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OPTIONS PRICES IN INCOMPLETE MARKETS*

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Abstract. In this paper we consider the valuation of an option with time to expiration T and pay-off function g which is a convex function (as is a European call option), and constant interest rate $r = 0$, for a variety of underlying price process models constructed from two independent Poisson processes, and an independent Brownian motion. This gives rise to incomplete market models with an infinite number of risk neutral measures. The collection of risk neutral measures gives rise to different prices, which comprise intervals that we calculate. The intervals can vary dramatically depending on the model parameters.

1. Introduction

One might naively think that the subject of Mathematical Finance is a mature subject, but in fact the subject has fundamental topics that remain vastly unexplored. The largest elephant in the room is the topic of incomplete markets. Here one has an infinite number of possible choices for a risk neutral measure. While there are some topics where such a situation can be seen as an aid in the theory, in particular the theory of mathematical models of financial bubbles, most of the time an infinite choice of risk neutral measures presents serious obstacles to a coherent theory.

Quite a bit of published research exists addressing the topic of how to choose a risk neutral probability measure, using different ideas. For example, giving a typical paper in each rubric, there is the idea of minimizing quadratic hedging risk error [8], the idea of choosing a risk neutral measure via indifference pricing [2], or the idea of minimizing the entropy between the objective measure and the risk neutral measure [10, 11]. More recently there is the idea that the market can choose a risk neutral measure for pricing, as proposed in [13, 19] and implemented in the form of calibration methods in the industry. But the issue is far from settled.

It therefore seems reasonable to address what might be the range of prices under either all the risk neutral measures, or under a reasonable subset of them. One of the tools developed for this purpose is that of super hedging.

In this article we address the range of option prices in some simple models of incomplete markets. The now classic papers in this direction are that of Eberlein and Jacod [7] for the purely discontinuous case and infinite activity jumps and that of Frey and Sin [9] for the continuous case with stochastic volatility. Here we consider various manifestations of combinations of two independent compensated Poisson processes combined with a Brownian motion, or not. In view of the afore-mentioned papers the results, especially concerning the ranges of option prices, are perhaps not very surprising, but point toward the fact that almost anything can happen, even in such a simple case.

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After some general results, in order to get more tractable results, and following the current prevalent customs, we restrict ourselves to risk neutral measures that preserve the property that the price process is a stochastic exponential of a Lévy process, or at least an additive process (a process with independent, but not necessarily stationary, increments). The advantage of the Lévy and Additive frameworks is that they allow some explicit calculations, which allows one to calculate the ranges of option prices for this collection of models.

2. The Basics

In this section we review the basics. Even for these well known topics, there might be some subtleties not yet realized by some potential readers of this article. We shall attempt to point them out.

We let $T > 0$ and we work on the compact time interval $[0, T]$. This is known as a *finite horizon paradigm*. We also assume throughout that the interest rate is $r = 0$: when the instantaneous rate r_t is not identically vanishing at all times t we can simply replace below the price S_t at time t of any asset by the discounted price R_tS_t, where $R_t = \exp(-\int_0^t r_s\,ds)$, so assuming $r = 0$ is really not a restriction.

Definition 1 (Financial Model). *A Financial Model is a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ *with* $\mathcal{F} = \mathcal{F}_T$ *and where* $\mathcal{F}_0$ *is the trivial* $\sigma-$*algebra a.s. In addition we have a collection* $S = (S^i)_{1\leq i\leq d}$ *of adapted positive price processes (the prices of* d *assets). We denote such a model by* $(\Omega, (\mathcal{F}_t), (S^i), P)$.

There are a variety of conditions implying that the model does not allow for some kind of arbitrage. They typically state that some class of probability measures on $(\Omega, \mathcal{F})$ is not empty. We let $\mathcal{Q}_{LMG}$, respectively $\mathcal{Q}_{MG}$, respectively $\mathcal{Q}_{\sigma MG}$, denote the set of all probability measures Q which are equivalent to P and for which all of the S^i are local martingales, respectively martingales, respectively sigma martingales, and obviously $\mathcal{Q}_{MG} \subset \mathcal{Q}_{LMG} \subset \mathcal{Q}_{\sigma MG}$. The most often used notion of no-arbitrage is the now classic and standard condition called NFLVR for "No Free Lunch with Vanishing Risk". This has been introduced by Delbaen and Schachermayer, who proved in [4–6] that NFLVR is equivalent to having $\mathcal{Q}_{\sigma MG} \neq \emptyset$.

Note that, by virtue of the classic argument of Dellacherie, this condition implies that each S^i has to be a semimartingale. Note also that any positive sigma martingale is a local martingale, so in our setting where $S^i_t > 0$ for all i we have $\mathcal{Q}_{\sigma MG} = \mathcal{Q}_{LMG}$. For applications such as futures, which can be negatively priced, or for example electricity prices where in some markets prices can go negative, $\mathcal{Q}_{\sigma MG}$ might come into play. But here we are excluding those possibilities.

The main interest in the subject is to find the correct prices for contingent claims, and also a replicating strategy, or at least a reasonable hedging strategy. By a *contingent claim* we mean a nonnegative random variable $U \in \mathcal{F}_T$. Among these, of particular interest are the puts and calls written on a single price process, say S^1: the claim is then $U = g(S^1_T)$, with a payoff function g defined on $\mathbb{R}_+$, and we restrict our attention to call options only. The pay-off function g is assumed to satisfy

$$g \text{ is convex}, \;\; 0 \leq g(x) \leq x, \quad \lim_{x\to\infty} \frac{g(x)}{x} = 1, \quad \text{hence } g \text{ is non-decreasing.} \tag{1}$$

Among all call functions, the identity function $g(x) \equiv x$ is called the *trivial one*, and it is generally excluded in what follows. However note that $g(x) = x$ permits the underlying price process itself to be considered as a call option.

For the European version of the option, the price of the option is generally taken to be

$$\Pi_e(g)^Q := E_Q(g(S^1_T)) \qquad \text{for some } Q \in \mathcal{Q}_{LMG}. \tag{2}$$

with the subscript e standing for European. Similarly, the subscript a stands for American and the price of the American version, which can be exercised at an arbitrary time τ in the set $\mathcal{T}$ of all stopping times τ satisfying $\tau \leq T$, is usually taken to be

$$\Pi_a(g)^Q := \sup_{\tau\in\mathcal{T}} E_Q(g(S^1_\tau)) \qquad \text{for some } Q \in \mathcal{Q}_{LMG}. \tag{3}$$

There are well known relationships (some of them really obvious), using in particular Doob's stopping theorem and the fact that under $Q \in \mathcal{Q}_{LMG}$ the process S^1_t is a supermartingale, and is a martingale if $Q \in \mathcal{Q}_{MG}$, implying (in this case) that $g(S^1_t)$ is a submartingale because of Jensen's inequality; we also denote by R the convex support of S^1_T, that is the smallest closed interval such that $P(S^1_T \in R) = 1$; saying that g is "not affine on R" implies that R has positive length:

$$
\begin{array}{l}
\bullet\ \ \Pi_e(g)^Q \leq \Pi_a(g)^Q \leq S^1_0, \qquad \Pi_a(g)^Q \geq g(S^1_0) \\
\bullet\ \ \Pi_e(g)^Q = S^1_0 \ \ \Leftrightarrow \ \ g \text{ is trivial and } Q \in \mathcal{Q}_{MG} \\
\bullet\ \ \text{if } Q \in \mathcal{Q}_{MG}: \ \left\{ \begin{array}{l} \Pi_e(g)^Q = \Pi_a(g)^Q \geq g(S^1_0) \\ \Pi_e(g)^Q = g(S^1_0) \Leftrightarrow \text{if } g \text{ is affine on } R \end{array} \right. \\
\bullet\ \ \text{if } Q \in \mathcal{Q}_{LMG} \backslash \mathcal{Q}_{MG} \text{ and } g \text{ is affine on } R: \ \ \Pi_e(g)^Q < g(S^1_0).
\end{array}
\tag{4}
$$

Since we always have $\Pi_e(g)^Q \leq S^1_0$, it is natural to ask if S^1_0 is the supremum of all possible prices with Q ranging over all of $\mathcal{Q}_{MG}$ (hence over all of $\mathcal{Q}_{LMG}$ as well). There is a simple, and useful, criterion for this property (it is basically due to Frey and Sin in [9], however we reprove it here because our setting is more general than in that paper):

Proposition 1. *Let g be a non trivial call pay-off function and Q_n a sequence of measures in $\mathcal{Q}_{MG}$. Then $\Pi_e(g)^{Q_n} \to S^1_0$ if and only if $S^1_T \xrightarrow{Q_n} 0$.*

Proof. For any $\varepsilon > 0$ we have $g(x) \leq x - (\varepsilon - g(\varepsilon))\, 1_{\{x>\varepsilon\}}$. Thus, since $E_{Q_n}(S^1_T) = S^1_0$,

$$\Pi_e(g)^{Q_n} \leq S^1_0 - (\varepsilon - g(\varepsilon))\, Q_n(S^1_T > \varepsilon).$$

If $\Pi_e(g)^{Q_n} \to S^1_0$, and since $g(\varepsilon) < \varepsilon$, we must have $Q_n(S^1_T > \varepsilon) \to 0$, hence $S^1_T \xrightarrow{Q_n} 0$.

Conversely, assume $S^1_T \xrightarrow{Q_n} 0$. For any $\varepsilon > 0$ there is $A < \infty$ such that $g(x) \geq (1-\varepsilon)x$ when $x > A$, hence $\Pi_e(g)^{Q_n} \geq (1-\varepsilon) E_{Q_n}(S^1_T\, 1_{\{S^1_T > A\}})$. Moreover, $E_{Q_n}\big(S^1_T\, 1_{\{S^1_T \leq A\}}\big) \leq \eta + A\, Q_n(S^1_T > \eta)$ for any $\eta > 0$, so $E_{Q_n}(S^1_T\, 1_{\{S^1_T > A\}}) \to 0$ because of our hypothesis. Since $E_{Q_n}(S^1_T) = S^1_0$, we deduce that

$$E_{Q_n}\big(S^1_T\, 1_{\{S^1_T > A\}}\big) = S^1_0 - E_{Q_n}\big(S^1_T\, 1_{\{S^1_T \leq C\}}\big) \to S^1_0,$$

hence $\liminf \Pi_e(g)^{Q_n} \geq (1-\varepsilon) S^1_0$. Since $\varepsilon > 0$ is arbitrarily small, whereas $\Pi_e(g)^{Q_n} \leq S^1_0$ by (4), we deduce $\Pi_e(g)^{Q_n} \to S^1_0$. □

2.1. Pricing via hedging

Recall that a strategy (relative to the asset price) is any predictable process θ such that the stochastic integral process $\theta \cdot S_t = \int_0^t \theta_s\, dS_s$ is well defined. The strategy is called *admissible* if the process $\theta \cdot S$ is bounded from below, and $\mathcal{A}$ is the set of admissible strategies. If $\theta \in \mathcal{A}$ and $x \in \mathbb{R}$, we say that the pair (x, θ) *replicates* U if $x + \theta \cdot S_T = U$ a.s., and it *super-replicates* U if $x + \theta \cdot S_T \geq U$ a.s. The super-replication price of the European option with pay-off function g is, with the usual convention $\inf \emptyset = +\infty$:

$$\Pi_e(g)^{sup} = \inf\big(x: \text{ there is } \theta \in \mathcal{A} \text{ such that } (x,\theta) \text{ super-replicates } g(S^1_T)\big). \tag{5}$$

For an American option the replication should hold at any time, that is, (x, θ) *strongly replicates* $g(S^1_\cdot)$ as a process if we have $x + \theta \cdot S^1_\tau = g(S^1_\tau)$ a.s. for any stopping time $\tau \in \mathcal{T}$, or equivalently (by P.A. Meyer's section theorem) the process $x + \theta \cdot S^1 - g(S^1)$ is a.s. vanishing. *Strong super-replication* is defined analogously, with $x + \theta \cdot S^1 - g(S^1)$ being a.s. a nonnegative process. The super-replication price is then

$$\Pi_a(g)^{sup} = \inf\big(x: \text{ there is } \theta \in \mathcal{A} \text{ such that } (x,\theta) \text{ strongly super-replicates } g(S^1_\cdot)\big). \tag{6}$$

2.2. **The complete case**

The model $(\Omega, (\mathcal{F}_t), (S^i), P)$ is called *complete* if any bounded claim can be replicated. This notion depends on S, obviously, and also *a priori* on P and on the filtration $(\mathcal{F}_t)$, and below we denote by $(\mathcal{F}_t^S)$ the sub-filtration generated by S.

As far as P is concerned, and since the stochastic integral processes are unchanged under an absolutely continuous change of measure, it follows that if
$(\Omega, (\mathcal{F}_t), (S^i), P)$ is complete, then so is $(\Omega, (\mathcal{F}_t), (S^i), Q)$ for any $Q << P$.

The following result is well known, see e.g. Corollary (11.4) in [12]:

Theorem 1. *Assuming $\mathcal{Q}_{LMG} \neq \emptyset$, the following statements are equivalent:*
(a) The model is complete
(b) $\mathcal{Q}_{LMG}$ is a singleton.
(c) The martingale representation property relative to the Q-local martingale S holds for some $Q \in \mathcal{Q}_{LMG}$.
(d) The martingale representation property relative to the Q-local martingale S holds for all $Q \in \mathcal{Q}_{LMG}$.

Assume that the model is complete, and let g be as in (1). Since $g(S_T^1)$ is Q-integrable by (1), there is $(x, \theta) \in \mathbb{R} \times \mathcal{A}$ such that $x + \theta \cdot S_T = g(S_T^1)$, and $x = E_Q(g(S_T^1))$ and $x + \theta \cdot S_\tau = E_Q(g(S_T^1) \mid \mathcal{F}_\tau)$ for any $\tau \in \mathcal{T}$. If $(x', \theta') \in \mathbb{R} \times \mathcal{A}$ is another replicating strategy, the process $x' + \theta' \cdot S$ is a local martingale bounded from below, hence a supermartingale and thus $x' \geq E_Q(x' + \theta' \cdot S_T) = E_Q(g(S_T^1)) = x$ (in fact, $x' = x$ implies $\theta' = \theta$ up to a suitable null set, and otherwise $\theta' \cdot S$ is a strict local martingale). Exactly the same argument shows that if (x', θ') super-replicates $g(S_T^1)$ we also have $x' \geq x$. Therefore, we have

$$\Pi_e(g)^{sup} = \Pi_e(g)^Q \qquad \text{if} \;\; \mathcal{Q}_{LMG} = \{Q\}. \tag{7}$$

On the other hand, the strategy (x, θ) defined above does *not* strongly replicate $g(S_\cdot^1)$, unless the process $g(S_\cdot^1)$ is itself a Q-martingale, which implies that $Q \in \mathcal{Q}_{MG}$ and that g is affine on the union $\cup_{t \in [0,T]} R_t$ of all convex supports R_t of S_t. If we are not in this very special case and nevertheless $Q \in \mathcal{Q}_{MG}$, we have

$$\Pi_e(g)^{sup} = \Pi_e(g)^Q = \Pi_a(g)^Q < \Pi_a(g)^{sup}. \tag{8}$$

This simple inequality casts some doubt about the use of strong super-replicating strategies for pricing an American option.

Note also that the submodel obtained by taking a smaller time horizon $T' < T$ is also complete: indeed, if U is bounded $\mathcal{F}_{T'}$-measurable, it can be replicated as $U = x + \phi \cdot S_T$ with $\phi \cdot S$ being a Q-martingale. Then we have $x + \phi \cdot S_{T'} = E(x + \phi \cdot S_T \mid \mathcal{F}_{T'}) = E(U \mid \mathcal{F}_{T'}) = U$.

Finally, observe that completeness is a notion relative to the process S and the measure P, of course, but it is also to the filtration $(\mathcal{F}_t)$, in the sense that the replicating strategy is $(\mathcal{F}_t)$-predictable, whereas the claim U is $\mathcal{F}_T$-measurable. A natural question arises, namely, what happens if we change the filtration, and in particular if we restrict it to be $(\mathcal{F}_t^S)$? In this direction, we have:

(1) A first natural conjecture would be that completeness implies $\mathcal{F}_t^S = \mathcal{F}_t$ (up to null sets), because adding "extra randomness" on top of S should impair the martingale representation property. However, this is *wrong* in general, as seen in the following example.[1].

Take a Brownian motion W and $\mathcal{F}_t = \mathcal{F}_t^W$, set $B_t = \int_0^t \mathrm{sign}(W_w)\, dW_s$, which is again an $(\mathcal{F}_t)$-Brownian motion, and consider the one-dimensional price process $S_t = e^{B_t - t/2}$ which is a geometric Brownian motion, so (by the martingale representation property) the model $(\Omega, (\mathcal{F}_t^S), S, P)$ is complete, see Section 3. Since $W_t = \int_0^t \mathrm{sign}(W_s)\, dB_s$, again the martingale representation property (with respect to W this time) easily yields that the model $(\Omega, (\mathcal{F}_t), S, P)$ is also complete. However, the filtration $(\mathcal{F}_t^S)$ is the filtration generated by the process $|W_t|$ and is *strictly smaller* than $(\mathcal{F}_t)$.

[1]The authors thank Monique Jeanblanc for helpful discussions on this point

(2) Another conjecture is that completeness of $(\Omega, (\mathcal{F}_t), S, P)$ implies completeness of $(\Omega, (\mathcal{F}^S_t), S, P)$. The answer to this question seems to be unknown (the previous example does not contradict the conjecture).
The fact that the answer to such a simple question remains unknown indicates the subtleties of the subject.

2.3. The incomplete case

Now we suppose that the model is incomplete.

Suppose that (x, θ) super-replicates $U = g(S^1_T)$, and let $Q \in \mathcal{Q}_{LMG}$. Under Q the process $V = x + \theta \cdot S$ is bounded from below, hence necessarily also a local martingale and a supermartingale. Since $Y_t = E_Q(g(S^1_T) \mid \mathcal{F}_t)$ is a Q-martingale, the difference $V - Y$ is again a Q-supermartingale, which satisfies $V_T - Y_T \geq 0$ Q-a.s., whereas $V_0 - Y_0 = x - E_Q(U)$. Therefore we have

$$Q \in \mathcal{Q}_{LMG} \quad \Rightarrow \quad \Pi_e(g)^Q \leq \Pi(g)^{sup}. \tag{9}$$

We also have the following result of Kramkov [16, Theorem 3.2], unfortunately under an additional assumption on the price process S:

Theorem 2. *Suppose that the process S is locally bounded and that $\mathcal{Q}_{LMG} \neq \emptyset$. Then $\Pi_e(g)^{\text{sup}} = \sup(\Pi_e(g)^Q : Q \in \mathcal{Q}_{LMG})$ and there exists a super-hedging strategy with price $\Pi_e(g)^{\text{sup}}$.*

2.4. Traded options

From now on we only consider European options, so $\Pi_e(g)^Q$ is abbreviated as $\Pi(g)^Q$.

We have J European options with pay-off functions $g_1, \ldots, g_J$, all written on the same asset S^1 and with the same maturity T, for simplicity. We suppose that all these options are traded, the market price of the jth one at time t being Π^j_t. For no-arbitrage reasons the set $\mathcal{Q}_{LMG}(J)$ of all equivalent measures Q under which S and all Π^j are local martingales must be non-empty. We will also denote by $\mathcal{Q}_{LMG,MG}(J)$ and $\mathcal{Q}_{MG,MG}(J)$, respectively, the (possibly empty) subsets of all Q in $\mathcal{Q}_{LMG}$, resp. $\mathcal{Q}_{MG}$, such that all Π^j are martingales under Q.

If we believe that (2) gives us a price for a European option at time 0, for obvious no-arbitrage reasons we should use the same risk neutral measure Q when we price all the options with pay-off functions g_j; moreover, the price at time t of these options should be

$$\Pi(g_j)^Q_t = E_Q(g_j(S^1_T) \mid \mathcal{F}_t), \tag{10}$$

exactly as $S_t = E_Q(S_T \mid \mathcal{F}_t)$ under a risk neutral measure Q (at least for a *martingale* measure Q).

Now, it seems that in (10) nothing prevents us from taking a measure $Q_{t,\omega}$ in $\mathcal{Q}_{LMG}$ (or $\mathcal{Q}_{MG}$) which varies with time, and possibly with ω (in an $\mathcal{F}_t$-measurable way, of course). Since at time t the market price Π^j_t is known, for consistency we must have

$$\Pi^j_t = E_{Q_{t,\omega}}(g_j(S^1_T) \mid \mathcal{F}_t) \tag{11}$$

However, it is customary to use a single fixed measure Q. Although the choice of a Q independent of t, ω is not mathematically founded in a rigorous way, there are good reasons for this, as explained below.

Indeed, suppose that for some $Q \in \mathcal{Q}_{LMG}$ the "theoretical" price of our options is given by $E_Q(g_j(S^1_T))$, hence the "theoretical" price at time t should be given by (10). For consistency with the market price, we must also have $\Pi^j_t = E_Q(g_j(S^1_T) \mid \mathcal{F}_t)$: in other words, Π^j is a martingale under Q and we have (11) for the measure Q as well as for $Q_{t,\omega}$. Therefore, it does not seem to be a restriction that the market prices Π^j are given by

$$\Pi^j_t = E_Q(g_j(S^1_T) \mid \mathcal{F}_t) \qquad \text{for some} \ \ Q \in \mathcal{Q}_{LMG} \tag{12}$$

(then automatically $Q \in \mathcal{Q}_{LMG,MG}$).

The situation is different when considering the birth of financial bubbles, since in that case one can change the risk neutral measure and the market price, formerly a martingale under some $Q \in \mathcal{Q}_{MG}$ becomes a strict local martingale under a different equivalent probability measure $Q' \in \mathcal{Q}_{LMG} \backslash \mathcal{Q}_{MG}$, and we do not necessarily have a global absence of arbitrage. Instead we have an absence of arbitrage piecewise on sequential time intervals. See for example [1, 15].

3. Illustrative Examples of Incomplete Markets

From now on, $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}, P)$ is a filtered probability space endowed with a one-dimensional càdlàg processes S with $S_0 = 1$, and $\mathcal{F} = \mathcal{F}_T$ and $\mathcal{F}_t = \mathcal{F}_t^S$. Our examples of incomplete markets are mixtures of the two simplest complete models, which we quickly describe, mainly for notational purposes.

The simplest continuous complete market is the *Black-Scholes model*

$$S_t = S_t^\sigma = \mathcal{E}(\sigma W)_t = e^{\sigma W_t - t\sigma^2/2} \tag{13}$$

where W is a Brownian motion and $\sigma > 0$. (the Black-Scholes model usually incorporates a constant drift which is added to σW, yielding a probability measure P' which is equivalent to our measure P; hence for our concern the form (13) is not a restriction.) We have $\mathcal{Q}_{LMG} = \mathcal{Q}_{MG} = \{P\}$ and the price of a European option with pay-off function g is

$$\Pi_{BS}(g;\sigma) = E\big(g(e^{\sigma U\sqrt{T} - \sigma^2 T/2})\big), \qquad U \sim \mathcal{N}(0,1). \tag{14}$$

For models with jumps, the simplest example of a complete model is the stochastic exponential of a compensated Poisson process with jump size $b \in (-1,0) \cup (0,\infty)$, that is

$$S_t = S_t^{b,\alpha} = e^{-\alpha b t}\,(1+b)^{N_t^\alpha}, \tag{15}$$

where N^α is a Poisson process with intensity $\alpha > 0$. Then S is a positive martingale with $S_0 = 1$, and the price of a European option with pay-off function g is

$$\Pi(g; b, \alpha) = E(g(S_T^{b,\alpha})) = \sum_{n\geq 0} e^{-\alpha T}\,\frac{(\alpha T)^n}{n!}\,g\big(e^{-\alpha b T}\,(1+b)^n\big). \tag{16}$$

3.1. The setting

We will consider three different situations, with the notation (13) and (15) and with independent factors below and $b \neq b'$:

$$\textbf{(A):}\ \ S_t = S_t^\sigma S_t^{b,\alpha}, \quad \textbf{(B):}\ \ S_t = S_t^{b,\alpha} S_t^{b',\alpha'}, \quad \textbf{(C):}\ \ S_t = S_t^\sigma S_t^{b,\alpha} S_t^{b',\alpha'}.$$

In order to accommodate these three cases at once, it is convenient to rewrite S as

$$S_t = \mathcal{E}(Y + b\overline{N} + b'\overline{N}')_t = e^{Y_t - \zeta t}\,(1+b)^{N_t}(1+b')^{N'_t}, \qquad \zeta = \frac{\sigma^2}{2} + \alpha b + \alpha' b'. \tag{17}$$

In this formula, we assume the jump sizes $b, b' \in (-1,0) \cup (0,\infty)$ with $b < b'$, and $\sigma, \alpha' \geq 0$ and $\alpha > 0$; then we take $Y = \sigma W$ with W is a Brownian motion, and N and N' are two independent Poisson process with intensities α and α' (so $N'_t \equiv 0$ if $\alpha' = 0$), independent of W, and $\overline{N}_t = N_t - \alpha t$ and $\overline{N}'_t = N'_t - \alpha' t$. The three cases above correspond to:

$$\textbf{(A)}: \quad \sigma > 0,\ \alpha' = 0, \quad \textbf{(B)}: \quad \sigma = 0,\ \alpha' > 0, \quad \textbf{(C)}: \quad \sigma > 0,\ \alpha' > 0. \tag{18}$$

The filtration $(\mathcal{F}_t) = (\mathcal{F}_t^S)$ is also the filtration generated by (Y, N, N'), or equivalently by $(Y, \overline{N}, \overline{N}')$.

Note that S is a martingale with $S_0 = 1$ and $\mathcal{F}_0$ is P-a.s. trivial. So $P \in \mathcal{Q}_{MG}$, and obviously $\log S$ is a Lévy process under P. It will be convenient to single out the sub-classes $\mathcal{Q}^L_{LMG}$ (resp. $\mathcal{Q}^A_{LMG}$) of all $Q \in \mathcal{Q}_{LMG}$ under which $\log S$ is still a Lévy process (resp. is an additive process, that is, has independent increments). The inclusions $\mathcal{Q}^L_{LMG} \subset \mathcal{Q}^A_{LMG} \subset \mathcal{Q}_{LMG}$ are obvious, but, since S has no fixed times of discontinuity under P, hence under any $Q \in \mathcal{Q}_{LMG}$ as well, we even have $\mathcal{Q}^A_{LMG} \subset \mathcal{Q}_{MG}$ by the following result of independent interest:

Proposition 2. *Let X be an additive process with $X_0 = 0$ and without fixed times of discontinuities. If e^X is a local martingale, it is necessarily a martingale.*

Proof. Since e^X is a semimartingale, X is also a semimartingale whose characteristics are denoted as (B, C, ν). Since X is additive without fixed times of discontinuities, these characteristics are non random and continuous in time, and X_t is infinitely divisible with characteristic triple (B_t, C_t, F_t), where $F_t(dx) = \nu((0, t], dx)$. Since $E(e^{X_t}) \leq 1$, standard results on infinitely divisible variables yield

$$E(e^{X_t}) = e^{\psi_t}, \tag{19}$$

where

$$\psi_t = B_t + \frac{1}{2} C_t + \int_{\mathbb{R}} \left(e^x - 1 - x\, 1_{\{|x| \leq 1\}}\right) F_t(dx).$$

In particular the last two terms above are finite and non-decreasing in t, hence $t \mapsto \psi_t$ is of finite variation. Finally, Itô's formula implies that $e^{X_t} - \int_0^t e^{X_s}\, d\psi_s$ is a local martingale, and since e^{X_t} is a local martingale, we have that $\int_0^t e^{X_s}\, d\psi_s$ must also be a local martingale. This makes it a continuous local martingale with paths of finite variation, hence it must be constant. Therefore we must have $\int_0^t e^{X_s}\, d\psi_s = 0$ for all t, implying in turn $\psi_t = 0$, since it has continuous paths of finite variation. Therefore $E(e^{X_t}) = 1$ for all t using (19), and we deduce that e^X is a martingale. □

3.2. The range of European call option prices

We consider a European call with pay-off function g satisfying (1). By (4) we have $\Pi(g)^Q \leq S_0 = 1$ for all $Q \in \mathcal{Q}_{LMG}$, and $\Pi(g)^Q \geq g(S_0) = g(1)$ if further $Q \in \mathcal{Q}_{MG}$, and these are strict inequalities unless g is trivial, or affine on the range R. Note that no lower bound except $\Pi(g)^Q \geq 0$ is known when $Q \in \mathcal{Q}_{LMG} \backslash \mathcal{Q}_{MG}$). so we will focus on the set $\mathcal{Q}_{LMG}$ only. We have here

$$R = \begin{cases} \mathbb{R}_+ & \text{in cases A,B and in case C with } b < 0 < b' \\ \left[e^{-\zeta T}, \infty\right) & \text{in case C with } b > 0 \\ \left[0, e^{-\zeta T}\right] & \text{in case C with } b' < 0. \end{cases} \tag{20}$$

Actually, if g is affine on R we have $\Pi(g)^Q = g(1)$ for all $P \in \mathcal{Q}_{MG}$. If we exclude this case we have $g(1) < \Pi(g)^Q < 1$ and it seems natural to ask ourselves whether these bounds are sharp. Since $\mathcal{Q}_{MG}$ is a convex set of probability measures, the range $J = \{\Pi(g)^Q : Q \in \mathcal{Q}_{MG}\}$ is necessarily a subinterval of the open interval $(g(1), 1)$.

More specifically we fully characterize J below, and show in particular that it is the same as the range $J^L = \{\Pi(g)^Q : Q \in \mathcal{Q}^L_{MG}\}$: *the price ranges are the same for Lévy property preserving risk neutral measures as they are for arbitrary risk neutral measures.*

Theorem 3. *Assume that g is not affine on R.*

a) In cases A and C we have $J = J^L = (\Pi_{BS}(g; \sigma), 1)$.

b) In case B and with $\beta = \alpha + \alpha' b'/b$ and $\beta' = \alpha' + \alpha b/b'$ we have $J^L = D \subset J \subset \overline{D}$, where

$$D = \begin{cases} \big(\Pi(g; b, \beta), \Pi(g; b', \beta')\big), & \text{if } 0 < b < b' \\ \big(\Pi(g; b', \beta'), \Pi(g; b, \beta)\big), & \text{if } b < b' < 0 \\ \big(\Pi(g; b, \beta), 1\big), & \text{if } b < 0 < b',\ \alpha b + \alpha' b' < 0 \\ \big(\Pi(g; b', \beta'), 1\big), & \text{if } b < 0 < b',\ \alpha b + \alpha' b' \geq 0 \end{cases} \tag{21}$$

Although not obvious at first glance the interval for the first two cases in (21) is never empty. The proof of the theorem is given in Subsection 3.4.

This result, which is completely explicit but only about a very special class of models, should be compared to the results in [7] and [9]. In both these papers, but under very different circumstances (infinite activity jumps and no Brownian in the first paper, no jumps and a stochastic volatility in the second one), the typical range is the full $(g(1), 1)$.

3.3. **Completing the Model in Cases A and B**

Now we suppose that the call option with some non-trivial pay-off function g is traded, with market price Π_t at time t. In the cases A and B, for which we have only two "independent sources" of randomness (the Brownian motion and the Poisson process), one would expect the market model $(\Omega, (\mathcal{F}_t), (S, \Pi), P)$ to be complete, whereas it should not be true in case C because of a third independent source of randomness (the second Poisson process).

However, even in the cases A and B we have not been able to prove this intuitively obvious result in full generality. The problem arises in the specification of the risk neutral measure Q that the market implicitly uses for pricing the option: recall that $\Pi_t = \Pi(g)_t^Q$ is given by (10) for some (*a priori* unknown) $Q \in \mathcal{Q}_{LMG}$, and we only have a partial result, when this "pricing measure" belongs to the set $\mathcal{Q}_{LMG}^A$ where S is still an additive process under Q. The space of measures $\mathcal{Q}_{LMG}^A$ is introduced before Proposition 2, itself in Subsection 3.1

For dealing with case B we need an additional notation, for nonnegative integers n, m, and with x^+ and x^- denoting the positive and negative parts of a real x:

$$I_{n,m} = \big[(1 - b^-) e^{-\zeta T} (1 + b)^n (1 + b')^m, (1 + b'^+) e^{-\zeta T} (1 + b)^n (1 + b')^m\big]. \tag{22}$$

Theorem 4. *Assume that g is not trivial and that the pricing measure belongs to $\mathcal{Q}_{LMG}^A$, and consider the extended model $(\Omega, (\mathcal{F}_t), (S_t, \Pi_t), P)$*

a) In case A and in case B with $b < 0 < b'$, the extended model is complete.

b) In case B with $0 < b < b'$ or $b < b' < 0$, the extended model is complete if and only if we have

$$\begin{array}{l} \text{there are two sequences } n_k, m_k \text{ going to } \infty \text{ such that} \\ \text{for all } k, \text{ the function } g \text{ is not affine on the interval } I_{n_k, m_k}, \end{array} \tag{23}$$

In Case A and when the pricing measure belongs to $\mathcal{Q}_{LMG}^L$, this result is due to Mancini [17].

The condition (23) looks awkward, and perhaps slightly strange. It is of course satisfied if g is strictly convex, and also in case B when $b < 0 < b'$: indeed, we then have $I_{n,m} = \big[e^{-\zeta T} (1+b)^{n+1} (1+b')^m, e^{-\zeta T} (1+b)^n (1+b')^{m+1}\big]$ and $1 + b < 1 < 1 + b'$, and there exists $x > 0$ such that g is not affine on $[x - \varepsilon, x + \varepsilon]$ for any $\varepsilon > 0$ (otherwise g would be trivial). It is then easily seen that x belongs to the interior of infinitely many $I_{n,m}$, so (23) holds.

To get some insight on why (23) is necessary for the completeness of the extended model, assume the most standard form $g(x) = (x - K)^+$ for a call function. This g does not satisfy the condition, because it is affine on $[0, K]$ and on $[K, \infty)$, whereas the $I_{n,m}$'s are included into $(0, K)$ (resp. (K, ∞) when $b < b' < 0$ (resp. $0 < b < b'$), *for all* n, m large enough. Now if $0 < b < b'$ for example, when $K \leq e^{-\zeta T}$ we necessarily have $S_T \geq K$, so $\Pi_t = S_t - K$ and the model (S_t, Π_t) reduces to S_t itself. But, even when $K > e^{-\zeta T}$ the model

(S_t, Π_t) is not complete, because on the set $\{S_s \geq Ke^{\zeta(T-s)}\}$ which has a positive probability we have $S_T \geq K$ and thus $\Pi_t = S_t - K$ for all $t \in [s,T]$ on this set, implying that the models (S_t, Π_t) and S_t are identical on the time interval $[s,T]$, in restriction to the set $\{S_s \geq Ke^{\zeta(T-s)}\}$.

When the pricing measure Q implicitly used by the market is in $\mathcal{Q}_{MG} \backslash \mathcal{Q}^A_{LMG}$ but preserves the Markov property of S, then the result of Theorem 4 still holds under appropriate regularity assumptions on the transition semi-group of S under Q, and this can be extended even to some non-homogeneous Markov risk neutral measures. Without any assumption other than $Q \in \mathcal{Q}_{MG}$ we are unable to prove the result of Theorem 4, and it is not even clear to us that it should hold.

3.4. **The proofs**

1) Preliminaries. We start with some general considerations about the construction of risk neutral measures.

If $Q \in \mathcal{Q}_{LMG}$ the Radon-Nikodym derivative process $Z_t = \frac{dQ}{dP}_{|\mathcal{F}_t}$ has the form

$$\begin{array}{c} Z = \mathcal{E}(M), \qquad M_t = \int_0^t \big(\phi_s\, dY_s + (\overline{\phi}_s - 1)\, d\overline{N}_s + (\overline{\phi}'_s - 1)\, d\overline{N}'_s\big) \\ \text{where } \ \sigma^2 \phi_s + b\alpha(\overline{\phi}_s - 1) + b'\alpha'(\overline{\phi}'_s - 1) = 0, \ \text{ and } \overline{\phi}_s,\ \overline{\phi}'_s > 0, \end{array} \tag{24}$$

for some predictable processes $\phi, \overline{\phi}, \overline{\phi}'$. We denote by $\mathcal{H}_{LMG}$ and $\mathcal{H}_{MG}$ the sets of those triples $(\phi, \overline{\phi}, \overline{\phi}')$ associated with some Q in $\mathcal{Q}_{LMG}$ or $\mathcal{Q}_{MG}$. More correctly, $\mathcal{H}_{LMG}$ and $\mathcal{H}_{MG}$ are the sets of equivalence classes of such triples, where $\psi \equiv \psi'$ if $E\big(\int_0^T |\psi_s - \psi'_s|\, ds\big) = 0$, and there are one-to-one correspondences between $\mathcal{H}_{LMG}$ and $\mathcal{Q}_{LMG}$, and between $\mathcal{H}_{MG}$ and $\mathcal{Q}_{MG}$. Note that

$$\begin{array}{c} (\phi, \overline{\phi}, \overline{\phi}') \in \mathcal{H}_{LMG} \quad \Rightarrow \quad \int_0^T \phi_s^2\, ds + \int_0^T \overline{\phi}_s\, ds < \infty \quad \big(\text{hence } \int_0^T \overline{\phi}'_s\, ds < \infty\ \big) \\ \int_0^T \phi_s^2\, ds + \int_0^T \overline{\phi}_s\, ds \leq K \ \ \big(\text{hence } \ \int_0^T \overline{\phi}'_s\, ds \leq K\big) \quad \Rightarrow \quad (\phi, \overline{\phi}, \overline{\phi}') \in \mathcal{H}_{MG}. \end{array} \tag{25}$$

Under the measure $Q \in \mathcal{Q}_{LMG}$ associated with $(\phi, \overline{\phi}, \overline{\phi}') \in \mathcal{H}_{LMG}$, we have

$$\begin{array}{ll} Y_t^Q := Y_t - \sigma^2 \int_0^t \phi_s\, ds & \text{is } \sigma \text{ times a Brownian motion, when } \sigma > 0 \\ \overline{N}_t^Q := N_t - \alpha \int_0^t \overline{\phi}_s\, ds & \text{is a local martingale} \\ \overline{N}_t'^Q := N_t - \alpha' \int_0^t \overline{\phi}'_s\, ds & \text{is a local martingale,} \end{array} \tag{26}$$

and also $S = S^Q \overline{S}^Q \overline{S}'^Q$, where

$$S_t^Q = e^{Y_t^Q - \sigma^2 t/2}, \qquad \overline{S}_t^Q = e^{-\alpha b \int_0^t \overline{\phi}_s\, ds}\, (1+b)^{N_t} \qquad \overline{S}_t'^Q = e^{-\alpha' b' \int_0^t \overline{\phi}'_s\, ds}\, (1+b')^{N'_t}. \tag{27}$$

Then $S^Q, \overline{S}^Q, \overline{S}'^Q$ are positive Q-local martingales, mutually orthogonal.

Moreover, a measure Q is in $\mathcal{Q}^L_{LMG}$ if and only if the associated processes $\phi, \overline{\phi}, \overline{\phi}'$ can be chosen identically constant (c.f., e.g. [14], Corollary II.4.19), say $\gamma, \overline{\gamma}, \overline{\gamma}'$, with the constraint

$$\sigma^2 \gamma + b\alpha(\overline{\gamma} - 1) + b'\alpha'(\overline{\gamma}' - 1) = 0, \qquad \overline{\gamma}, \overline{\gamma}' > 0. \tag{28}$$

Finally, for any $y > 0$ we set $g_y(x) = \frac{1}{y}\, g(yx)$, which defines another call function g_y in the sense that it satisfies (1).

2) Proof of (b) of Theorem 3. *Step 1.* We are in case B. We begin with a proof for $J^L = D$.

Since $\sigma = 0$, by virtue of (28) and Proposition 2 a measure Q in $\mathcal{Q}_{LMG}$ belongs to $\mathcal{Q}_{MG}$ and is entirely specified by a single number $\overline{\gamma}$, so we write it as $Q^{\overline{\gamma}}$, and further $\overline{\gamma}$ ranges through the interval $\overline{I} = (u, u')$ where

$$\begin{array}{llll} 0 < b < b' \text{ or } b < b' < 0 & \Rightarrow & u = 0, & u' = 1 + \alpha' b'/\alpha b \\ b < 0 < b' & \Rightarrow & u = (1 + \alpha' b'/\alpha b)^+, & u' = \infty. \end{array}$$

Under $Q^{\overline{\gamma}}$ the two independent Poisson variables N_T and N'_T have the parameters $v(\overline{\gamma}) = \overline{\gamma}\alpha T$ and $v(\overline{\gamma})' = (\alpha' + \alpha b(1-\overline{\gamma})/b')T$, hence

$$f(\overline{\gamma}) := \Pi(g)^{Q^{\overline{\gamma}}} = \sum_{n,m\geq 0} e^{-v(\overline{\gamma})-v(\overline{\gamma})'}\,\frac{v(\overline{\gamma})^n}{n!}\,\frac{v(\overline{\gamma})'^m}{m!}\,g\big(e^{-\zeta T}\,(1+b)^n\,(1+b')^m\big). \tag{29}$$

The functions v and v' are affine with derivatives αT and $-\alpha T b/b'$, so f is C^∞ on (u,u'), and with $x_{n,m} = e^{\zeta T}\,(1+b)^n\,(1+b')^m$ we see after some calculations that its derivative is

$$\begin{array}{c} f'(\overline{\gamma}) = \frac{\alpha T}{b'} \sum\limits_{n,m\geq 0} e^{-v(\overline{\gamma})-v(\overline{\gamma})'}\,\frac{v(\overline{\gamma})^n}{n!}\,\frac{v(\overline{\gamma})'^m}{m!}\,\eta(n,m), \quad \text{where} \\ \eta(n,m) = (b-b')\,g(x_{n,m}) + b'\,g(x_{n,m}(1+b)) - b\,g(x_{n,m}(1+b')). \end{array}$$

The convexity of g implies $\eta_{n,m} \leq 0$ when $b>0$ and $\eta_{n,m} \geq 0$ when $b<0$ and, since g is not affine on R, these inequalities are strict for at least one value of (n,m). Thus f is strictly decreasing when $b>0$ and strictly increasing when $b<0$, implying $J^L = (a',a)$ when $b>0$ and $J^L = (a,a')$ when $b<0$, where a and a' are the limits of $f(\overline{\gamma})$ as $\overline{\gamma}$ decreases to u or increases to u'.

Suppose either $b>0$ or $b'<0$. When $\overline{\gamma}\downarrow u=0$ we have $v(\overline{\gamma})\to 0$ and $v(\overline{\gamma})'\to\beta' T$. So the variables N_T and N'_T under $Q^{\overline{\gamma}}$ converge in law to 0 and to a Poisson variable with parameter $\beta' T$, respectively, implying that S_T converges in law to the process of (15) at time T, with $(\alpha b,\alpha)$ replaced by $(\alpha b+\alpha' b',\beta')$, and the uniform integrability of S_T under $Q^{\overline{\gamma}}$ when $\overline{\gamma}$ varies is easy to check. We then readily deduce that $a = \Pi(g;b',\beta')$. Similarly, when $\overline{\gamma}\uparrow u' = 1+\alpha' b'/\alpha b$ we have $v(\overline{\gamma})\to\beta$ and $v(\overline{\gamma})'\to 0$, so the same argument implies $a' = \Pi(g;b,\beta)$. This proves $J^L = D$ if $b>0$ or if if $b'<0$.

Now, suppose $b<0<b'$. Again the same argument shows us that if $\alpha b+\alpha' b'<0$, by letting $\overline{\gamma}\downarrow u = 1+\alpha' b'/\alpha b$ we get $a = \Pi(g;b,\beta)$, whereas if $\alpha b+\alpha' b'\geq 0$ and by letting $\overline{\gamma}\downarrow u=0$ we get $a=\Pi(g;b',\beta')$.

When $b<0<b'$ it remains to find the limit of $f(\overline{\gamma})$ when $\overline{\gamma}\to u'=\infty$. We have $v(\overline{\gamma})/\overline{\gamma}\alpha = T$ and $v(\overline{\gamma})'/\overline{\gamma}\alpha \to -bT/b' > 0$, so $v(\overline{\gamma})$ and $v(\overline{\gamma})'$ go to ∞, implying the convergence $N_T/v(\overline{\gamma})\to 1$ and $N'_T/v(\overline{\gamma})'\to 1$ in $Q^{\overline{\gamma}}$-probability. In turn, this gives us

$$\frac{\log S_T}{T\overline{\gamma}\alpha} = -\frac{\alpha b+\alpha' b'}{\overline{\gamma}\alpha} + \frac{N_T\,\log(1+b)}{T\overline{\gamma}\alpha} + \frac{N'_T\,\log(1+b')}{T\overline{\gamma}\alpha} \xrightarrow{Q^{\overline{\gamma}}} \log(1+b) - \frac{b}{b'}\,\log(1+b').$$

The limit above is negative, so $S_T \xrightarrow{Q^{\overline{\gamma}}} 0$, hence Proposition 1 yields $f(\overline{\gamma})\to a' = 1$. This completes the proof of $J^L = D$ when $b<0<b'$.

Step 2: Since $J^L\subset J'$ it remains to prove that $J'\subset\overline{D}$. Any $Q\in\mathcal{Q}_{MG}$ is characterized by a predictable $\overline{I}$-valued process $\overline{\phi}$, since here $\sigma=0$ and $\overline{\phi}' = 1+\alpha b(1-\overline{\phi})/\alpha' b'$, and we write it as $Q^{\overline{\phi}}$.

This step is devoted to some preliminaries. We extend the definition (16) by setting $\Pi(g;b,\alpha;t) = E(g(S_t^{b,\alpha}))$ for any $t\in[0,T]$ so $\Pi(g;b,\alpha) = \Pi(g;b,\alpha,T)$. Recalling $g_y(x) = \frac{1}{y}\,g(yx)$, for any integer $k\geq 2$ we define the functions $g^{\alpha,j}$ for $j=0,\dots,k$ by downward induction, as follows: set $g^{\alpha,k}=g$ and, for $0\leq j<k$,

$$g^{\alpha,j}(x) = x\,\Pi(g_x^{j+1};b,\alpha;T/k) = E\big(g^{\alpha,j+1}(xS_{T/k})\big).$$

If $g^{\alpha,j+1}$ is a call pay-off function then $g^{\alpha,j}$ is the same: the convexity of $g^{\alpha,j}$ and $0\leq g^{\alpha,j}(x)\leq x$ are obvious, and $g^{\alpha,j}(x)/x\to 1$ as $x\to\infty$ follows from the dominated convergence theorem and $E(S_{T/k}^{b,\alpha}) = 1$. Then indeed all $g^{\alpha,j}$ are call pay-off functions.

Now, we prove that for any $j=0,\dots,k$ we have

$$\Pi(g;b,\alpha) = \Pi(g^{\alpha,j};b,\alpha;jT/k). \tag{30}$$

This holds by definition if $j = k$, and if it holds for $j+1$, the Lévy property of the process $\log S^{b,\alpha}$ in (15) implies

$$\begin{aligned}\Pi(g;b,\alpha) &= E\big(g^{\alpha,j+1}(S^{b,\alpha}_{(j+1)T/k})\big) = E\big(S^{b,\alpha}_{jT/k}\,E\big(g^{\alpha,j+1}_{S^{b,\alpha}_{jT/k}}(S^{b,\alpha}_{(\alpha,j+1)T/k}/S^{b,\alpha}_{jT/k})\mid \mathcal{F}_{jT/k}\big)\\ &= E\big(S_{jT/k}\,\Pi(g^{\alpha,j+1}_{S^{b,\alpha}_{jT/k}};b,\alpha;T/k)\big) = E\big(g^{\alpha,j}(S^{b,\alpha}_{jT/k})\big) = \Pi(g^{\alpha,j};b,\alpha;jT/k).\end{aligned}$$

Then a downward induction yields (30) for all j.

Step 3: In this step we suppose that $\overline{\phi}$ is a "simple process" of the form

$$\overline{\phi}_t = \sum_{i=1}^{k} \overline{\gamma}_i\, 1_{\{(i-1)T/k<t\leq iT/k\}}, \qquad \text{where } \overline{\gamma}_i \text{ is } \mathcal{F}_{(i-1T/k}\text{-measurable, bounded, } \overline{I}\text{-valued,} \tag{31}$$

for some $k \geq 2$, and we will show that the price $\Pi(g)^{Q^{\overline{\phi}}}$ belongs to L.

Fix j between 0 and $k-1$. Conditionally on $\mathcal{F}_{jT/k}$ and under the measure $Q^{\overline{\phi}}$, the variable $\overline{\gamma}_{j+1}$ is non random, and the law of $S_{(j+1)T/k}/S_{jT/k}$ is the same as the law of the law of S_T/k under $Q'^{\overline{\gamma}_{j+1}}$ (notation of Step 1, where again we consider $\overline{\gamma}_{j+1}$ as non random). Thus, for any call pay-off function h we have

$$E_{Q^{\overline{\phi}}}\big(h(S_{(j+1)T/k})\big) = E_{Q^{\overline{\phi}}}\big(S_{jT/k}\,E_{Q^{\overline{\gamma}_{j+1}}}\big(h_{S_{jT/k}}(S_{T/k})\big)\big).$$

Suppose that $b>0$. By Step 1 with options having maturity t instead of T, we have $E_{Q^x}(h(S_t)) > \Pi(h;b,\beta;t)$ for any $x \in \overline{I}$. Plugging this into the previous equality, we get

$$E_{Q^{\overline{\phi}}}\big(h(S_{(j+1)T/k})\big) > E_{Q^{\phi}}\big(S_{jT/k}\,\Pi(h_{S_{jT/k}};b,\beta;T/k)\big).$$

Using the functions $g^{\beta,j}$ (with β instead of α), the previous inequality with $h = g^{\beta,j+1}$ yields

$$E_{Q^{\overline{\phi}}}\big(g^{\beta,j+1}(S_{(j+1)T/k})\big) > E_{Q^{\overline{\phi}}}\big(g^{\beta,j}(S_{jT/k})\big).$$

In turn, applying this successively with $j = k-1, k-2, \dots, 0$ readily gives us

$$\Pi(g)^{Q^{\overline{\phi}}} = E^{Q^{\overline{\phi}}}(g^{\beta,k}(S_T)) > E_{Q^{\overline{\phi}}}\big(g^{\beta,0}(S_0)\big) = g^{\beta,0}(1) = \Pi(g;b,\beta), \tag{32}$$

where the last equality follows from (30) with $j=0$. The upper bound $\Pi(g)^{Q^{\overline{\phi}}} < \Pi(g;b',\beta')$ is obtained in exactly the same way, so indeed $\Pi(g)^{Q^{\overline{\phi}}} \in D$. The same argument shows that this result also hold when $b'<0$ and when $b<0<b'$.

Step 4: Finally, we consider an arbitrary $Q \in \mathcal{Q}_{MG}$, which by (24) is associated with a predictable process $\overline{\phi}$ taking its values in the interval $\overline{I}$. The sequence of stopping times $\theta_p = \inf\big(t : \int_0^t |\overline{\phi}_s|\,ds \geq p\big)$ increases to ∞ as p does. A classical density argument shows the existence of a sequence $\overline{\phi}^n$ of simple processes of the form (31) with convergence to $\overline{\phi}$ in the sense that $E\big(\int_0^{T\wedge\theta_p} |\overline{\phi}^n_s - \overline{\phi}_s|\,ds\big) \to 0$ for each p (the k_n's associated with $\overline{\phi}^n$ go to ∞ in general).

We let M^n and M be associated with $\overline{\phi}^n$ and $\overline{\phi}$ by (24) (recall that ϕ can be taken equal to 0 here, and $\overline{\phi}'$ and $\overline{\phi}'^n$ are fully determined by $\overline{\phi}$ and $\overline{\phi}^n$). The processes $Z^n = \mathcal{E}(M^n)$ and $Z = \mathcal{E}(M)$ are P-martingales, and we have $Q^{\overline{\phi}^n} = Z^n_T \cdot P$ and $Q^{\overline{\phi}} = Z_T \cdot P$. The processes M^n and M are of finite variation, and our hypothesis on the sequence $\overline{\phi}^n$ implies that the total variation of $M^n - M$ on the interval $[0, T\wedge\theta_p]$ goes to 0 in $L^1(P)$. In turn, by classical stability theorems for integro-differential equations, this implies that the stopped processes

$(Z^n)^{\theta_p}$ converge in probability, uniformly in time, to Z^{θ_p}. This being true for all p, we deduce $Z^n_T \xrightarrow{P} Z_T$, and thus for any $A > 0$ we have

$$\begin{array}{l} E\big(Z^n_T g(S_T)\, 1_{\{S_T \le A\}}\big) \to E\big(Z_T g(S_T)\, 1_{\{S_T \le A\}}\big) \\ E\big(Z^n_T S_T\, 1_{\{S_T \le A\}}\big) \to E\big(Z_T S_T\, 1_{\{S_T \le A\}}\big). \end{array} \tag{33}$$

Moreover, we have $E(Z^n_T S_T) = E_{Q^{\overline{\phi}_n}}(S_T) = 1$, hence

$$\big|\, E\big(Z^n_T g(S_T)\big) - E\big(Z^n_T g(S_T)\, 1_{\{S_T \le A\}}\big)\big| \le E\big(Z^n_T S_T\, 1_{\{S_T > A\}}\big) = 1 - E\big(Z^n_T S_T\, 1_{\{S_T \le A\}}\big),$$

and the same holds with Z instead of Z^n because $\phi \in \mathcal{H}'$ by hypothesis. Then (33) implies

$$\begin{array}{l} E\big(Z_T g(S_T)\big) - E\big(Z_T S_T\, 1_{\{S_T > A\}}\big) \le \liminf_n\, E\big(Z^n_T g(S_T)\big) \\ \limsup_n\, E\big(Z^n_T g(S_T)\big) \le E\big(Z_T g(S_T)\big) + E\big(Z_T S_T\, 1_{\{S_T > A\}}\big). \end{array}$$

Since $E\big(Z_T S_T\, 1_{\{S_T > A\}}\big) \to 0$ as $A \to \infty$, we deduce that

$$\Pi(g)^{Q^{\overline{\phi}^n}} = E_P(Z^n_T g(S_T)) \;\to\; E_P(Z_T g(S_T)) = \Pi(g)^Q.$$

Since $\Pi(g)^{Q^{\phi^n}} \in D$ by Step 3, we deduce $\Pi(g)^Q \in \overline{D}$ and the proof of (b) of Theorem 3 is complete.

3) Proof of (a) of Theorem 3. *Step 1.* In our first step we prove that $\Pi(g)^Q > \Pi_{BS} := \Pi_{BS}(g;\sigma)$ for any $Q \in \mathcal{Q}_{MG}$. We focus on case C, since case A is analogous (and slightly simpler).

Set $S' = S^Q$ and $S'' = \overline{S}^Q \overline{S}'^Q$, so $S = S'S''$ and S' is a Q-martingale and S'' a Q-local martingale and we have $E_Q(g(S'_T)) = \Pi_{BS}$. We consider a localizing sequence of stopping times (θ_n), such that each stopped process $S''^n = S''^{\theta_n}$ is a Q-martingale.

Let $Q' = Q'(\omega, d\omega')$ be a regular version of the Q-conditional probability knowing the σ-field $\mathcal{F}^{Y^Q}_T$. If $C \in \mathcal{F}^{Y^Q}_T$, the martingale representation for $(\mathcal{F}^{Y^Q}_t)$-martingales implies that the process $M^C_t = Q(C \mid \mathcal{F}^{Y^Q}_t)$ has the form $M^C_t = Q(C) + \int_0^t H_s\, dW^Q_s$ for an $(\mathcal{F}^{Y^Q}_t)$-predictable process H, hence M^C is also an $(\mathcal{F}_t)$-martingale under Q, and it is further orthogonal to S''. Therefore if $s \ge t \ge 0$ and $B \in \mathcal{F}_t$ we have

$$\begin{array}{rcl} E_Q\big(1_C\, E_{Q'}((S''^n_s - S''^n_t)1_B)\big) &=& E_Q\big(M^C_T\,(S''^n_s - S''^n_t)1_B\big) \\ &=& E_Q\big((M^C_s - M^C_t)(S''^n_s - S''^n_t)1_B\big) \;=\; 0, \end{array}$$

where the second equality comes from the fact that M^C and S''^n are martingales and $B \in \mathcal{F}_t$, and the third one from the fact that M^C and S''^{θ_n} are orthogonal. Since C is arbitrary in $\mathcal{F}^{Y^Q}_T$ we deduce that $E_{Q'(\omega,.)}((S''^{\theta_n}_s - S''^{\theta_n}_t)1_B) = 0$ for Q-almost all ω, and by a classical separability argument it follows that one can find a version of Q' such that $E_{Q'(\omega,.)}(S''^{\theta_n}_s - S''^{\theta_n}_t \mid \mathcal{F}_t) = 0$ for all ω and n, hence S''^{θ_n} is a $Q'(\omega,.)$-martingale for all ω and n, and S'' is a $Q'(\omega,.)$-local martingale.

Now, S is a Q-martingale, so $E_Q(S_T) = 1$. We have $V(\omega) := E_{Q'(\omega,.)}(S''_T) \le 1$ and $E_Q(S_T) = E_Q(S'_T\, V)$ and $E_Q(S'_T) = 1$. So we necessarily have $V = 1$ Q-a.s., hence $E_Q(S''_T) = 1$, implying that S'' is a $Q'(\omega,.)$-martingale. We can then use the first strict inequality in (4) with S'' and $Q'(\omega,.)$ and the function $g_{S'(\omega)_T}$, instead of S, Q and g, to get

$$\begin{array}{rcl} \Pi(g)^Q \;=\; E_Q(g(S_T)) &=& \int \Big(\int g(S'_T(\omega) S''_T(\omega'))\, Q'(\omega, d\omega')\Big)\, Q(d\omega) \\ &=& \int S'_T(\omega) \Big(\int g_{S'(\omega)_T}(S''_T(\omega'))\, Q'(\omega, d\omega')\Big)\, Q(d\omega) \\ &>& \int S'_T(\omega)\, g_{S'(\omega)_T}(1)\, Q(d\omega) \;=\; E_Q\big(g(S'_T)\big) \;=\; \Pi_{BS}. \end{array}$$

Step 2: Since $J^L \subset J \subset (\Pi_{BS}, 1)$ by (4) and Step 1, we are left needing to prove $J^L = (\Pi_{BS}, 1)$.

By (28) any measure in $\mathcal{Q}^L_{LMG}$ is entirely specified by two constants $\gamma, \overline{\gamma}$, so it is written as $Q^{\gamma,\overline{\gamma}}$, and the pair $(\gamma, \overline{\gamma})$ ranges through the set $\widehat{I} = \{(\gamma, \overline{\gamma}) : \gamma \in I, \overline{\gamma} \in \overline{I}_\gamma\}$, where (recall $\sigma, \alpha, \alpha' > 0$ here, and $b < b'$)

$$\begin{array}{lll} I = \big(-\infty, \frac{\alpha b+\alpha' b'}{\sigma^2}\big), & \overline{I}_\gamma = \big(0, 1 + \frac{\alpha' b' - \sigma^2\gamma}{\alpha b}\big) & \text{if } b > 0 \\ I = \mathbb{R}, & \overline{I}_\gamma = \big((1 + \frac{\alpha' b' - \sigma^2\gamma}{\alpha b})^+, \infty\big) & \text{if } b < 0 < b' \\ I = \big(\frac{\alpha b+\alpha' b'}{\sigma^2}, \infty\big), & \overline{I}_\gamma = \big(0, 1 + \frac{\alpha' b' - \sigma^2\gamma}{\alpha b}\big) & \text{if } b' < 0, \end{array} \tag{34}$$

and we also set $\overline{\gamma} = 1 - \big(\sigma^2\gamma + b\alpha(\gamma - 1)\big)/b'\alpha'$, so that the constraint (28) is satisfied.

Under the measure $Q^{\gamma,\overline{\gamma}}$, the variable S_T is still given by (17) (with $t = T$), with Y_T an $\mathcal{N}(0, \sigma^2 T)$-variable and N_T, N'_T are two independent Poisson variables with parameters $v(\gamma, \overline{\gamma})) = \alpha\overline{\gamma}T$ and $v(\gamma, \overline{\gamma})' = \alpha' T + (\alpha b - \alpha b\overline{\gamma} - \sigma^2\gamma)T/b'$, provided we replace ζ by $\zeta(\gamma) = \zeta - \sigma^2\gamma$. Then, as in (29) we have $\Pi(g)^{Q^{\gamma,\overline{\gamma},\overline{\gamma}'}} = f(\gamma, \overline{\gamma})$, where

$$f(\gamma, \overline{\gamma}) = \sum_{n,m\geq 0} e^{-v(\gamma,\overline{\gamma}) - v(\gamma,\overline{\gamma})'}\, \frac{v(\gamma, \overline{\gamma})^n}{n!}\, \frac{v(\gamma, \overline{\gamma})'^m}{m!}\, E\Big(g\big(e^{Y_T - \zeta(\gamma)T}\,(1 + b)^n\,(1 + b')^m\big)\Big).$$

Since v and v' are affine functions on $\widehat{I}$, whereas the expectations above are smaller than $K(1+b)^n\,(1+b')^m$, we easily see that f is C^∞ on $\widehat{I}$. Since the set $\widehat{I}$ is connected, it is thus enough to show the following two properties:

$$\inf_{(\gamma,\overline{\gamma})\in\widehat{I}} f(\gamma, \overline{\gamma}) = \Pi_{BS}, \qquad \sup_{(\gamma,\overline{\gamma})\in\widehat{I}} f(\gamma, \overline{\gamma}) = 1. \tag{35}$$

The first property above is simple to prove. Namely, in all three cases of (34), it is possible to find a sequence $(\gamma_n, \overline{\gamma}_n) \in \widehat{I}$ such that $\overline{\gamma}_n \to 0$ and $\gamma_n \to (\alpha b+\alpha' b)/\sigma$. Then $v(\gamma_n, \overline{\gamma}_n) \to 0$ and $v(\gamma_n, \overline{\gamma}_n)' \to 0$ and $\zeta(\gamma_n) \to c/2$, so by the dominated convergence theorem we get

$$f(\gamma_n, \overline{\gamma}_n) \;\to\; E\big(g\big(e^{Y_T - cT/2}\big)\big) \;=\; \Pi_{BS}(g).$$

For the second property (35), we argue as in Step 1 of the proof of Theorem 3-(b). We first consider the case $b > 0$ (resp. $b' < 0$): we can find a sequence $(\gamma_n, \overline{\gamma}_n) \in \widehat{I}$ such that $\gamma_n \to -\infty$ (resp. $\gamma_n \to \infty$) and $\overline{\gamma}_n \to 0$. Then $v(\gamma_n, \overline{\gamma}_n) \to 0$ and $v'_n := v(\gamma_n, \overline{\gamma}_n)' \to \infty$, so $N_T \to 0$ and $N'_T/v'_n \to 1$ in $Q^{\gamma_n,\overline{\gamma}_n}$-probability, whereas $\sigma\gamma_n/v'_n \to -b'$. Therefore we have the following convergence in $Q^{\gamma_n,\overline{\gamma}_n}$-probability

$$\frac{\log S_T}{v'_n} = \frac{Y_T - (\zeta - \sigma\gamma_n)T}{v'_n} + \frac{N_T\,\log(1+b)}{v'_n} + \frac{N'_T\,\log(1+b')}{v'_n} \;\to\; \log(1+b') - b' \;<\; 0.$$

Thus $S_T \to 0$ in $Q'^{\gamma_n,\overline{\gamma}_n}$-probability, hence $f(\gamma_n, \overline{\gamma}_n) \to 1$ by Proposition 1, and the second part of (35) holds.

Finally, suppose that $b < 0 < b'$. We take a sequence $(\gamma_n, \overline{\gamma}_n) \in \widehat{I}$ such that $\overline{\gamma}_n \to \infty$ and $\gamma_n \to -\infty$, with further $\sigma^2\gamma_n + \alpha b\overline{\gamma}_n \to \alpha b + \alpha' b'$. Then $v_n = v(\gamma, \overline{\gamma}_n) \to \infty$ and $v(\gamma, \overline{\gamma}_n)' \to 0$ and $\sigma\gamma_n/v_n \to -1$, and also $N'_T \to 0$ and $N_T/v_n \to 1$ in $Q^{\gamma_n,\overline{\gamma}_n}$-probability. Hence

$$\frac{\log S_T}{v_n} = \frac{Y_T - (\zeta - \sigma\gamma_n)T}{v_n} + \frac{N_T\,\log(1+b)}{v_n} + \frac{N'_T\,\log(1+b')}{v_n} \;\overset{Q^{\gamma_n,\overline{\gamma}_n}}{\longrightarrow}\; \log(1+b) - 1 \;<\; 0,$$

and we conclude the second part of (35) as above.

4) Proof of Theorem 4. *Step 1.* The (fixed) pricing measure $Q \in \mathcal{Q}^A_{LMG}$ is associated with a non random triple $(\phi, \overline{\phi}, \overline{\phi}') \in \mathcal{H}'$, via its density process Z in (24), and we use the processes $Y^Q, \overline{N}^Q, \overline{N}'^Q$ of (26). We treat the two cases A and B together, as far as we can, and recall that $N' = \overline{N}'^Q = 0$ in case A and $Y = Y^Q = 0$ in case B. We denote as $\overline{S}$ the two-dimensional process with components $S^1_t = S_t$ and $S^2_t = \Pi_t$.

Under P the three terms Y, N, N' are independent, so the martingale representation theorem tells us that any martingale M equals M_0 plus three stochastic integrals of predictable processes with respect to Y and to the compensated processes $\overline{N}$ and $\overline{N}'$. Since Q is equivalent to P, the same holds for Q-martingales: any Q-martingale M has the form

$$M = M_0 + \delta' \cdot Y^Q + b\delta \cdot \overline{N}^Q + b'\delta' \cdot \overline{N}'^Q \tag{36}$$

for suitable predictable processes δ, δ' (since either $Y^Q = 0$ or $\overline{N}^Q = 0$, taking the same integrand δ' for these two processes is not a restriction). In particular there are two predictable processes ψ, ψ' such that

$$\Pi_t = \Pi_0 + \psi' \cdot Y^Q_t + b\psi \cdot \overline{N}^Q_t + b'\psi' \cdot \overline{N}'^Q_t. \tag{37}$$

Since $S = \mathcal{E}(Y^Q + b\overline{N}^Q + b'\overline{N}'^Q)$, the property $M = M_0 + \overline{\beta} \cdot \overline{S}_t$ for a Q-martingale M, where here we use the “joint” integral of the two-dimensional process $\overline{\beta} = (\beta, \beta')$ with respect to $\overline{S}$, can be expressed as

$$M = M_0 + (\beta S_- + \beta'\psi') \cdot Y^Q_t + b(\beta S_- + \beta'\psi) \cdot \overline{N}^Q_t + b'(\beta S_- + \beta'\psi') \cdot \overline{N}^Q_t.$$

In (36) the two processes δ, δ' are uniquely determined by M, up to a ν-null set, where $\nu(d\omega, dt) = Q(d\omega) \otimes dt$, because $\overline{N}^Q$ is orthogonal to either Y^Q or $\overline{N}'^Q$. Thus the above is equivalent to having

$$\beta_t S_{t-} + \beta'_t\psi_t = \delta_t, \quad \beta'_t(\psi_t - \psi'_t) = \delta_t - \delta'_t \qquad \text{outside a } \nu\text{-null set.}$$

At this stage, we show that the model $\overline{S}$ is complete if and only if we have

$$\psi_t \neq \psi'_t \qquad \text{outside a } \nu\text{-null set.} \tag{38}$$

Indeed, if this holds, any Q-martingale M as in (36) takes the form $M = M_0 + \overline{\beta} \cdot \overline{S}_t$, where $\overline{\beta} = (\beta, \beta')$ with $\beta'_t = \frac{\delta_t - \delta'_t}{\psi_t - \psi'_t} 1_{\{\psi_t \neq \psi'_t\}}$ and $\beta_t = \frac{\delta_t - \beta'_t\psi_t}{S_{t-}}$, and $\overline{\beta}$ clearly is predictable and jointly integrable with respect to $\overline{S}$, so by Theorem 1 the model is complete. Conversely if (38) fails, the Q-martingale M with $\delta_t = 1_{\{\psi_t \neq \psi'_t\}}$ and $\delta'_t = 0$ is non trivial, and cannot be written as $M = M_0 + \overline{\beta} \cdot \overline{S}_t$, so the model is not complete.

We will also use the following fact: let μ_t be the law of the ratio S_T/S_t. Since $Q \in \mathcal{Q}^A_{LMG}$ this ratio is independent of $\mathcal{F}_t$ under Q, which implies

$$\Pi_t = E_Q\Big(g\big(S_t \frac{S_T}{S_t}\big)) \mid \mathcal{F}_t\Big) = k_t(S_t), \qquad k_t(x) = \int_0^\infty g(xy)\, \mu_t(dy). \tag{39}$$

Note that k_t is convex nondecreasing because g is such, but we have even more: if J_t denotes the support of the measure μ_t and L is any interval of $\mathbb{R}_+$, we have

$$k_t \text{ is affine on } J \;\Rightarrow\; g \text{ is affine on the interval } \{xy : x \in L\}, \text{ for each } y \in J_t. \tag{40}$$

Step 2: In this step we study case A. We set $u(t) = \alpha \int_t^T \overline{\phi}_s\, ds$ and $v(t) = bu(t) + \sigma^2 \frac{T-t}{2}$ and $w(t) = \sigma\sqrt{T-t}$. Recalling (27) with $b' = 0$, under Q the ratio S_T/S_t has the same law as $e^{w(t)U - v(t)}\,(1+b)^{\Phi_t}$, where U and Φ_t are independent, respectively $\mathcal{N}(0,1)$ and Poisson with parameter $u(t)$. Therefore, if f denotes the density of $\mathcal{N}(0,1)$, we see that μ_t admits the following density on $(0,\infty)$:

$$h_t(x) = e^{-u(t)} \sum_{n\geq 0} \frac{u(t)^n}{n!}\, \frac{1}{w(t)x}\, f\Big(\frac{v(t) + \log x - n\log(1+b)}{w(t)}\Big).$$

The functions u, v, w have (generalized) derivatives $u'(t) = -\alpha\overline{\phi}_t$ and $v'(t) = -\alpha b\overline{\phi}_t - \sigma^2/2$ and $w'(t) = \sigma^2/2w(t)$, whereas $f'(x) = -xf(x)$, so we have $h_t(x) = h_0(x) + \int_0^t \overline{h}_s(x)\,ds$ for $x > 0$, where

$$\begin{aligned}\overline{h}_t(x) = \tfrac{e^{-u(t)}}{2w(t)^5\,x} \sum_{n\geq 0} f\big(\tfrac{v(t)+\log x - n\log(1+b)}{w(t)}\big)\Big(2w(t)^4\overline{\phi}_t\big(\tfrac{u(t)^n}{n!} - \tfrac{u(t)^{n-1}}{(n-1)!}\,1_{\{n\geq 1\}}\big)\\ -\tfrac{u(t)^n}{n!}\,\sigma^2 w(t)^2 + \tfrac{u(t)^n}{n!}\,\big(w(t)^2(2\alpha b\overline{\phi}_t + \sigma^2)\big(v(t) + \log x - n\log(1+b)\big)\\ +\big(v(t) + \log x - n\log(1+b)\big)^2\Big).\end{aligned}$$

It is straightforward to deduce from the properties of the normal density f that h_0 and all $\overline{h}_t$ are C^∞ on $(0,\infty)$ and, for all integer $n \geq 0$ and $s \in [0, T-\varepsilon]$ the functions $|x^{n+1}h_0^{(n)}(x)|$ and $|x^{n+1}\overline{h}_s^{(n)}(x)|$ are smaller than some Lebesgue-integrable function (depending on n and ε). Therefore

$$\begin{aligned}k_t(x) = k_0(x) + \textstyle\int_0^t \overline{k}_s(x)\,ds, \qquad \text{with}\\ k_0(y) = \tfrac{1}{y}\textstyle\int_0^\infty g(x)h_0(x/y)\,dx, \quad \overline{k}_t(y) = \tfrac{1}{y}\int_0^\infty g(x)\overline{h}_s(x/y)\,dx,\end{aligned}$$

and since $g(x) \leq x$ the previous properties of $h_0, \overline{h}_t$ imply that all $\overline{k}_s$ are C^∞ with derivatives $\overline{k}_s^{(n)}$ bounded on $(0,\infty)$, uniformly in $s \in [0, T-\varepsilon]$.

At this stage, we deduce from (39) and Itô's formula and $\Delta S_s = bS_{s-}\Delta N_s$ that

$$\begin{aligned}\Pi_t &= \Pi_9 + \textstyle\int_0^t k_s'(S_{s-})\,dS_s + \int_0^t \overline{k}_s(S_{s-})\,ds + \frac{1}{2}\int_0^t k_s''(S_{s-})\,d\langle S^c, S^c\rangle_s\\ &\qquad + \textstyle\sum_{s\leq t}\big(k_s(S_{s-} + \Delta S_s) - k_s(S_{s-}) - k_s'(S_{s-})\Delta S_s\big)\\ &= \Pi_0 + \textstyle\int_0^t k_s'(S_{s-})S_{s-}\,dY_s^Q + \int_0^t \big(k_s((1+b)S_{s-}) - k_s(S_{s-})\big)\,d\overline{N}_s^Q\\ &\qquad + \textstyle\int_0^t \big(\overline{k}_s(S_{s-}) + \frac{\sigma^2}{2}\,k_s''(S_{s-})S_{s-}^2 + b\big(k_s((1+b)S_{s-}) - k_s(S_{s-})\big)\overline{\phi}_s\big)\,ds.\end{aligned}$$

Since Π_t is a Q-martingale the continuous finite-variation terms disappear. Hence (37) holds with $\psi_t' = k_t'(S_{t-})$ and $\psi_t = \frac{k_t((1+b)S_{t-}) - k_t(S_{t-})}{b}$. Now, in case A we have $J_t = \mathbb{R}_+$ for all $t \in [0,T)$, so (40) and the fact that g is not the identity imply that k_t is not affine on any interval of positive length, hence is strictly convex. Therefore $\psi_t < \psi_t'$ (resp. $\psi_t > \psi_t'$) identically when $b < 0$ (resp. $b' < 0$), implying (38), and the proof in case A is complete.

Step 3: We now turn to case B. Set $u(t) = \alpha\int_t^T \overline{\phi}_s\,ds$ again, and $u'(t) = \alpha'\int_t^T \overline{\phi}_s'\,ds = \alpha'\int_t^T \big(1 - \frac{\alpha b}{\alpha' b'}\,(\overline{\phi}_s - 1)\big)\,ds$. In view of (27), under Q the ratio S_T/S_t has the same law as $e^{-\zeta(T-t)}\,(1+b)^{\Phi_t}(1+b')^{\Phi_t'}$, where Φ_t and Φ_t' are independent Poisson variables with parameters $u(t)$ and $u'(t)$. Therefore,

$$k_t(x) = \sum_{n,m\geq 0} e^{-v(t)-v'(t)}\,\frac{v(t)^n}{n!}\,\frac{v'(t)^m}{m!}\,g\big(xe^{-\zeta(T-t)}\,(1+b)^n\,(1+b')^m\big).$$

The map $t \mapsto k_t(x)$ is absolutely continuous with (generalized) derivative called $\overline{k}_t(x)$, and since k_t is convex and non-decreasing it has right and left derivatives $k_t'^{+}$ and $k_t'^{-}$. Since S is of finite variation, with jumps occurring at times having an absolutely continuous law, we have $k_\tau'^{+}(S_{\tau-}) = k_\tau'^{-}(S_{\tau-})$ and $k_\tau'^{+}(S_\tau) = k_\tau'^{-}(S_\tau)$ a.s. for any jump time τ and we get by (39) and integration by parts:

$$\begin{aligned}\Pi_t &= \Pi_9 + \textstyle\int_0^t k_s'^{+}(S_{s-})\,dS_s + \int_0^t \overline{k}_s(S_{s-})\,ds\\ &\qquad + \textstyle\sum_{s\leq t}\big(k_s(S_{s-} + \Delta S_s) - k_s(S_{s-}) - k_s'^{+}(S_{s-})\Delta S_s\big)\\ &= \Pi_9 + \textstyle\int_0^t \big(k_s((1+b)S_{s-}) - k_s(S_{s-})\big)\,d\overline{N}_s^Q + \int_0^t \big(k_s((1+b')S_{s-}) - k_s(S_{s-})\big)\,d\overline{N}_s'^Q\\ &\quad + \textstyle\int_0^t \big(\overline{k}_s(S_{s-}) + \big(k_s((1+b)S_{s-}) - k_s(S_{s-})\big)\overline{\phi}_s + \big(k_s((1+b')S_{s-}) - k_s(S_{s-})\big)\overline{\phi}_s'\big)\,ds.\end{aligned}$$

As in Step 2, the last line above vanishes because Π_t is a Q-martingale, so (37) holds with $\psi_t = \frac{k_t((1+b)S_{t-})-k_t(S_{t-})}{b}$ and $\psi'_t = \frac{k_t((1+b')S_{t-})-k_t(S_{t-})}{b'}$, and thus $\psi'_t - \psi_t = \ell_t(S_{t-})/bb'$, where

$$\ell_t(x) = bk_t((1+b')x) - b'k_t((1+b)x) + (b'-b)k_t(x).$$

The variable S_{t-} is discrete, with support $R_t = \bigcup_{n,m\in\mathbb{N}} \{e^{-\zeta t}x_{n,m}\}$, where $x_{n,m} = (1+b)^n(1+b')^m$. Therefore, in view of Step 1, it remains to prove the equivalence of (23) with the following condition:

$$\ell_t(e^{-\zeta t}x_{n,m}) \neq 0 \quad \text{for all } n,m \in \mathbb{N} \text{ and Lebesgue-almost all } t \in [0,T). \tag{41}$$

Since k_t is convex, $\ell_t(e^{-\zeta t}x_{n,m}) = 0$ if and only if the function k_t is affine on the interval $L_{n,m}(t) = [e^{-\zeta t}x_{n,m}(1-b^-), e^{-\zeta t}x_{n,m}(1+b'^+)]$. By (40), and since the support of μ_t is R_{T-t}, this is equivalent to having g affine on the intervals $I_{n',m'}$ for all $n' \geq n$ and $m' \geq m'$, a property which does not depend on t: at this stage, the result follows easily.

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