2 = \sigma^2(1-\varrho)\frac{|I|\varrho + 1}{|I|\varrho + 1 - \varrho} \ \text{ and } \ \rho^I = \frac{\varrho}{|I|\varrho + 1}. \ \blacksquare \tag{2.6}$$

Lemma 2.2. *For $t \geq 0$, $I = N \setminus J$ and $u_j > 0, j \in J$, it holds*

$$\mathbb{E}[\mathbb{1}_{\{u_j < \tau_j, j\in J\}}|\mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)] = \Phi_{J,\rho^I,\sigma^I}(Z_t^{j,I}(u_j), j \in J). \tag{2.7}$$

Proof. For $j \in J$ and $u_j \in \mathbb{R}$, the condition $u_j < \tau_j$ is equivalent to

$$Z_t^{j,I}(u_j) = \frac{h_j(u_j) - m_t^j}{\alpha(t)} - \lambda^I \sum_{i\in I}\frac{\bar{m}_t^i}{\alpha(t)} \ \ < \ \ \frac{\bar{m}_t^j}{\alpha(t)} - \lambda^I \sum_{i\in I}\frac{\bar{m}_t^i}{\alpha(t)} \tag{2.8}$$

Noting that $m^j_t \in \mathcal{B}_t, \bar{m}^i_t \in \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I), i \in I$, the desired result follows by an application of Lemma 2.1. ∎

Lemma 2.3. *For $t > 0$ and $I = N \setminus J$, we have*

$$\{\tau_i \le t < \tau_j : i \in I, j \in J\} \cap \mathcal{G}_t = \{\tau_i \le t < \tau_j : i \in I, j \in J\} \cap (\mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)). \tag{2.9}$$

Proof. Let the $\tau_{(i)}$ be the increasing ordering of the τ_i, with also $\tau_{(0)} = 0$ and $\tau_{(n+1)} = \infty$. According to the optional splitting formula which holds in any multivariate density model of default times (see Song (2014b)), for any $\mathbb{G}$ optional process Y, there exists a $\mathcal{O}(\mathbb{B}) \otimes \mathcal{B}([0,\infty]^n)$-measurable functions $Y^{(i)},\ i \in N$, such that

$$Y = \sum_{i=0}^{n} Y^{(i)}(\tau_{-1} \dagger \tau_{(i)}, \dots, \tau_n \dagger \tau_{(i)}) \mathbb{1}_{[\tau_{(i)}, \tau_{(i+1)})}, \tag{2.10}$$

where $a \dagger b$ denotes a if $a \le b$ and ∞ if $a > b$, for $a, b \in [0,\infty]$. Since $\mathcal{G}_t = \sigma(Y_t)$ and $Y^{(i)}(\tau_1 \dagger \tau_{(i)}, \dots, \tau_n \dagger \tau_{(i)}) \mathbb{1}_{[\tau_{(i)}, \tau_{(i+1)})}$ is a function of $\mathcal{B}_t$ and $\boldsymbol{\tau}_I$ on $\{\tau_i \le t < \tau_j : i \in I, j \in J\}$, this implies (2.9). ∎

Theorem 2.2. *For any $l \in N$, τ_l admits a $(\mathbb{G}, \mathbb{Q})$ intensity given by*

$$\gamma^l_t = \mathbb{1}_{\{t<\tau_l\}} \frac{\dot{h}_l(t)}{\alpha(t)} \psi^l_{\mathcal{J}_t, \rho_t, \sigma_t}\big(Z^{j,\mathcal{I}_t}_t(t), j \in \mathcal{J}_t\big),\ \ t > 0. \tag{2.11}$$

Proof. For bounded $\mathcal{B}_t$ measurable functions F, for measurable bounded function f, for $0 \le t \le s < \infty$, we look at

$$\mathbb{E}[F f(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \le t < \tau_j : i \in I, j \in J\}} \mathbb{1}_{\{t < \tau_l \le s\}}].$$

We need only to consider $l \in J$. Then, using (2.7) to pass to the third line and conditioning in conjunction with the tower rule to pass to the fourth line:

$$\begin{array}{cl} & \mathbb{E}[F f(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \le t < \tau_j, i \in I, j \in J\}} \mathbb{1}_{\{s < \tau_l\}}] \\ = & \mathbb{E}[F f(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \le t, i \in I\}} \mathbb{E}[\mathbb{1}_{\{t < \tau_j, j \in J\}} \mathbb{1}_{\{s < \tau_l\}} | \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)]] \\ = & \mathbb{E}[F f(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \le t, i \in I\}} \Phi_{J, \rho^I, \sigma^I}(Z^{j,I}_t(u_j), j \in J)] \\ & \text{where } u_j = t \text{ except } u_l = s, \\ = & \mathbb{E}[F f(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \le t < \tau_j, i \in I, j \in J\}} \frac{\Phi_{J,\rho^I,\sigma^I}(Z^{j,I}_t(u_j), j \in J)}{\Phi_{J,\rho^I,\sigma^I}(Z^{j,I}_t(t), j \in J)}] \\ = & \mathbb{E}[F f(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \le t < \tau_j, i \in I, j \in J\}} \frac{\Phi_{\mathcal{J}_t,\rho_t,\sigma_t}(Z^{j,\mathcal{I}_t}_t(u_j), j \in \mathcal{J}_t)}{\Phi_{\mathcal{J}_t,\rho_t,\sigma_t}(Z^{j,\mathcal{I}_t}_t(t), j \in \mathcal{J}_t)}]. \end{array} \tag{2.12}$$

With the formula (2.9), we conclude

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\{t < \tau_l \le s\}} | \mathcal{G}_t] &= \mathbb{E}[\mathbb{1}_{\{t < \tau_l\}} | \mathcal{G}_t] - \mathbb{E}[\mathbb{1}_{\{s < \tau_l\}} | \mathcal{G}_t] \\ &= \mathbb{1}_{\{t < \tau_l\}} \Big(1 - \frac{\Phi_{\mathcal{J}_t,\rho_t,\sigma_t}(Z^{j,\mathcal{I}_t}_t(u_j), j \in \mathcal{J}_t)}{\Phi_{\mathcal{J}_t,\rho_t,\sigma_t}(Z^{j,\mathcal{I}_t}_t(t), j \in \mathcal{J}_t)}\Big). \end{aligned}$$

The stated result follows by an application of the Laplace formula of Dellacherie (1972, Chapter V, Theorem T54) (see also Dellacherie and Doléans-Dade (1971) or Knight (1991)). ∎

2.4. Computation of the Drift of the Brownian Motion

Next we study the processes $B^i, i \in N$, in the filtration $\mathbb{G}$. Thanks to Theorem 2.1, the DGC model is a multivariate density model. According to Jacod (1987), this implies the following:

Lemma 2.4. *The processes $B^i, i \in N$, are $\mathbb{G}$ semimartingales.* ∎

By virtue of Jeanblanc and Song (2013, Theorem 6.4), another consequence of the multivariate density property is the martingale representation property.

Theorem 2.3. *Let W^i, for $i \in N$, denote the martingale part in $\mathbb{G}$ of B^i. Let*

$$dM_t^i = d\mathbb{1}_{\tau_i \le t} - \gamma_t^i dt, \quad t > 0,$$

where the process γ^i is defined in (2.11). Then, the martingale representation property holds in $\mathbb{G}$ with respect to $(W^i, M^i, i \in N)$.

This section is devoted to the computation of the martingales W^i. We begin with the following remark on the Gaussian processes B^i (cf. Lemma 2.1).

Lemma 2.5. *For $J \subseteq N$ and $I = N \setminus J$, the family of processes*

$$\left(B^j - \frac{\varrho}{(|I|-1)\varrho + 1} \sum_{i \in I} B^i \right)_{j \in J} \tag{2.13}$$

is a continuous Lévy process (multivariate Brownian motion with drift) independent of $\sigma(B^i, i \in I)$, of homogeneous marginal variances $(\sigma^I)^2$ and pairwise correlations ρ^I. ∎

Proof. This follows by computing the brackets of the continuous local martingales and applying the Lévy processes characterization. ∎

Lemma 2.6. *For $k \in I$ and $0 \le t \le s \le s'$,*

$$\mathbb{E}[B_{s'}^k - B_s^k | \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)] = (\tfrac{1}{\alpha_t^2} \textstyle\int_s^{s'} \varsigma_u du)\ \bar{m}_t^k.$$

Proof. For $k \in I$, for $0 \le t \le s \le s' < \infty$,

$$B_{s'}^k - B_s^k - (\frac{1}{\alpha_t^2} \int_s^{s'} \varsigma_u du)\ \bar{m}_t^k \tag{2.14}$$

is a centered Gaussian random variable, independent of $\bar{m}_t^k$, with variance

$$(s' - s) - 2\frac{1}{\alpha_t^2}(\int_s^{s'} \varsigma_u du)^2 + \frac{1}{\alpha_t^2}(\int_s^{s'} \varsigma_u du)^2 = (s' - s) - \frac{1}{\alpha_t^2}(\int_s^{s'} \varsigma_u du)^2.$$

Hence, for $k \in I$,

$$\begin{aligned}
& \mathbb{E}[B_{s'}^k - B_s^k | \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)] = \mathbb{E}[B_{s'}^k - B_s^k | \mathcal{B}_t \vee \sigma(\bar{m}_t^i, i \in I)] \\
= \ & \mathbb{E}[B_{s'}^k - B_s^k | \mathcal{B}_t \vee \sigma(\bar{m}_t^k) \vee \sigma(\xi_i^I, i \in I \setminus \{k\})] \\
& \text{where the } \xi_i^I \text{ form a Gaussian family independent of } \mathcal{B}_t \vee \sigma(B^k) \text{, constructed with (2.13),} \\
= \ & \mathbb{E}[B_{s'}^k - B_s^k | \sigma(\bar{m}_t^k)] = (\tfrac{1}{\alpha_t^2} \textstyle\int_s^{s'} \varsigma_u du)\ \bar{m}_t^k \text{ because of (2.14).}
\end{aligned}$$ ∎

In the sequel we find it sometimes convenient to denote stochastic integration (or integration against measures) by $\centerdot$ and the Lebesgue measure on the half-line by $\boldsymbol{\lambda}$.

Theorem 2.4. *For $J \subseteq N$ and $k \in J$, define the function*

$$\mathfrak{b}_J^k(\mathbf{z}, x) = \mathbb{E}[\mathbb{1}_{\{z_j < \xi_j^I, j \in J\}}(\xi_k^I + x)], \quad x \in \mathbb{R}, \mathbf{z} = (z_j, j \in J),$$

where $(\xi^I_j, j\in J)$ *is a Gaussian family of homogeneous marginal variances* $(\sigma^I)^2$ *and pairwise correlations* ρ^I*. For any* $k\in N$*, define the process*

$$\beta^k_t=\frac{\varsigma(t)}{\alpha(t)}\left(\mathbb{1}_{\{k\in\mathcal{I}_t\}}\ \frac{\bar{m}^k_t}{\alpha(t)}+\mathbb{1}_{\{k\notin\mathcal{I}_t\}}\frac{\mathfrak{b}^k_{\mathcal{J}_t}((Z^{j,\mathcal{I}_t}_t(t),j\in\mathcal{J}_t),\ \lambda^{\mathcal{I}_t}\sum_{i\in\mathcal{I}_t}\frac{\bar{m}^i_t}{\alpha(t)})}{\Phi_{\mathcal{J}_t,\rho_t,\sigma_t}(Z^{j,\mathcal{I}_t}_t(t),j\in\mathcal{J}_t)}\right),\ \ t\in\mathbb{R}_+.$$

Then, $W^k=B^k-\beta^k\boldsymbol{.}\boldsymbol{\lambda}$*.*

Proof. For $0\le t\le s\le s'<\infty$, for any bounded $\mathcal{B}_t$ measurable function F and measurable bounded function f, we compute

$$\begin{array}{rl} & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t<\tau_j,i\in I,j\in J\}}(B^k_{s'}-B^k_s)]\\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t,i\in I\}}\mathbb{1}_{\{t<\tau_j,j\in J\}}\mathbb{E}[(B^k_{s'}-B^k_s)|\mathcal{B}_t\vee\sigma(\boldsymbol{\tau}_N)]]\\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t,i\in I\}}\mathbb{1}_{\{t<\tau_j,j\in J\}}(\frac{1}{\alpha^2_t}\int_s^{s'}\varsigma_u du)\ \bar{m}^k_t].\end{array}$$

If $k\in I$, $\bar{m}^k_t\in\mathcal{B}_t\vee\sigma(\boldsymbol{\tau}_I)$. If $k\notin I$,

$$\begin{array}{rl} & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t,i\in I\}}\mathbb{1}_{\{t<\tau_j,j\in J\}}(\frac{1}{\alpha^2_t}\int_s^{s'}\varsigma_u du)\ \bar{m}^k_t]\\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t,i\in I\}}(\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du)\ \mathbb{E}[\mathbb{1}_{\{t<\tau_j,j\in J\}}\frac{\bar{m}^k_t}{\alpha(t)}|\mathcal{B}_t\vee\sigma(\boldsymbol{\tau}_I)]]\\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t,i\in I\}}(\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du)\\ & \mathbb{E}[\mathbb{1}_{\{Z^{j,I}_t(t)<\frac{\bar{m}^j_t}{\alpha(t)}-\lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)},j\in J\}}(\frac{\bar{m}^k_t}{\alpha(t)}-\lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)}+\lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)})|\mathcal{B}_t\vee\sigma(\boldsymbol{\tau}_I)]]\\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t,i\in I\}}(\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du)\ \mathbb{E}[\mathbb{1}_{\{Z^{j,I}_t(t)<\xi^I_j,j\in J\}}(\xi^I_k+\lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)})]]\\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t,i\in I\}}(\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du)\ \mathfrak{b}^k_J((Z^{j,I}_t(t),j\in J),\ \lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)})]\\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t,i\in I\}}\mathbb{1}_{\{t<\tau_j,j\in J\}}(\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du)\ \frac{\mathfrak{b}^k_J((Z^{j,I}_t(t),j\in J),\ \lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)})}{\Phi_{J,\rho^I,\sigma^I}(Z^{j,I}_t(t),j\in J)}]\end{array}$$

This, combined with the formula (2.9), implies

$$\mathbb{E}[(B^k_{s'}-B^k_s)|\mathcal{G}_t]=(\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du)\times$$
$$\left(\mathbb{1}_{\{k\in\mathcal{I}_t\}}\ \frac{\bar{m}^k_t}{\alpha(t)}+\mathbb{1}_{\{k\notin\mathcal{I}_t\}}\frac{\mathfrak{b}^k_{\mathcal{J}_t}((Z^{j,\mathcal{I}_t}_t(t),j\in\mathcal{J}_t),\ \lambda^{\mathcal{I}_t}\sum_{i\in\mathcal{I}_t}\frac{\bar{m}^i_t}{\alpha(t)})}{\Phi_{\mathcal{J}_t,\rho_t,\sigma_t}(Z^{j,\mathcal{I}_t}_t(t),j\in\mathcal{J}_t)}\right).$$

The $\mathbb{G}$ drift of B^k is obtained as the differential of the above with respect to Lebesgue measure, i.e.

$$\frac{\varsigma(t)}{\alpha(t)}\left(\mathbb{1}_{\{k\in\mathcal{I}_t\}}\ \frac{\bar{m}^k_t}{\alpha(t)}+\mathbb{1}_{\{k\notin\mathcal{I}_t\}}\frac{\mathfrak{b}^k_{\mathcal{J}_t}((Z^{j,\mathcal{I}_t}_t(t),j\in\mathcal{J}_t),\ \lambda^{\mathcal{I}_t}\sum_{i\in\mathcal{I}_t}\frac{\bar{m}^i_t}{\alpha(t)})}{\Phi_{\mathcal{J}_t,\rho_t,\sigma_t}(Z^{j,\mathcal{I}_t}_t(t),j\in\mathcal{J}_t)}\right)dt.\ \blacksquare$$

The function $\mathfrak{b}^k_J(\mathbf{z},x)$ is closely linked with the functions $\psi^j_{J,\rho,\sigma}(\mathbf{z})$.

Lemma 2.7. *We have the identity:*

$$\mathfrak{b}^k_J(\mathbf{z},x)=\Big(x+(\sigma^I\sqrt{\rho^I})^2\ \ \sum_{l\in J}\psi^l_{J,\rho^I,\sigma^I}(\mathbf{z})+(\sigma^I\sqrt{1-\rho^I})^2\psi^k_{J,\rho^I,\sigma^I}(\mathbf{z})\Big)\,\Phi_{J,\rho^I,\sigma^I}(\mathbf{z}).$$

Proof. We write the function $\mathfrak{b}_J^k$ with the Gaussian density functions:

$$\begin{aligned}
\mathfrak{b}_J^k(\mathbf{z},x) &= \int_{\mathbb{R}}\phi(y)dy\int_{\mathbb{R}^{|J|}}\prod_{j\in J}\phi(y_j)\prod_{j\in J}\mathbb{1}_{\{z_j<\sigma^I\sqrt{\rho^I}y+\sigma^I\sqrt{1-\rho^I}y_j\}}(\sigma^I\sqrt{\rho^I}y+\sigma^I\sqrt{1-\rho^I}y_k+x)\prod_{j\in J}dy_j\\
&= x\int_{\mathbb{R}}\phi(y)dy\int_{\mathbb{R}^{|J|}}\prod_{j\in J}\phi(y_j)\prod_{j\in J}\mathbb{1}_{\{z_j<\sigma^I\sqrt{\rho^I}y+\sigma^I\sqrt{1-\rho^I}y_j\}}\prod_{j\in J}dy_j\\
&\quad+\sigma^I\sqrt{\rho^I}\int_{\mathbb{R}}y\phi(y)dy\int_{\mathbb{R}^{|J|}}\prod_{j\in J}\phi(y_j)\prod_{j\in J}\mathbb{1}_{\{z_j<\sigma^I\sqrt{\rho^I}y+\sigma^I\sqrt{1-\rho^I}y_j\}}\prod_{j\in J}dy_j\\
&\quad+\sigma^I\sqrt{1-\rho^I}\int_{\mathbb{R}}\phi(y)dy\int_{\mathbb{R}^{|J|}}y_k\prod_{j\in J}\phi(y_j)\prod_{j\in J}\mathbb{1}_{\{z_j<\sigma^I\sqrt{\rho^I}y+\sigma^I\sqrt{1-\rho^I}y_j\}}\prod_{j\in J}dy_j.
\end{aligned}$$

Then, applying the integration by parts formula to replace the appearance of y and y_k by the Gaussian functions:

$$\begin{aligned}
\mathfrak{b}_J^k(\mathbf{z},x) &= x\int_{\mathbb{R}}\phi(y)dy\prod_{j\in J}\Phi(\tfrac{z_j-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\\
&\quad+\sigma^I\sqrt{\rho^I}\int_{\mathbb{R}}d(-\phi(y))\prod_{j\in J}\Phi(\tfrac{z_j-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\\
&\quad+\sigma^I\sqrt{1-\rho^I}\int_{\mathbb{R}}\phi(y)dy\prod_{j\in J,j\neq k}\Phi(\tfrac{z_j-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\int_{\mathbb{R}}\phi(y_k)y_k\mathbb{1}_{\{z_k<\sigma^I\sqrt{\rho^I}y+\sigma^I\sqrt{1-\rho^I}y_k\}}dy_k\\
&= x\int_{\mathbb{R}}\phi(y)dy\prod_{j\in J}\Phi(\tfrac{z_j-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\\
&\quad+\sigma^I\sqrt{\rho^I}\int_{\mathbb{R}}\phi(y)dy\frac{\partial}{\partial y}\left(\prod_{j\in J}\Phi(\tfrac{z_j-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\right)\\
&\quad+\sigma^I\sqrt{1-\rho^I}\int_{\mathbb{R}}\phi(y)dy\prod_{j\in J,j\neq k}\Phi(\tfrac{z_j-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\phi(\tfrac{z_k-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\\
&= x\int_{\mathbb{R}}\phi(y)dy\prod_{j\in J}\Phi(\tfrac{z_j-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\\
&\quad+\sigma^I\sqrt{\rho^I}\int_{\mathbb{R}}\phi(y)dy\frac{\sigma^I\sqrt{\rho^I}}{\sigma^I\sqrt{1-\rho^I}}\sum_{l\in J}\left(\prod_{j\in J,j\neq l}\Phi(\tfrac{z_j-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\phi(\tfrac{z_l-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\right)\\
&\quad+\sigma^I\sqrt{1-\rho^I}\int_{\mathbb{R}}\phi(y)dy\prod_{j\in J,j\neq k}\Phi(\tfrac{z_j-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\phi(\tfrac{z_k-\sigma^I\sqrt{\rho^I}y}{\sigma^I\sqrt{1-\rho^I}})\\
&= x\,\Phi_{J,\rho^I,\sigma^I}(\mathbf{z})+(\sigma^I\sqrt{\rho^I})^2\sum_{l\in J}\psi^l_{J,\rho^I,\sigma^I}(\mathbf{z})\Phi_{J,\rho^I,\sigma^I}(\mathbf{z})+(\sigma^I\sqrt{1-\rho^I})^2\psi^k_{J,\rho^I,\sigma^I}(\mathbf{z})\Phi_{J,\rho^I,\sigma^I}(\mathbf{z}).\ \blacksquare
\end{aligned}$$

3. Reduction Structure

We now study the invariance properties of the DGC model. In this perspective, the market information before the default event of the bank or of its counterparty is modeled by the filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$, where

$$\mathcal{F}_t=\cap_{s>t}\big(\mathcal{B}_s\vee\bigvee_{i\in N^\star}\big(\sigma(\tau_i\wedge s)\big),\tag{3.1}$$

augmented so as to satisfy the usual conditions.

Because of the multivariate density property of the family of $(\tau^j, j\in N^\star)$ with respect to the filtration $\mathbb{B}$ (same proof as Theorem 2.1), the computations we have made in $\mathbb{G}$ in the previous section can be made similarly in $\mathbb{F}$. In particular, the following splitting formula holds (cf. (2.9)): for any $t>0$ and $I\subseteq N^\star$, writing $J=N^\star\setminus I$,

$$\{\tau_i\leq t<\tau_j: i\in I, j\in J\}\cap\mathcal{F}_t=\{\tau_i\leq t<\tau_j: i\in I, j\in J\}\cap\mathcal{B}_t\vee\sigma(\boldsymbol{\tau}_I).\tag{3.2}$$

Moreover, the so-called condition (H') holds, i.e. the processes $B^k, k \in N$, are $\mathbb{F}$ semimartingales, and the random times $\tau_j, j \in N^\star$, are $\mathbb{F}$ totally inaccessible stopping times, as stated in the following lemma. For $t > 0$, let

$$\mathcal{I}_t^\star = \{i \in N^\star : \tau_i \le t\},\ \mathcal{J}_t^\star = N^\star \setminus \mathcal{I}_t^\star,\ \rho_t^\star = \rho^{\mathcal{I}_t^\star},\ \sigma_t^\star = \sigma^{\mathcal{I}_t^\star}.$$

Lemma 3.1. *For any $k \in N$, the process $\bar{W}_t^k = B_t^k - \int_0^t \bar{\beta}_s^k ds,\ t \ge 0$ is an $\mathbb{F}$ local martingale, where*

$$\bar{\beta}_t^k = \frac{\varsigma(t)}{\alpha(t)} \left(\mathbb{1}_{\{k \in \mathcal{I}_t^\star\}} \frac{\bar{m}_t^k}{\alpha(t)} + \mathbb{1}_{\{k \notin \mathcal{I}_t^\star\}} \frac{\mathfrak{b}^k_{\mathcal{J}_t^\star}((Z_t^{j,\mathcal{I}_t^\star}(t), j \in \mathcal{J}_t^\star),\ \lambda^{\mathcal{I}_t^\star} \sum_{i \in \mathcal{I}_t^\star} \frac{\bar{m}_t^i}{\alpha(t)})}{\Phi_{\mathcal{J}_t^\star,\rho_t^\star,\sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t), j \in \mathcal{J}_t^\star)} \right),\ \ t \in \mathbb{R}_+.$$

For $j \in N^\star$, τ_j is an $\mathbb{F}$ totally inaccessible stopping time and the process $d\bar{M}_t^j = d\mathbb{1}_{\tau_j \le t} - \bar{\gamma}_t^j dt,\ \ t > 0$, is an $\mathbb{F}$ local martingale, where

$$\bar{\gamma}_t^j = \mathbb{1}_{\{t < \tau_j\}} \frac{\dot{h}_j(t)}{\alpha(t)} \psi^j_{\mathcal{J}_t^\star,\rho_t^\star,\sigma_t^\star}\big(Z_t^{j,\mathcal{I}_t^\star}(t), j \in \mathcal{J}_t^\star\big),\ \ t > 0. \tag{3.3}$$

The family of processes $\bar{W}^k, k \in N$ and $\bar{M}^j, j \in N^\star$, has the martingale representation property in the filtration $\mathbb{F}$. ∎

3.1. The Azéma Supermartingale

Our next aim is to compute the Azéma supermartingale of the random time $\tau_{-1} \wedge \tau_0$ in the filtration $\mathbb{F}$, i.e.,

$$\mathbb{E}[\mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}} | \mathcal{F}_t],\ t \ge 0.$$

Lemma 3.2. *The Azéma supermartingale of the random time $\tau_{-1} \wedge \tau_0$ in the filtration $\mathbb{F}$ is given by*

$$S_t = \frac{\Phi_{\mathcal{J}_t^\star \cup \{-1,0\},\rho_t^\star,\sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t), j \in \mathcal{J}_t^\star \cup \{-1,0\})}{\Phi_{\mathcal{J}_t^\star,\rho_t^\star,\sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t), j \in \mathcal{J}_t^\star)},\ t \ge 0. \tag{3.4}$$

In particular, the Azéma supermartingale S is positive.

***Proof*.** For any bounded $\mathcal{B}_t$ measurable functions F and measurable bounded function f, we compute (cf. (2.12))

$$\begin{aligned}
& \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \le t < \tau_j, i \in I, j \in J\}}\mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}}] \\
= \ & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \le t : i \in I\}}\mathbb{E}[\mathbb{1}_{\{t < \tau_j : j \in J\}}\mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}} | \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)]] \\
= \ & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \le t : i \in I\}}\Phi_{J \cup \{-1,0\},\rho^I,\sigma^I}(Z_t^{j,I}(t) : j \in J \cup \{-1,0\})] \\
= \ & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \le t < \tau_j : i \in I, j \in J\}}\tfrac{\Phi_{J \cup \{-1,0\},\rho^I,\sigma^I}(Z_t^{j,I}(t) : j \in J \cup \{-1,0\})}{\Phi_{J,\rho^I,\sigma^I}(Z_t^{j,I}(t) : j \in J)}] \\
= \ & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \le t < \tau_j : i \in I, j \in J\}}\tfrac{\Phi_{\mathcal{J}_t^\star \cup \{-1,0\},\rho_t^\star,\sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t) : j \in \mathcal{J}_t^\star \cup \{-1,0\})}{\Phi_{\mathcal{J}_t^\star,\rho_t^\star,\sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t) : j \in \mathcal{J}_t^\star)}],
\end{aligned}$$

where conditioning and the tower rule are used in the next-to-last identity. With the formula (3.2), we conclude

$$\mathbb{E}[\mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}} | \mathcal{F}_t] = \frac{\Phi_{\mathcal{J}_t^\star \cup \{-1,0\},\rho_t^\star,\sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t) : j \in \mathcal{J}_t^\star \cup \{-1,0\})}{\Phi_{\mathcal{J}_t^\star,\rho_t^\star,\sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t) : j \in \mathcal{J}_t^\star)}. \ ∎$$

Let $\nu = \frac{1}{S} \centerdot S^c$, where S^c denotes the continuous martingale component of the $(\mathbb{F}, \mathbb{Q})$ Azéma supermartingale S.

Lemma 3.3. *We have*

$$\begin{aligned} d\nu_t &= \textstyle\sum_{j\in\mathcal{J}^\star_{t-}\cup\{-1,0\}} \psi^j_{\mathcal{J}^\star_{t-}\cup\{-1,0\},\rho^\star_t,\sigma^\star_t}\big(Z^{j,\mathcal{I}^\star_{t-}}_t(t) : j\in\mathcal{J}^\star_t\cup\{-1,0\}\big)d\zeta^{j,\mathcal{I}^\star_t}_t \\ &\quad -\textstyle\sum_{j\in\mathcal{J}^\star_{t-}} \psi^j_{\mathcal{J}^\star_{t-},\rho^\star_t,\sigma^\star_t}\big(Z^{j,\mathcal{I}^\star_{t-}}_t(t) : j\in\mathcal{J}^\star_t\big)d\zeta^{j,\mathcal{I}^\star_t}_t, \end{aligned}$$

where, for $I\subseteq N^\star$, $\zeta^{j,I}_t$ denotes the martingale part of $\Big(-\frac{1}{\alpha(t)}dm^j_t+\frac{\varrho}{(|I|-1)\varrho+1}\sum_{i\in I}\frac{1}{\alpha(t)}dm^i_t\Big)$ in $\mathbb{F}$.

Proof. To obtain dS^c_t (which is then divided by S_t), it suffices to apply Itô calculus to the expression (3.4) of S on every random interval where $\mathcal{I}^\star_{t-}$ is constant. Note that, knowing t is in such an interval, $\boldsymbol{\tau}_{\mathcal{I}^\star_{t-}}$ is in $\mathcal{F}_t$. Also note that the jumps of S_t triggered by the jumps of $\mathcal{I}^\star_{t-}$ have no impact here, because S^c is a continuous local martingale. ∎

3.2. $\mathbb{F}$ **Reductions of** β^k,γ^j

Lemma 3.4. *The triplet $(\tau_{-1}\wedge\tau_0,\mathbb{F},\mathbb{G})$ satisfies the condition (B).*

Proof. To check the condition (B), by the monotone class theorem, we only need consider the elementary $\mathbb{G}$ predictable processes of the form $U=\nu f(\tau_{-1}\wedge s,\tau_0\wedge s)\mathbb{1}_{(s,t]}$, for an $\mathcal{F}_s$ measurable random variable F and a Borel function f. Since $U\mathbb{1}_{(0,\tau]}=Ff(s,s)\mathbb{1}_{(s,t]}\mathbb{1}_{(0,\tau]}$, we may take $U'=Ff(s,s)\mathbb{1}_{(s,t]}$ in the condition (B). ∎

Next we consider the reduction of the processes β^k,γ^j in the filtration $\mathbb{F}$. Notice that, for $t<\tau_{-1}\wedge\tau_0$,

$$\mathcal{I}_t=\mathcal{I}^\star_t,\ \mathcal{J}_t=\mathcal{J}^\star_t\cup\{-1,0\}.$$

Therefore, the following lemma holds.

Lemma 3.5. *The $\mathbb{F}$ reduction of $\gamma^j, j\in N^\star$, is*

$$\widetilde{\gamma}^j_t=\mathbb{1}_{\{t<\tau_j\}}\frac{\dot{h}_j(t)}{\alpha(t)}\psi^j_{\mathcal{J}^\star_t\cup\{-1,0\},\rho^\star_t,\sigma^\star_t}\big(Z^{j,\mathcal{I}^\star_t}_t(t),j\in\mathcal{J}^\star_t\cup\{-1,0\}\big),\ \ t>0. \tag{3.5}$$

Similarly, the $\mathbb{F}$ reduction of $\beta^k, k\in N$, is

$$\widetilde{\beta}^k_t=\frac{\varsigma(t)}{\alpha(t)}\times\left(\mathbb{1}_{\{k\in\mathcal{I}^\star_t\}}\frac{\bar{m}^k_t}{\alpha(t)}+ \mathbb{1}_{\{k\notin\mathcal{I}^\star_t\}}\frac{\mathfrak{b}^k_{\mathcal{J}^\star_t\cup\{-1,0\}}((Z^{j,\mathcal{I}^\star_t}_t(t),j\in\mathcal{J}^\star_t\cup\{-1,0\}),\ \lambda^{\mathcal{I}^\star_t}\sum_{i\in\mathcal{I}^\star_t}\frac{\bar{m}^i_t}{\alpha(t)})}{\Phi_{\mathcal{J}^\star_t\cup\{-1,0\},\rho^\star_t,\sigma^\star_t}(Z^{j,\mathcal{I}^\star_t}_t(t),j\in\mathcal{J}^\star_t\cup\{-1,0\})}\right),\ \ t\in\mathbb{R}_+.\ ∎$$

Note that the processes $\gamma^j,\widetilde{\gamma}^j$ and $\beta^k,\widetilde{\beta}^k$ are càdlàg. The next result shows that the process β^k (and consequently $\widetilde{\beta}$) is linked with $\bar{\beta}^k$ through the process ν.

Lemma 3.6. *For $k\in N$,*

$$\int_0^t\widetilde{\beta}^k_s ds=\int_0^t\bar{\beta}^k_s ds+\langle\bar{W}^k,\nu\rangle_t,\ \ t\in[0,\tau_{-1}\wedge\tau_0].$$

Proof. Notice that B^k is a continuous process. By the Jeulin–Yor formula (see e.g. Dellacherie, Maisonneuve, and Meyer (1992, no 77 Remarques b))),

$$B^k_t-\int_0^t\bar{\beta}^k_s ds-\langle\bar{W}^k,\nu\rangle_t,\ \ t\in[0,\tau_{-1}\wedge\tau_0],$$

defines a $\mathbb{G}$ local martingale. But, acccording to Theorem 2.4, the drift of B^k in $\mathbb{G}$ is $\int_0^t \beta^k_s ds, t \geq 0$. We conclude that

$$\int_0^t \widetilde{\beta}^k_s ds = \int_0^t \beta^k_s ds = \int_0^t \bar{\beta}^k_s ds + \langle \bar{W}^k, \nu \rangle_t$$

for $t \in [0, \tau_{-1} \wedge \tau_0]$. ∎

4. The Condition (A) and the Invariance Probability Measure

In this section we prove two results regarding the condition (A) in the DGC model. Technically this section is the most significant part of the paper. The Gaussian correlation computations in the DGC model (cf. Lemma 2.5), which was the key in the computations of the previous sections, becomes useless, , used in the previous sections, turn to be helpless, while Gaussian tail moment estimates become crucial.

Here is the first result to be proved.

Theorem 4.1. *There exists a $\varrho_0 > 0$ such that, if the correlation coefficient ϱ is inferior to ϱ_0, the condition (A) holds in the DGC model $(\tau_{-1} \wedge \tau_0, \mathbb{F}, \mathbb{G}, \mathbb{Q})$ on $[0, T]$ for any finite constant $T > 0$.*

Remark 4.1. In the univariate case where $n = -1$ (case of a single default time τ_{-1}), all the terms involving ϱ disappear in the computations. Hence, by revisiting all the computations in this (much simpler) case, the condition (A) always holds.

When the condition (A) holds, we also have an explicit expression of the invariance probability measure.

Theorem 4.2. *Under the condition of Theorem 4.1, the invariance probability has a density process in $(\mathbb{F}, \mathbb{Q})$ given by $\mathcal{E}(\mu)$, where*

$$\mu = \nu + \sum_{j \in N^*} (\frac{\widetilde{\gamma}^j_-}{\bar{\gamma}^j_-} - 1) \centerdot \bar{M}^j.$$

Theorem 4.2 can be proved with a semimartingale calculus. In contrast, the proof of Theorem 4.1 is much more involved. The proof relies on the sufficiency condition given in Crépey and Song (2017, Theorem 3.5). To compute the quantities in this sufficiency condition, we begin with the following lemma.

Lemma 4.2. *The $(\mathbb{G}, \mathbb{Q})$ compensator of the stopping time $\tau_{-1} \wedge \tau_0$ has an intensity γ, which is given by*

$$\gamma = \mathbb{1}_{[0, \tau_{-1} \wedge \tau_0]}(\gamma^{-1} + \gamma^0).$$

Proof. This follows from, for example, Crépey and Song (2016, Lemma 6.2). ∎
As a consequence of this lemma and of the formula (A.4) in Crépey and Song (2017), the sufficiency condition of Crépey and Song (2017, Theorem 3.5) reduces to the positivity condition $S_T > 0$, which has been established in Lemma 3.2, and to the $\mathbb{Q}$ integrability of $e^{\int_0^T \mathbb{1}_{\{s \leq \tau_{-1} \wedge \tau_0\}}(\gamma^{-1}_s + \gamma^0_s) ds}$. The latter, however, necessitates a careful study of the intensity processes γ^{-1} and γ^0. A major technical difficulty stems from the explosion of the functions h_i and their derivatives at 0, for which the following Gaussian estimates are required.

4.1. **Gaussian Estimates**

Let $\sigma > 0$. We denote by ϕ_σ the density function of a centered Gaussian random variable of variance σ^2, i.e.,

$$\phi_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}, \; x \in \mathbb{R}.$$

Let Φ_σ denote the survival function associated with ϕ_σ, i.e., $\Phi_\sigma(y) = \int_y^\infty \phi_\sigma(t) dt$. Notice that, for any $j \in N$,

$$\Phi_{\{j\}, \rho, \sigma} = \Phi_\sigma$$

(for every $0 \le \rho < 1$).

Lemma 4.3. *Let $l \in J$. For $\mathbf{z} = (z_j, j \in N) \in \mathbb{R}^N$ we have*

$$\psi^l_{J,\rho,\sigma}(\mathbf{z}_J) = -\frac{\partial_{z_l}\Phi_{J,\rho,\sigma}(\mathbf{z})}{\Phi_{J,\rho,\sigma}(\mathbf{z})} \le \frac{\phi_\sigma(z_l)}{\Phi_\sigma(z_l)} = -\frac{d}{dt}\ln\Phi_\sigma(t)_{t=z_l}.$$

Remark 4.4. The merit of this lemma is provide a majorant of a function of $\mathbf{z}$ by a function of z_l, which is, moreover, an exact differential.

Proof. Consider the Gaussian vector ξ introduced in (2.5). Separating ξ_l from the other Gaussian random variables $\xi_j, j \neq l$, we can write

$$\begin{aligned} &\Phi_{J,\rho,\sigma}(\mathbf{z}_J) = \mathbb{P}[z_l < \xi_l, z_j - \rho\xi_l < \xi_j - \rho\xi_l, j \in J, j \neq l] \\ = \;& \mathbb{E}[\mathbb{1}_{\{z_l<\xi_l\}}\mathbb{P}[z_j - \rho y < \xi_j - \rho\xi_l, j \in J, j \neq l]_{y=\xi_l}] = \mathbb{E}[\mathbb{1}_{\{z_l<\xi_l\}}\Phi_{J\setminus\{l\},\frac{\rho}{1+\rho},\sigma\sqrt{1-\rho^2}}(\mathbf{z}_{J\setminus\{l\}} - \rho\xi_l)] \\ = \;& \int_{z_l}^\infty \phi_\sigma(y)\Phi_{J\setminus\{l\},\frac{\rho}{1+\rho},\sigma\sqrt{1-\rho^2}}(\mathbf{z}_{J\setminus\{l\}} - \rho y)\, dy. \end{aligned}$$

From that, we deduce two estimates:

$$\begin{aligned} \partial_{z_l}\Phi_{J,\rho,\sigma}(\mathbf{z}_J) &= -\phi_\sigma(z_l)\Phi_{J\setminus\{l\},\frac{\rho}{1+\rho},\sigma\sqrt{1-\rho^2}}(\mathbf{z}_{J\setminus\{l\}} - \rho z_l), \\ \Phi_{J,\rho,\sigma}(\mathbf{z}_J) &\ge \Phi_\sigma(z_l)\;\Phi_{J\setminus\{l\},\frac{\rho}{1+\rho},\sigma\sqrt{1-\rho^2}}(\mathbf{z}_{J\setminus\{l\}} - \rho z_l), \end{aligned}$$

because the function $\Phi_{J\setminus\{l\},\frac{\rho}{1+\rho},\sigma\sqrt{1-\rho^2}}(\mathbf{z}_{J\setminus\{l\}} - \rho y)$ is increasing in y. As a consequence,

$$-\frac{\partial_{z_l}\Phi_{J,\rho,\sigma}(\mathbf{z}_J)}{\Phi_{J,\rho,\sigma}(\mathbf{z}_J)} \le \frac{\phi_\sigma(z_l)\Phi_{J\setminus\{l\},\frac{\rho}{1+\rho},\sigma\sqrt{1-\rho^2}}(\mathbf{z}_{J\setminus\{l\}} - \rho z_l)}{\Phi_\sigma(z_l)\Phi_{J\setminus\{l\},\frac{\rho}{1+\rho},\sigma\sqrt{1-\rho^2}}(\mathbf{z}_{J\setminus\{l\}} - \rho z_l)} = \frac{\phi_\sigma(z_l)}{\Phi_\sigma(z_l)}. \blacksquare \tag{4.1}$$

Lemma 4.5. *The function $\frac{\phi_\sigma}{\Phi_\sigma}$ is increasing.*

Proof. We notice that, for $t > 0$,

$$\Phi_\sigma(t) = \int_t^\infty \phi_\sigma(s)ds \le \frac{1}{t}\int_t^\infty s\phi_\sigma(s)ds = \frac{1}{t}\sigma^2\phi_\sigma(t).$$

Hence

$$\frac{\phi_\sigma(t)}{\Phi_\sigma(t)} \ge \frac{\phi_\sigma(t)}{\phi_\sigma(t)\frac{\sigma^2}{t}} = \frac{t}{\sigma^2},\ t > 0.$$

We compute now the derivative, on $t > 0$,

$$\tfrac{d}{dt}\big(\tfrac{\phi_\sigma(t)}{\Phi_\sigma(t)}\big) = -\tfrac{t}{\sigma^2}\tfrac{\phi_\sigma(t)}{\Phi_\sigma(t)} + \big(\tfrac{\phi_\sigma(t)}{\Phi_\sigma(t)}\big)^2 = \big(-\tfrac{t}{\sigma^2} + \tfrac{\phi_\sigma(t)}{\Phi_\sigma(t)}\big)\tfrac{\phi_\sigma(t)}{\Phi_\sigma(t)} \ge \big(-\tfrac{t}{\sigma^2} + \tfrac{t}{\sigma^2}\big)\tfrac{\phi_\sigma(t)}{\Phi_\sigma(t)} = 0.$$

This proves the lemma for $t > 0$. But the lemma is trivially valid for $t \le 0$. ∎

Lemma 4.6. *We have $\Phi_\sigma(t) \le \frac{\sigma^2}{t}\phi_\sigma(t)$ for $t > 0$, and $\Phi_\sigma(t) \ge \frac{\sigma^2}{2}\frac{1}{t}\phi_\sigma(t)$ for $t \ge \sigma$.*

Proof. In the proof of the previous lemma, we have already proved $\Phi_\sigma(t) \le \frac{\sigma^2}{t}\phi_\sigma(t),\ t > 0$. Let $\Gamma = \phi_\sigma$ in Lemma A.1. As $\phi'_\sigma(t) = -\frac{t}{\sigma^2}\phi(t), t > 0$, we have $g(t) = \frac{t}{\sigma^2}$. Hence Lemma A.1(ii) applied with $\beta' = 0$ and $\epsilon = \frac{\sigma^2}{2}$ yields

$$\Phi_\sigma(t) \ge \frac{\sigma^2}{2}\frac{1}{t}\phi_\sigma(t),\ t > \sigma,$$

and then also for $t = \sigma$ by continuity.∎

4.2. The Exponential Integrability of the Integrals of the γ^l

For $j \in J \subseteq N$, we define the process $G_{j,t}^{\mathcal{I}_t}$ through

$$Z^{j,\mathcal{I}_t}(t) \quad = \quad \frac{h_j(t)}{\alpha(t)} - \frac{m_t^j}{\alpha(t)} + \lambda^{\mathcal{I}_t} \sum_{i\in\mathcal{I}_t} \frac{m_t^i - h_i(\tau_i)}{\alpha(t)} =: \frac{h_j(t)}{\alpha(t)} + \frac{G_t^{j,\mathcal{I}_t}}{\alpha(t)},\ t > 0.$$

We define

$$\sigma_k = \sigma^I \text{ when } |I| = k, \text{ and } \ U_t = \sup_{0<v\leq t} \frac{G_v^{l,\mathcal{I}_v}}{\alpha(v)}.$$

The result that follows is the key that allows absorbing the explosion of the functions h_l and their derivatives at time 0.

Lemma 4.7. *For $l \in N$, let $t > 0$ be such that $h_l(t) \leq 0$. We have*

$$\int_0^t \gamma_s^l ds \leq \sum_{k=0}^{2+n-1} \frac{1}{\alpha(t)} \left[ln2 + \ln(U_t \vee \sigma_k) - \ln \sigma_k + 1/2 \ln 2 + 1/2 \ln \pi + \frac{U_t^2 \vee \sigma_k^2}{2\sigma_k^2} \right].$$

Proof. Using Theorem 2.2 as well as Lemmas 4.3 and 4.6, we can compute

$$\begin{aligned} \int_0^u \gamma_u^l du &= \int_0^t \mathbb{1}_{\{t\leq\tau_l\}} \frac{\dot h_l(u)}{\alpha(u)} \psi^l_{\mathcal{J}_u,\rho_u,\sigma_u}(Z_u^{j,\mathcal{I}_u}(u), j \in \mathcal{J}_u) du \\ &= \int_{-\infty}^{h_l(t)} \mathbb{1}_{\{t\leq\tau_l\}} \frac{1}{\alpha(k(s))} \psi^l_{\mathcal{J}_{k(s)},\rho_{k(s)},\sigma_{k(s)}}(Z_{k(s)}^{j,\mathcal{I}_{k(s)}}(k(s)), j \in \mathcal{J}_{k(s)}) ds, \end{aligned}$$

where $k(\cdot) = h_l^{-1}$. Hence,

$$\begin{aligned} \int_0^u \gamma_u^l du &\leq \int_{-\infty}^{h_l(t)} \mathbb{1}_{\{k(s)\leq\tau_l\}} \frac{1}{\alpha(k(s))} \frac{\phi_{\sigma_{k(s)}}}{\Phi_{\sigma_{k(s)}}} (Z_{k(s)}^{l,\mathcal{I}_{k(s)}}(k(s))) ds \\ &= \int_{-\infty}^{h_l(t)} \mathbb{1}_{\{k(s)\leq\tau_l\}} \frac{1}{\alpha(k(s))} \frac{\phi_{\sigma_{k(s)}}}{\Phi_{\sigma_{k(s)}}} \left(\frac{s}{\alpha(k(s))} + \frac{G_{k(s)}^{l,\mathcal{I}_{k(s)}}}{\alpha(k(s))}\right) ds \\ &\leq \frac{1}{\alpha(t)} \int_{-\infty}^{h_l(t)} \mathbb{1}_{\{k(s)\leq\tau_l\}} \frac{\phi_{\sigma_{k(s)}}}{\Phi_{\sigma_{k(s)}}} \left(\frac{s}{\alpha(0)} + \sup_{0<v\leq t} \frac{G_v^{l,\mathcal{I}_v}}{\alpha(v)}\right) ds, \end{aligned}$$

because $s \leq h_l(t) \leq 0$. Therefore,

$$\begin{aligned} \int_0^u \gamma_u^l du &= \frac{1}{\alpha(t)} \int_{-\infty}^{h_l(t)} \mathbb{1}_{\{k(s)\leq\tau_l\}} \frac{\phi_{\sigma_{k(s)}}}{\Phi_{\sigma_{k(s)}}} (s + U_t) ds \\ &\leq \frac{1}{\alpha(t)} \int_{-\infty}^{h_l(t)} \sum_{k=0}^{2+n-1} \frac{\phi_{\sigma_k}}{\Phi_{\sigma_k}} (s + U_t) ds \\ &= -\sum_{k=0}^{2+n-1} \frac{1}{\alpha(t)} \ln \Phi_{\sigma_k}(h_l(t) + U_t)) \\ &\leq -\sum_{k=0}^{2+n-1} \frac{1}{\alpha(t)} \ln \Phi_{\sigma_k}(U_t) \quad \text{because } h_l(t) \leq 0 \\ &\leq \sum_{k=0}^{2+n-1} \frac{1}{\alpha(t)} \ln \left[\frac{2(U_t \vee \sigma_k)}{\sigma_k^2} \sqrt{2\pi\sigma_k^2} e^{\frac{U_t^2 \vee \sigma_k^2}{2\sigma_k^2}} \right] \\ &= \sum_{k=0}^{2+n-1} \frac{1}{\alpha(t)} \left[ln2 + \ln(U_t \vee \sigma_k) - \ln \sigma_k + 1/2 \ln 2 + 1/2 \ln \pi + \frac{U_t^2 \vee \sigma_k^2}{2\sigma_k^2} \right]. \blacksquare \end{aligned}$$

Lemma 4.8. *Let $t > 0$ such that $h(t) \leq 0$. The random variable U_t defined in the preceding lemma is bounded by*

$$\frac{\varrho}{\alpha(t)} \sum_{j\in N} (h_j(\tau_j))^- + \frac{1}{\alpha(t)} \sum_{j\in N} \sup_{0<v\leq t} |m_v^j|.$$

Therefore, for any $p \geq 1$, if $t_0 > 0$ and ϱ are small enough, $e^{\int_0^{t_0} \gamma_s^l ds}$ is p times $\mathbb{Q}$ integrable.

Proof. The process $G_t^{j,\mathcal{I}_t}$ is composed, on the one hand, of elements $-h_j(\tau_j), j \in \mathcal{I}_t$ multiplied by $\lambda^{\mathcal{I}_t}$, and, on the other hand, of elements m_t^j multiplied by $\lambda^{\mathcal{I}_t}$ or by -1. We notice that $0 \leq \lambda^{\mathcal{I}_t} \leq \varrho \leq 1$. We also notice that, for $j \in \mathcal{I}_t$, $\tau_j \leq t$ i.e., $h_j(\tau_j) \leq h_j(t) \leq 0$. Hence, $-h_j(\tau_j) = (h_j(\tau_j))^-$. The bound on U_t is now clear.

As $h_j(\tau_j)$ takes the form $h_j(\tau_j) = \sqrt{\varrho}\xi + \sqrt{1-\varrho}\xi_j$ where the random variables ξ, ξ_j are independent Gaussian random variables with unit variance, we have

$$\begin{array}{rl} & (\sum_{j\in N}(h_j(\tau_j))^-)^2 \leq (\sum_{j\in N}(\sqrt{\varrho}|\xi| + \sqrt{1-\varrho}|\xi_j|))^2 = ((2+n)\sqrt{\varrho}|\xi| + \sqrt{1-\varrho}\sum_{j\in N}|\xi_j|)^2 \\ \leq & (3+n)(2+n)^2\varrho\xi^2 + (3+n)(1-\varrho)\sum_{j\in N}\xi_j^2. \end{array} \tag{4.2}$$

We compute

$$\begin{array}{l} \sum_{k=0}^{2+n-1} \ln(U_t \vee \sigma_k) \leq (2+n)\ln(U_t \vee 1), \quad \text{because } \sigma_k \leq 1, \\ \sum_{k=0}^{2+n-1} \frac{U_t^2 \vee \sigma_k^2}{2\sigma_k^2} \leq \sum_{k=0}^{2+n-1} \frac{U_t^2 + \sigma_k^2}{2\sigma_k^2} = U_t^2 \sum_{k=0}^{2+n-1} \frac{1}{2\sigma_k^2} + \sum_{k=0}^{2+n-1} \frac{\sigma_k^2}{2\sigma_k^2}. \end{array}$$

By the estimate in Lemma 4.7, for any $p > 0$,

$$\mathbb{E}[e^{p\int_0^{t_0}\gamma_s^l ds}] \leq C\mathbb{E}[(U_t \vee 1)^{\frac{p(2+n)}{\alpha(t)}} e^{\frac{p}{\alpha(t)}U_t^2\sum_{k=0}^{2+n-1}\frac{1}{2\sigma_k^2}}] \leq C\mathbb{E}[(U_t \vee 1)^{\frac{2p(2+n)}{\alpha(t)}}]^{1/2}\mathbb{E}[e^{\frac{2p}{\alpha(t)}U_t^2\sum_{k=0}^{2+n-1}\frac{1}{2\sigma_k^2}}]^{1/2}$$

(where C denotes a generic constant which may change from place to place). By the Gaussian nature of $h_j(\tau_j)$ and of m_v^j, the first expectation $\mathbb{E}[(U_t \vee 1)^{\frac{2p(2+n)}{\alpha(t)}}]^{1/2}$ is always finite (cf. Lemma B.1). As for the second expectation

$$\begin{array}{rl} & \mathbb{E}[e^{\frac{2p}{\alpha(t)}U_t^2\sum_{k=0}^{2+n-1}\frac{1}{2\sigma_k^2}}] \\ \leq & \mathbb{E}[e^{\frac{2p}{\alpha(t)}(\frac{\varrho^2}{\alpha(t)^2}2A^2 + \frac{1}{\alpha(t)^2}2B^2)\sum_{k=0}^{2+n-1}\frac{1}{2\sigma_k^2}}] \leq \mathbb{E}[e^{\frac{2p}{\alpha(t)}\frac{\varrho^2}{\alpha(t)^2}4A^2\sum_{k=0}^{2+n-1}\frac{1}{2\sigma_k^2}}]^{1/2}\mathbb{E}[e^{\frac{2p}{\alpha(t)}\frac{1}{\alpha(t)^2}4B^2\sum_{k=0}^{2+n-1}\frac{1}{2\sigma_k^2}}]^{1/2}, \end{array}$$

where

$$A = \sum_{j\in N}(h_j(\tau_j))^-, \quad B = \sum_{j\in N}\sup_{0<v\leq t}|m_v^j|.$$

From the inequality (4.2), for ϱ sufficiently small, the expectation $\mathbb{E}[e^{\frac{2p}{\alpha(t)}\frac{\varrho^2}{\alpha(t)^2}4A^2\sum_{k=0}^{2+n-1}\frac{1}{2\sigma_k^2}}]$ is finite. Next we apply Lemma B.1 to control the terms with $B = \sum_{j\in N}\sup_{0<v\leq t}|m_v^j|$. We see that, for $t > 0$ small enough, the last expectation $\mathbb{E}[e^{\frac{2p}{\alpha(t)}\frac{1}{\alpha(t)^2}4B^2\sum_{k=0}^{2+n-1}\frac{1}{2\sigma_k^2}}]$ is also finite. The lemma is proved. ∎

Lemma 4.9. *For any $0 < t < t'$, for any $p \geq 1$, the random variable $e^{\int_t^{t'}\gamma_s^l ds}$ is p times $\mathbb{Q}$ integrable.*

Proof. We have

$$\begin{array}{rcl} \int_t^{t'}\gamma_s^l ds & = & \int_0^t \mathbb{1}_{\{t\leq\tau_h\}}\frac{\dot{h}_l(s)}{\alpha(s)}\psi^l_{\mathcal{J}_s,\rho_s,\sigma_s}(Z_s^{j,\mathcal{I}_s}(s), j\in\mathcal{J}_s)ds \\ & \leq & C\int_t^{t'}\psi^l_{\mathcal{J}_s,\rho_s,\sigma_s}(Z_s^{j,\mathcal{I}_s}(s), j\in\mathcal{J}_s)ds \leq C\int_t^{t'}\frac{\phi_{\sigma_s}}{\Phi_{\sigma_s}}(Z_s^{l,\mathcal{I}_s}(s))ds \leq C\int_t^{t'}\frac{\phi_{\sigma_s}}{\Phi_{\sigma_s}}(\sup_{t<v\leq t'}\frac{h_l(v)}{\alpha(v)} + U_{t'})ds \\ & = & C(t'-t)\sum_{k=0}^{2+n-1}\frac{\phi_{\sigma_k}}{\Phi_{\sigma_k}}(\sup_{t<v\leq t'}\frac{h_l(v)}{\alpha(v)} + U_{t'}) \\ & \leq & C(t'-t)(\sum_{k=0}^{2+n-1}\frac{2}{\sigma_k^2})(\sup_{t<v\leq t'}\frac{h_l(v)}{\alpha(v)} + U_{t'}), \end{array}$$

thanks to Lemma 4.6. The random variables $h_j(\tau_j)$ and m_t^j are Gaussian random variables. So (cf. Lemma 4.8), $e^{U_{t'}}$ has any integer level of $\mathbb{Q}$ integrability. The lemma is proved. ∎

4.3. **Proof of Theorem 4.1**

We want to prove the $\mathbb{Q}$ integrability of $e^{\int_0^T \mathbb{1}_{\{s\leq\tau_{-1}\wedge\tau_0\}}(\gamma_s^{-1}+\gamma_s^0)ds}$ when ϱ is small enough. But

$$e^{\int_0^T \mathbb{1}_{\{s\leq\tau_{-1}\wedge\tau_0\}}(\gamma_s^{-1}+\gamma_s^0)ds} = e^{\int_0^{t_0} \mathbb{1}_{\{s\leq\tau_{-1}\wedge\tau_0\}}(\gamma_s^{-1}+\gamma_s^0)ds}e^{\int_{t_0}^T \mathbb{1}_{\{s\leq\tau_{-1}\wedge\tau_0\}}(\gamma_s^{-1}+\gamma_s^0)ds},$$

for the t_0 of Lemma 4.8. Combining Lemmas 4.8 and 4.9 proves the theorem.

4.4. **Proof of Theorem 4.2**

The density process of the invariance probability $\mathbb{P}$ is a Dolean-Dade exponential $\mathcal{E}(X)$ for some $(\mathbb{F},\mathbb{Q})$ local martingale X ($X_0=0$). The Girsanov theorem gives the $(\mathbb{F},\mathbb{P})$ semimartingale decompositions of the B^k and of the $\mathbb{1}_{[\tau_i,\infty)}$ in terms of X. The drift parts of these decompositions are determined by the martingale invariance property, i.e., for $k\in N$,

$$\int_0^t \bar{\beta}^k_s ds + \langle \bar{W}^k, X^c\rangle_t = \int_0^t \widetilde{\beta}^k_s ds = \int_0^t \bar{\beta}^k_s ds + \langle \bar{W}^k, \nu\rangle_t,\ t\in[0,T],$$

(cf. Lemma 3.6) and, for $i\in N^\star$,

$$\int_0^t \bar{\gamma}^i_s ds + \langle \bar{M}^i, X^d\rangle_t = \int_0^t \widetilde{\gamma}^i_s ds = \int_0^t \bar{\gamma}^i_s ds + \langle \bar{M}^i, \sum_{j\in N^\star}(\frac{\widetilde{\gamma}^j_-}{\bar{\gamma}^j_-}-1)\centerdot \bar{M}^j\rangle_t,\ t\in[0,T],$$

as follows from a direct computation. Recall that the martingale representation property holds in $(\mathbb{F},\mathbb{Q})$ with respect to $\bar{W}^i$ and to $\bar{M}^j$. The above equations imply that X and μ have the same martingale representation. Consequently, $X=\mu$.

Remark 4.10. As a consequence of the above theorem, the Dolean-Dade exponential $\mathcal{E}(\mu)$ is a true martingale, which is hard to prove directly.

A. Tail Moment Estimates

The following tail moment estimates are of independent interest.

Lemma A.1. *Consider a positive decreasing continuously differentiable function $\Gamma(y)$ defined on $\mathbb{R}_+$. Set $g(y)=-\frac{\Gamma'(y)}{\Gamma(y)}$. Let $k\geq 0$ be an integer such that $\lim_{t\uparrow\infty} t^{k-1}\Gamma(t)\to 0$ and $\int_{\mathbb{R}_+} t^k\Gamma(t)dt<\infty$. Set $G(y)=\int_y^\infty t^k\Gamma(t)dt, y\geq 0$ (the tail moment function).*
(i) *Suppose that there exist $\alpha>0,\alpha'\geq 0,\bar{y}\geq 0$ such that*

$$g(y)\geq \alpha y+\alpha' \quad \text{for} \quad y\geq\bar{y}.$$

Then, in the case where $k\leq 1$, we have

$$G(y)-\frac{1}{\alpha}y^{k-1}\Gamma(y)\leq 0, \quad \text{for} \quad y>\bar{y}.$$

In the case where $k>1$ and $\alpha'>0$, we have

$$G(y)-\frac{1}{\alpha}y^{k-1}\Gamma(y)\leq 0, \quad \text{for} \quad y\geq\bar{y}\vee\frac{k-1}{\alpha'}.$$

In the case where $k>1$ and $\alpha'=0$, for any $\epsilon>0$, we have

$$G(y)-(\frac{1}{\alpha}+\epsilon)y^{k-1}\Gamma(y)\leq 0, \quad \text{for} \quad y\geq\bar{y}\vee\sqrt{\frac{(\frac{1}{\alpha}+\epsilon)(k-1)}{\epsilon\alpha}}.$$

(ii) *Suppose that there exist $\beta>0,\beta'\geq 0,\bar{y}\geq 0$ such that*

$$g(y)\leq \beta y-\beta' \quad \text{for} \quad y\geq\bar{y}.$$

Then, in the case where $k = 0$, if $\beta' > 0$, we have

$$G(y) - \frac{1}{\beta} y^{-1}\Gamma(y) \geq 0, \quad \textit{for} \quad y > \bar{y} \vee \frac{1}{\beta'},$$

while if $\beta' = 0$, for any $\epsilon > 0$, we have

$$G(y) - (\frac{1}{\beta} - \epsilon) y^{-1}\Gamma(y) \geq 0, \quad \textit{for} \quad y > \bar{y} \vee \sqrt{\frac{(\frac{1}{\beta} - \epsilon)^+}{\epsilon\beta}}.$$

In the case where $k \geq 1$, we have

$$G(y) - \frac{1}{\beta} y^{k-1}\Gamma(y) \geq 0, \quad \textit{for} \quad y \geq \bar{y}.$$

Proof. (i) Let φ be a positive continuously differentiable function on $\mathbb{R}_+^*$. We compute the derivative on $y > 0$:

$$\begin{aligned} (G(y) - \varphi(y)\Gamma(y))' &= -y^k\Gamma(y) - \varphi'(y)\Gamma(y) + \varphi(y)g(y)\Gamma(y) = (\varphi(y)g(y) - y^k - \varphi'(y))\Gamma(y) \\ &\geq (\varphi(y)(\alpha y + \alpha') - y^k - \varphi'(y))\Gamma(y), \quad \text{if} \quad y \geq \bar{y}. \end{aligned}$$

For $\epsilon \geq 0$ consider $\varphi(y) = (\frac{1}{\alpha} + \epsilon)y^{k-1}$. We have

$$\begin{aligned} & \varphi(y)(\alpha y + \alpha') - y^k - \varphi'(y) \\ = & (\tfrac{1}{\alpha} + \epsilon)y^{k-1}(\alpha y + \alpha') - y^k - (\tfrac{1}{\alpha} + \epsilon)(k-1)y^{k-2} \\ = & (1 + \epsilon\alpha)y^k + (\tfrac{1}{\alpha} + \epsilon)\alpha' y^{k-1} - y^k - (\tfrac{1}{\alpha} + \epsilon)(k-1)y^{k-2} \\ = & \epsilon\alpha y^k + (\tfrac{1}{\alpha} + \epsilon)\alpha' y^{k-1} - (\tfrac{1}{\alpha} + \epsilon)(k-1)y^{k-2} \\ = & (\epsilon\alpha y^2 + (\tfrac{1}{\alpha} + \epsilon)\alpha' y - (\tfrac{1}{\alpha} + \epsilon)(k-1))y^{k-2}. \end{aligned}$$

For $k = 0$ or $k = 1$, we take $\epsilon = 0$. Because $\frac{1}{\alpha}\alpha' y - \frac{1}{\alpha}(k-1) \geq 0$, the function $G(y) - \frac{1}{\alpha}y^{k-1}\Gamma(y)$ is not decreasing on $y > \bar{y}$. As $\lim_{y\uparrow\infty}\big(G(y) - \frac{1}{\alpha}y^{k-1}\Gamma(y)\big) = 0$, we conclude

$$G(y) - \frac{1}{\alpha}y^{k-1}\Gamma(y) \leq 0, \quad \text{for} \quad y > \bar{y}.$$

Suppose $k > 1$. If $\alpha' > 0$, we take $\epsilon = 0$. Because $\frac{1}{\alpha}\alpha' y - \frac{1}{\alpha}(k-1) \geq 0$ for $y \geq \frac{k-1}{\alpha'}$, the function $G(y) - \frac{1}{\alpha}y^{k-1}\Gamma(y)$ is not decreasing on $y \geq \bar{y} \vee \frac{k-1}{\alpha'}$. As $\lim_{y\uparrow\infty}\big(G(y) - \frac{1}{\alpha}y^{k-1}\Gamma(y)\big) = 0$, we conclude

$$G(y) - \frac{1}{\alpha}y^{k-1}\Gamma(y) \leq 0, \quad \text{for} \quad y \geq \bar{y} \vee \frac{k-1}{\alpha'}.$$

Suppose $k > 1$ and $\alpha' = 0$, we take an $\epsilon > 0$. Because $\epsilon\alpha y^2 - (\frac{1}{\alpha} + \epsilon)(k-1) \geq 0$ for $y \geq \sqrt{\frac{(\frac{1}{\alpha}+\epsilon)(k-1)}{\epsilon\alpha}}$, the function $G(y) - (\frac{1}{\alpha} + \epsilon)y^{k-1}\Gamma(y)$ is not decreasing on $y \geq \bar{y} \vee \sqrt{\frac{(\frac{1}{\alpha}+\epsilon)(k-1)}{\epsilon\alpha}}$. As $\lim_{y\uparrow\infty}\big(G(y) - (\frac{1}{\alpha} + \epsilon)y^{k-1}\Gamma(y)\big) = 0$, we conclude

$$G(y) - (\frac{1}{\alpha} + \epsilon)y^{k-1}\Gamma(y) \leq 0, \quad \text{for} \quad y \geq \bar{y} \vee \sqrt{\frac{(\frac{1}{\alpha} + \epsilon)(k-1)}{\epsilon\alpha}}.$$

The first part of the lemma is proved.

(ii) We begin with

$$\begin{aligned} (G(y) - \varphi(y)\Gamma(y))' &= (\varphi(y)g(y) - y^k - \varphi'(y))\Gamma(y) \\ &\leq (\varphi(y)(\beta y - \beta') - y^k - \varphi'(y))\Gamma(y), \quad \text{if} \quad y \geq \bar{y}. \end{aligned}$$

For $\epsilon \geq 0$ consider $\varphi(y) = (\frac{1}{\beta} - \epsilon)y^{k-1}$. We have

$$\begin{aligned}
& \varphi(y)(\beta y - \beta') - y^k - \varphi'(y) \\
= \; & (\tfrac{1}{\beta} - \epsilon)y^{k-1}(\beta y - \beta') - y^k - (\tfrac{1}{\beta} - \epsilon)(k-1)y^{k-2} \\
= \; & (1 - \epsilon\beta)y^k - (\tfrac{1}{\beta} - \epsilon)\beta' y^{k-1} - y^k - (\tfrac{1}{\beta} - \epsilon)(k-1)y^{k-2} \\
= \; & -\epsilon\beta y^k - (\tfrac{1}{\beta} - \epsilon)\beta' y^{k-1} - (\tfrac{1}{\beta} - \epsilon)(k-1)y^{k-2} \\
= \; & -(\epsilon\beta y^2 + (\tfrac{1}{\beta} - \epsilon)\beta' y + (\tfrac{1}{\beta} - \epsilon)(k-1))y^{k-2}.
\end{aligned}$$

Suppose $k = 0$. If in addition $\beta' > 0$, we take $\epsilon = 0$. Because $\frac{1}{\beta}\beta' y + \frac{1}{\beta}(-1) \geq 0$ for $y \geq \frac{1}{\beta'}$, the function $G(y) - \frac{1}{\beta}y^{-1}\Gamma(y)$ is not increasing on $y > \bar{y} \vee \frac{1}{\beta'}$. As $\lim_{y\uparrow\infty}\Big(G(y) - \frac{1}{\beta}y^{-1}\Gamma(y)\Big) = 0$, we conclude

$$G(y) - \frac{1}{\beta}y^{-1}\Gamma(y) \geq 0, \quad \text{for} \quad y > \bar{y} \vee \frac{1}{\beta'}.$$

If instead $\beta' = 0$, we take $\epsilon > 0$. Because $\epsilon\beta y^2 + (\frac{1}{\beta} - \epsilon)(-1) \geq 0$ for $y \geq \sqrt{\frac{(\frac{1}{\beta}-\epsilon)^+}{\epsilon\beta}}$, the function $G(y) - (\frac{1}{\beta} - \epsilon)y^{-1}\Gamma(y)$ is not increasing on $y > \bar{y} \vee \sqrt{\frac{(\frac{1}{\beta}-\epsilon)^+}{\epsilon\beta}}$. As $\lim_{y\uparrow\infty}\Big(G(y) - (\frac{1}{\beta} - \epsilon)y^{-1}\Gamma(y)\Big) = 0$, we conclude

$$G(y) - (\frac{1}{\beta} - \epsilon)y^{-1}\Gamma(y) \geq 0, \quad \text{for} \quad y > \bar{y} \vee \sqrt{\frac{(\frac{1}{\beta} - \epsilon)^+}{\epsilon\beta}}.$$

Suppose $k \geq 1$. We take $\epsilon = 0$. Because $\frac{1}{\beta}\beta' y + \frac{1}{\beta}(k-1) \geq 0$, the function $G(y) - \frac{1}{\beta}y^{k-1}\Gamma(y)$ is not increasing on $y \geq \bar{y}$. As $\lim_{y\uparrow\infty}\Big(G(y) - \frac{1}{\beta}y^{k-1}\Gamma(y)\Big) = 0$, we conclude

$$G(y) - \frac{1}{\beta}y^{k-1}\Gamma(y) \geq 0, \quad \text{for} \quad y \geq \bar{y}. \blacksquare$$

B. Gaussian Integrability

Lemma B.1. *Let $m_t = \int_0^t \varsigma(s)dB_s$, where B is a a univariate standard Brownian motion and ς is a square integrable function with unit L^2 norm. For any constant $q > 0$, $e^{q\sup_{0\leq s\leq t} m_s^2}$ is integrable for sufficiently small t.*

Proof. The process $(m_t)_{t\geq 0}$ is equal in law to a time changed Brownian motion $(W_{\bar{t}})_{t\geq 0}$, where W is a a univariate standard Brownian motion and $\bar{t} = \int_0^t \varsigma^2(s)ds$ goes to 0 with t. Thus, it suffices to show the result with m replaced by W. Let r_t be the density function of the law of $\sup_{0\leq s\leq t}|W_s|$ and let $R_t(y) = \int_y^\infty r_t(x)dx, y > 0$, so that

$$\mathbb{E}[e^{q\sup_{0\leq s\leq t} W_s^2}] = \int_0^\infty e^{qy^2} r_t(y)dy = -[R_t(y)e^{qy^2}]_0^\infty + 2q\int_0^\infty yR_t(y)e^{qy^2}dy \tag{B.1}$$

and (using the reflection principle of the Brownian motion)

$$\begin{aligned}
R_t(y) &= \mathbb{Q}[\sup_{0\leq s\leq t}(W_s^+ + W_s^-) > y] \leq \mathbb{Q}[\sup_{0\leq s\leq t} W_s^+ > \tfrac{y}{2}] + \mathbb{Q}[\sup_{0\leq s\leq t} W_s^- > \tfrac{y}{2}] \\
&= 2\mathbb{Q}[\sup_{0\leq s\leq t} W_s > \tfrac{y}{2}] = 2\mathbb{Q}[|W_t| > \tfrac{y}{2}] = 2\mathbb{Q}[|W_1| > \tfrac{y}{2\sqrt{t}}] = 4\Phi(\tfrac{y}{2\sqrt{t}}),
\end{aligned}$$

where by Lemma 4.6

$$\Phi(\frac{y}{2\sqrt{t}})\frac{y}{2\sqrt{t}} \leq \phi(\frac{y}{2\sqrt{t}}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{8t}}, \;\; y > 0.$$

Therefore, for $\frac{1}{8t} > q$, both terms are finite in the right hand side of (B.1), which shows the result. ∎

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