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REACHING COURNOT-WALRAS EQUILIBRIUM

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Abstract. Considered here is repeated interaction among economic agents. These must share privately held user rights to diverse production factors. The disparate features of the resulting economy motivate a solution concept which blends Cournot/Nash equilibrium with that of Walras. A novelty comes by showing that integrated equilibrium may emerge via adaptive behavior and repeated play.

1. Introduction

Large parts of economic theory revolve around various concepts of equilibrium. *Three* features have attracted much attention. *First*, absent good governance, many equilibria are utterly inefficient; they waste valuable resources.[1] *Second*, there are pressing issues about equitable sharing of benefits and burdens. *Third*, on a more technical note, it's often unclear how and which equilibrium will emerge.

Motivated by such concerns, this paper considers objects that reside at the intersection of game and market theory. Serving since long there are the models of Cournot and Walras. In the motivating instance, these models are linked as follows. Below a Cournot oligopoly for product *outputs* there is a Walrasian exchange economy for *user rights* to factor *inputs*. Accordingly, equilibrium acquires two disparate features. One is that producers, while conditioned by their acquired or rented user rights, choose a Cournot-Nash strategy profile. The other is that, having chosen such a profile, all agents should value marginal user rights in the same manner. That is, they should see equal factor prices.

Equilibrium of this sort can hardly be brought down by a single shot.[2] Agents need time; they must learn to play - and to do so without assistance of any central director. Moreover, as long as factor holdings (or user rights) change hands, so does the stage game as well.

It's presumed, here below, that producers be so competent and experienced that they quickly find a Cournot equilibrium. Also by presumption, each player values diverse production factors on line, always using personal bid-ask prices for their usage. Repeatedly, two random players are matched to contemplate exchange of production factors. If their valuations differ, they make a bilateral direct deal, facilitated maybe by side payments. Eventually, when valuations coincide across players, the factor market settles in a state of balance.

Will overall equilibrium finally obtain? That question is explored below. In doing so, this paper differs from many others studies on learning of strategic behavior (Hart and Mas-Colell 2013), (Peyton Young 2004). Notably, it goes beyond finite-strategy instances, and it allows coupling constraints. Neither fictitious play nor

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[1]Commercial fisheries are cases in point.

[2]Game theorists have long wrestled with equilibrium attainment; see Fudenberg (1998), Hart & Mas-Collel (2013), PeytonYoung (1998, 2004). Most studies deal with finite-strategy noncooperative games. It seems fair to say that there are no unifying results on learning to play Nah equilibrium.

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regret matching is an issue. No player needs know his rivals or their actions - present or past. And nobody must form forecasts or statistics. Instead, it suffices that everyone be driven by own margins. Broadly, an oligopolist should adapt until he sees zero margins for products and common margins for factor inputs[3].

Rosen's (1965) study is closest to ours in that players are guided by first-order differential data. Classical differentiability is, however, not needed. And most important: exchange makes coordination superfluous.

The paper is planned as follows. Setting the frames, *Section 2* recalls games with coupling constraints, characterizes what is called *normalized equilibrium*, and inquires about its stability. *Section 3* specializes the said constraints to be purely additive. Motivated by environmental games, many of which feature exchange and externalities, that section defines a solution concept which integrates Cournot-Nash equilibrium with that of Walras. To prepare for repeated play, *Section 4* deals with just *one* bilateral exchange. Section 5 models play as a discrete-time process driven by differences in agents valuations. *Section 6* provides numerical illustration by considering workhorse oligopolies. *Section 7* relates to the literature.

2. Games with Coupling Constraints

This section fixes the frames and the notation. It also recalls some material concerned with description and stability of solutions.

Henceforth consider a finite ensemble I of noncooperative economic agents. Member $i \in I$ makes a choice x_i, codified as a vector in some Euclidean space $\mathbb{X}_i$. When facing the profile $x_{-i} := (x_j)_{j \neq i}$ of his rivals, he gets pecuniary payoff $\pi_i(x_i, x_{-i})$.

Choice is subject to coupling constraints in that each profile $x = (x_i)$ must belong to a *non-rectangular* subset X of the product space $\mathbb{X} := \Pi_{i\in I}\mathbb{X}_i$.[4] Since payoffs are transferable, consider the bivariate function

$$\hat{x}, x \in \mathbb{X} \mapsto \pi(\hat{x}, x) := \sum_{i\in I} \pi_i(\hat{x}_i, x_{-i}).$$

Definition. *Declare $x = (x_i) \in X$ a Nash* ***normalized equilibrium*** *iff*

$$\pi(x, x) = \max \left\{ \sum_{i\in I} \pi_i(\hat{x}_i, x_{-i}) : \hat{x} \in X \right\}.$$

Proposition 2.1 (Existence of normalized equilibrium [10], [32]). *Suppose the constraint set X is non-empty compact convex. Also suppose the overall payoff $\pi(\hat{x}, x)$ is concave in $\hat{x}$, jointly upper semicontinuous in $(\hat{x}, x)$, and continuous in x. Then there exists a normalized equilibrium.* □

The hypotheses of Proposition 2.1 are henceforth in vigor. We shall need a differential characterization of normalized equilibrium. Recall that $\mathbb{X}$ is a Euclidean space, with some inner product $\langle \cdot, \cdot \rangle$. A vector $x^* \in \mathbb{X}$ is called a *supergradient* of a function $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ at x, if $f(x)$ is finite and

$$f(\hat{x}) \leq f(x) + \langle x^*, \hat{x} - x \rangle \quad \text{for all} \quad \hat{x} \in \mathbb{X}.$$

Then, write $x^* \in \partial f(x)$ or $x^* \in \frac{\partial}{\partial x} f(x)$. At any feasible point x, the subset $X \subseteq \mathbb{X}$ has a *normal cone*

$$N(x, X) := \{x^* \in \mathbb{X} : \langle x^*, X - x \rangle \leq 0\},$$

[3] Say, of permits to catch fish, tap water, or pollute commons.

[4] Most non-cooperative games are modelled without over-arching restrictions, interactions being felt only via individual payoffs. For an early and important exception., see Debreu (1952). In geometric terms, the strategy profiles belong to a "rectangular box." This optic does not fit environmental games in which players must share aggregate endowments.

of vectors which point right out of X. It's convenient to refer to members of the partial superdifferentials

$$M_i(x) := \frac{\partial}{\partial \hat{x}_i}\pi_i(\hat{x}_i, x_{-i})\,|_{\hat{x}_i = x_i} \quad \text{and} \quad M(x) := \frac{\partial}{\partial \hat{x}}\pi(\hat{x}, x)\,|_{\hat{x}=x} = \Pi_{i\in I} M_i(x)$$

as *margins.* Members of $M(x) - N(x, X)$ are called *essential margins.* In the spirit of neoclassical economics: *all essential margins should be nil in normalized equilibrium.* For succinct statement of this, let $P_C[\cdot]$ denote the orthogonal projection onto a non-empty closed convex subset C of a Euclidean space.

Proposition 2.2 (Characterizing normalized equilibrium [15]). *x is a normalized equilibrium iff the following three equivalent statements hold for one and the same margin $m \in M(x)$:*
- *The margin is nil or normal, meaning $m \in N(x, X)$.*
- *For each deviation, individual or joint, along any d in the cone of feasible directions*

$$D(x, X) := \mathbb{R}_+(X - x) \subseteq cl\mathbb{R}_+(X - x) =: T(x, X), \tag{1}$$

the margin indicates an overall loss. That is, $\langle m, d\rangle \leq 0$.
- *The orthogonal projection $P_{T(x,X)}[\cdot]$ onto the tangent cone $T(x, X)$* (1) *yields*

$$0 = P_{T(x,X)}[m]\,. \quad \square \tag{2}$$

Equilibrium condition (2) invites use of the differential system

$$\frac{dx(t)}{dt} := \dot{x} \in P_{T(x,X)}[M(x)]\,, \quad x(0) \in X, \tag{3}$$

as an idealized model of continuous-time, out-of-equilibrium play. A normalized equilibrium $\bar{x}$ is declared *asymptotically stable* if

$$x \in X \diagdown \bar{x} \Longrightarrow \langle M(x), x - \bar{x}\rangle < 0. \tag{4}$$

Proposition 2.3 (Uniqueness and asymptotic stability of normalized equilibrium [11], [15]). *Suppose $\bar{x}$ is asymptotically stable* (4)*. Then normalized equilibrium is unique and globally attractive for system* (3).[5] $\square$

Its stability not withstanding, system (3) can hardly depict practical play. Indeed, real agents make discrete steps in discrete time (as brought out in Section 5). No less important: how can players dispense with the joint projection in (3)? The next section addresses instances which render that operation superfluous, thereby restoring the fully decentralized nature of agents' behavior.

3. Games with Additive Coupling

This section specializes the joint constraints to have additive form, and it defines a solution concept which blends Cournot-Nash equilibrium with that of Walras.

Henceforth the strategy $x_i = (y_i, z_i)$ of each player $i \in I$ has two quite different components. The first, y_i, is seen as a production plan, chosen without any concern for overall efficiency or cooperation. In contrast, the second, z_i, is construed as a *perfectly transferable* commodity bundle, contingent claim, or natural resource vector.[6]

[5] Proposition 2.3 adds to the pioneering paper of Rosen's (1965). Related studies, all using various forms of monotonicity, include [10], [11] and [13].

[6] Important instances comprise transfers of fish quotas, production allowances, pollution permits, or rights to water usage. In either case, reallocation of initial endowments may improve payoffs and entail partial efficiency.

While purely additive, the coupling constraint affects only the transferable items. Specifically, players act in the domain

$$X := \left\{ x = (x_i) : x_i = (y_i, z_i) \in Y_i \times Z_i \ \& \ \sum_{i \in I} z_i \leq \sum_{i \in I} e_i \right\}.$$

Y_i, Z_i are non-empty compact convex subsets of Euclidean spaces $\mathbb{Y}_i$, $\mathbb{Z}$ respectively, and $e_i \in Z_i$ is the *endowment* of agent i. Most important, the endowment space $\mathbb{Z}$ is common and ordered in customary componentwise manner. By assumption, *transferable items create no externalities* in that

$$\pi_i(x) = \pi_i(y, z) = \pi_i(y, z_i) = \pi_i(y_i, y_{-i}, z_i),$$

and *at least one agent i has $\pi_i(y_i, y_{-i}, z_i)$ increasing in z_i*.[7]

As announced, the solution concept is a mixed one, embodying features found in the equilibria of Cournot, Nash and Walras:

Definition (Cournot-Nash-Walras equilibrium). *A profile $x = (x_i) \in X$, with $x_i = (y_i, z_i)$ alongside a price vector $p \in \mathbb{Z}$, constitutes an* ***equilibrium*** *if nobody regrets his choice. That is,*

$$\pi_i(y_i, y_{-i}, z_i) - \langle p, z_i \rangle = \max \left\{ \pi_i(\hat{y}_i, y_{-i}, \hat{z}_i) - \langle p, \hat{z}_i \rangle : \hat{y}_i \in Y_i, \ \hat{z}_i \in Z_i \right\} \quad \forall i. \tag{5}$$

Further, all factor markets clear in so far as

$$p \geq 0, \qquad \sum_{i \in I} z_i \leq \sum_{i \in I} e_i, \qquad \text{and} \quad \left\langle p, \sum_{i \in I} e_i - \sum_{i \in I} z_i \right\rangle = 0.$$

This definition would comply with the above on normalized equilibrium if some extra (fictitious) player chooses $p \geq 0$ so as to minimize the value of excess supply. In that optic, under reasonable assumptions, existence of a solution is guaranteed; see [15]. We shall side-step the attending mathematical issues and rather ask: *could the agents themselves - by way or repeated exchanges - make clearing prices emerge endogenously*? The subsequent two sections provide constructive and positive answers.

4. Bilateral Exchange

For preparation of the main model, this section isolates a recurrent episode in which two agents - by themselves and between themselves - exchange production factors, say, of fish quotas, rights to water usage, or pollution permits. Their direct deal is driven only by differences in their valuations of marginal transfers. For that reason, exchange is moderate and somewhat myopic. Recall that each factor bundle belongs to a common Euclidean space $\mathbb{Z}$, endowed with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$.

Fix two agents i, j who hold respective factor bundles $z_i \in Z_i$ and $z_j \in Z_j$. Suppose a vector $\Delta \in \mathbb{Z}$ be transferred to i from j. In case Δ is non-zero, define a unit direction $d := \Delta / \|\Delta\|$, and let $s := \|\Delta\|$ be the corresponding *step-size*. Consequently, with no loss of generality, transferring Δ yields updated holdings

$$z_i^{+1} := z_i + sd \in Z_i \quad \text{and} \quad z_j^{+1} := z_j - sd \in Z_j \tag{6}$$

where $s \|d\| > 0 \Longrightarrow \|d\| = 1$. The first inclusion in (6) implies that d belongs to the cone $D_i(z_i) := \mathbb{R}_+(Z_i - z_i)$ of feasible directions at z_i. Quite similarly, the second inclusion tells that $-d \in D_j(z_j)$. Hence, for the feasibility of exchange, it's necessary that

$$d \in D_{ij}(z_i, z_j) := D_i(z_i) \cap -D_j(z_j).$$

[7]Alternatively, absent such monotonicity, one may require that $\sum_{i \in I} z_i = \sum_{i \in I} e_i$ in the definition of X. In either case, π_i might depend on the aggregate holding $\sum_{i \in I} z_i$. Since the latter sum is constant, it has no effect on behavior or welfare.

Which directions in $D_{ij}(z_i, z_j)$ are desirable? To address that question, call the vector

$$p_i = g_i - n_i \text{ with } g_i \in \frac{\partial}{\partial z_i}\pi_i(y, z_i) \text{ and } n_i \in N(z_i, Z_i) \tag{7}$$

a *personal factor price* as it applies for agent i at (y, z_i). Since y is fixed here, temporarily let $\Pi_i(z_i) := \pi_i(y, z_i)$ and $\Pi_j(z_j) := \pi_j(y, z_j)$.

Proposition 4.1 (On bilateral exchange). *When agents i, j own respectively $z_i \in Z_i$ and $z_j \in Z_j$, they **cannot** make a proper trade in case the cone of feasible directions is degenerate, meaning $D_{ij}(z_i, z_j) = \{0\}$. Moreover, they **ought not** make any trade if some personal price vectors p_i and p_j are equal. The reason is that* (6) *and* (7) *then yield $\Pi_i(z_i^{+1}) + \Pi_j(z_j^{+1}) \le \Pi_i(z_i) + \Pi_j(z_j)$.*

Proof. For the second statement, suppose $g_i \in \partial\Pi_i(z_i)$, $g_j \in \partial\Pi_j(z_j)$ and $n_i \in N(z_i, Z_i)$, $n_j \in N(z_j, Z_j)$ satisfy

$$p_i = g_i - n_i = g_j - n_j = p_j.$$

Taken together, $z_i^{+1} = z_i + \sigma d$, $d \in D_i(z_i)$, and the concavity of $\Pi_i(\cdot)$ imply

$$\Pi_i(z_i^{+1}) \le \Pi_i(z_i) + \sigma \langle g_i, d\rangle \le \Pi_i(z_i) + \sigma \langle g_i, d\rangle - \sigma \langle n_i, d\rangle .$$

Quite likewise, $z_j^{+1} = z_j + \sigma(-d)$ and $-d \in D_j(z_j)$ imply

$$\Pi_j(z_j^{+1}) \le \Pi_j(z_j) - \sigma \langle g_j, d\rangle \le \Pi_j(z_j) - \sigma \langle g_j, d\rangle + \sigma \langle n_j, d\rangle .$$

Adding theses inequalities yields

$$\Pi_i(z_i^{+1}) + \Pi_j(z_j^{+1}) \le \Pi_i(z_i) + \Pi_j(z_j) + \sigma \langle g_i - n_i, d\rangle - \sigma \langle g_j - n_j, d\rangle = \Pi_i(z_i) + \Pi_j(z_j).$$

This completes the proof. □

The quite reasonable upshot is that i and j ought trade only when their factor valuations differ - that is, when

$$\left[\frac{\partial}{\partial z_i}\pi_i(y, z_i) - N(z_i, Z_i)\right] \cap \left[\frac{\partial}{\partial z_j}\pi_j(y, z_j) - N(z_j, Z_j)\right] = \emptyset.$$

On such an occasion, it appears reasonable that a transfer, to i from j, be aligned with a direction

$$d = p_i - p_j, \tag{8}$$

featuring personal prices p_i and p_j (7). To argue differently for format (8), note that the joint payoff $\Pi_i + \Pi_j$ has *steepest slope*

$$\mathfrak{S}_{ij}(z_i, z_j) := \sup\left\{\Pi_i'(z_i; d) + \Pi_j'(z_j; -d) : d \in D_{ij}(z_i, z_j) \ \& \ \|d\| \le 1\right\}, \tag{9}$$

$\Pi_i'(z_i; d)$ denoting the standard directional derivative. Let P_{ij} be the orthogonal projection (in the factor endowment space $\mathbb{Z}$) onto the closed convex tangent cone $T_{ij}(z_i, z_j) := clD_{ij}(z_i, z_j)$.

Proposition 4.2 (On the steepest payoff slope, best direction, and minimal deviation of margins [16]). *The steepest slope* (9) *equals*

$$\begin{aligned}\mathfrak{S}_{ij}(z_i, z_j) &= \min\{\|P_{ij}[g_i - g_j]\| : g_i \in \partial\Pi_i(z_i),\ g_j \in \partial\Pi_j(z_j)\} \\ &= \min\{\|p_i - p_j\| : p_i, p_j \text{ being personal prices (7)}\}.\ \square\end{aligned}$$

It's neither fully convincing nor quite practical that agents i, j - maybe of moderate competence - always secure the steepest payoff slope. More realistically, they could contend with achieving a fraction φ_{ij} of the said slope. Accordingly, let us say that they make a *real transfer* if (6) holds with

$$\Pi_i(z_i^{+1}) + \Pi_j(z_j^{+1}) - \Pi_i(z_i) - \Pi_j(z_j) \geq \varphi_{ij} s \mathfrak{S}_{ij}(z_i, z_j) > 0.$$

Proposition 4.3 (On real transfers [16]). *Whenever* $(z_i, z_j) \in Z_i \times Z_j$ *and* $\mathfrak{S}_{ij}(z_i, z_j) > 0$, *agents* i, j *may indeed make a real transfer. Then, the two inequalities*

$$\Pi_i(z_i^{+1}) + \mu_i > \Pi_i(z_i) \quad \text{and} \quad \Pi_j(z_j^{+1}) + \mu_j > \Pi_j(z_j)$$

are solvable with monetary side-payments μ_i, μ_j *that sum to zero.* □

Together, Propositions 4.2&3 open up for *many* exchanges at any stage - maybe in parallel or asynchronous, maybe governed by alternative protocols on who meets whom. In extremis, before reconsidering y, all steepest slopes $\mathfrak{S}_{ij}(z_i, z_j)$ could be nil. For more on this, and to relax on step-size regime (12), see [16].

5. Repeated Play

Using bilateral exchange as workhorse, this section casts adaptive play as a discrete-time process. Time flows on two different clocks. One ticks after attainment of Cournot type equilibrium, the other after completion of a bilateral exchange. The two clocks are both put in motion because condition (5), as it applies to agent i, involves *two* choices: y_i and z_i.

For motivation, suppose (the oligopolistic variable) y_i already be chosen, and that z_i be considered thereafter. More precisely, prior to any (re)consideration of $z = (z_i)$, let

$$\pi_i(y_i, y_{-i}, z_i) = \max\{\pi_i(\hat{y}_i, y_{-i}, z_i) : \hat{y}_i \in Y_i\} \quad \text{for all} \quad i. \tag{10}$$

(10) fixes a crucial feature of repeated play as modelled here. Specifically, by hypothesis, given any endowment profile (z_i), the agents need negligible time to find a Cournot-Nash equilibrium (10). The latter is contingent and partial, however, being restricted to (y_i).

In contrast, while still out of full equilibrium, players adapt their factor holdings stepwise - with notable caution and moderation. To capture this behavioral feature, suppose agents i, j, meet at some stage, then holding factor endowments z_i, z_j and using personal prices (7):

$$p_i \in \frac{\partial}{\partial z_i}\pi_i(y, z_i) - N(z_i, Z_i) \quad \text{and} \quad p_j \in \frac{\partial}{\partial z_j}\pi_j(y, z_j) - N(z_j, Z_j). \tag{11}$$

If $p_i = p_j$, inclusions (11) and Proposition 4.1 rationalize their respective holdings. Indeed, no direct exchange is worth their while. Otherwise, if $p_i \neq p_j$, a transfer to i from j appears justifiable - and that it be aligned with the price difference $d = p_i - p_j$ (8). To appreciate this last suggestion, consider an instance where $z_i \in intZ_i$ and $z_j \in intZ_i$. Then, both normal cones in (11) reduce to $\{0\}$. So, in case differentiability also be granted, the transfer direction

$$d = \frac{\partial}{\partial z_i}\pi_i(y, z_i) - \frac{\partial}{\partial z_j}\pi_j(y, z_j)$$

equals the difference in payoff margins (partial gradients). When z_i resides at the boundary of Z_i, agent i wants to suppress normal components of $\frac{\partial}{\partial z_i}\pi_i(y, z_i)$, if any, these pointing right out of Z_i. In short, d is aligned with the difference in (partial but) *essential margins* Proposition 4.2 already provided more detail about the direction d of transfer. And Proposition 4.3 clarified that side payments may lubricate the exchange mechanism. The upshot is that, for any two interlocutors, one receives/delivers positive amounts of those production factors he prices higher/lower.

Synthesizing these aspects, we cast **repeated play as a discrete-time process**:

• *Start* at any allocation $i \mapsto z_i \in Z_i$ of the transferable goods such that $\sum_{i\in I} z_i = \sum_{i\in I} e_i$ For example, let $z_i = e_i$.
• *Find a contingent equilibrium in* y (10), depending on the actual allocation $z = (z_i)$.
• *Randomly select two agents* i, j with uniform probabilities. Actually, these agents hold transferable endowments $z_i \in Z_i$ and $z_j \in Z_j$ respectively.
• *Given* y, *update their holdings* by (6), *using a step-size* $s \geq 0$ *and a direction* $d = p_i - p_j$ *with* p_i, p_j satisfying (11).
• *Return* to find a contingent equilibrium (10).
• *Continue* until convergence. $\Diamond$

At each stage, since y is a contingent equilibrium (10), each agent i sees a partial margin $\frac{\partial}{\partial y_i}\pi_i$ which is normal to Y_i at y_i - or it's nil. More precisely, every agent i invariably considers some margin

$$m_{iy} \in N(y_i, Y_i) \cap \frac{\partial}{\partial y_i}\pi_i(y, z_i).$$

in his variable y_i. It is tacitly understood that his other margin

$$m_{iz} \in \frac{\partial}{\partial z_i}\pi_i(y, z_i)$$

satisfies the compatibility condition

$$(m_{iy}, m_{iz}) \in \frac{\partial}{\partial(y_i, z_i)}\pi_i(y_i, y_{-i}, z_i).$$

This requirement is, of course, superfluous whenever the last superdifferential reduces to a singleton; that is, when agent i's payoff is differentiable in own variables.

Theorem 5.1 (Convergence of non-coordinated play [15]). *Suppose repeated play, as modelled above, proceeds at discrete stages* $k = 0, 1, ...$ *with step-sizes* $s_k \geq 0$, *chosen by the agents themselves, which satisfy*

$$\sum_{k=0}^{\infty} s_k = +\infty \quad \text{and} \quad \sum_{k=0}^{\infty} s_k^2 < +\infty. \tag{12}$$

Then, under asymptotic stability (4), *the generated sequence* (x^k) *converges to the unique normalized equilibrium.* $\Box$

Corollary 5.2 (On common factor pricing). *Besides the hypotheses in Theorem 5.1 suppose equilibrium* $x = (y, z)$ *is such that some agent* i *has* $z_i \in intZ_i$ *and* $\pi_i(y, \cdot)$ *Gâteaux differentiable at* z_i. *Then there is a unique equilibrium price in the factor market.*

Proof. Since $N(z_i, Z_i) = \{0\}$, posit

$$p := \frac{\partial}{\partial z_i}\pi_i(y, z_i) - N(z_i, Z_i).$$

From Proposition 4.2 follows that $\mathfrak{S}_{ij}(z_i, z_j) = 0$ iff p equals some personal price p_j of agent j. Proposition 4.3 implies that all $\mathfrak{S}_{ij}(z_i, z_j) = 0$. Consequently, the intersection $\cap_j \left[\frac{\partial}{\partial z_j}\pi_j(y, z_j) - N(z_j, Z_j)\right]$ reduces to a singleton, namely p. □

6. A Cournot Oligopoly

For illustration, this section considers *multi-output, multi-input Cournot oligopolies* which feature transferable production factors. Firm $i \in I$ produces marketable output y_i, using factor bundle z_i, at cost $c_i(y_i, z_i)$, to get payoff

$$\pi_i(y_i, y_{-i}, z_i) := \langle P(y_I), y_i \rangle - c_i(y_i, z_i).$$

All spaces $\mathbb{Y}_i$ coincide, $y_I := \sum_{i \in I} y_i$ denotes total supply, and $\langle \cdot, \cdot \rangle$ is the standard inner product. The "inverse demand curve" $y_I \mapsto P(y_I)$ reports the price vector at which output markets clear. If factor bundles were traded at price p, a price-taking firm i would take home overall profit $\pi_i(y_i, y_{-i}, z_i) + \langle p, e_i - z_i \rangle$.

For existence of equilibrium, note that rectangular domains of the type $Y_i = [0, \bar{y}_i] \subset \mathbb{Y}$ and $Z_i = [0, e_I] \subset \mathbb{Z}$ are compact convex. The vector $\bar{y}_i \geq 0$ then denotes the production capacity of firm i - whereas $e_I := \sum_{i \in I} e_i \geq 0$ is the aggregate endowment. For the other hypotheses in Proposition 2.1, let cost $c_i(y_i, z_i)$ be jointly convex, and output pricing $P(\cdot)$ be concave and differentiable with $P' + P'^T$ negative semidefinite. Under these conditions, $\pi_i(y_i, y_{-i}, z_i)$ becomes concave in $x_i = (y_i, z_i)$; see [32] or Lemma 7.5 in [17]. For a case with joint transportation, see [12].

As one might expect, cost sharing becomes (conditionally) efficient - as in Montgomery (1972):

Proposition 5.1 (Efficient sharing). *In equilibrium $x = (x_i)$ factors are efficiently allocated. That is, the factor price regime p satisfies*

$$\frac{\partial}{\partial z_i} c_i(y_i, z_i) \in -N_i(z_i, Z_i) - p \quad \forall i. \quad \square$$

Cournot and CO_2 emissions. Simplifying further, let there be just *one* marketable product and *one* transferable factor. Presume differentiable data and interior solutions. *In equilibrium,* there is

$$\left.\begin{array}{ll} \text{market balance:} & \sum_{i \in I} z_i = \sum_{i \in I} e_i, \\ \text{optimal outputs:} & P(y_I) + P'(y_I) y_i = \frac{\partial}{\partial y_i} c_i(y_i, z_i), \\ \text{and common factor pricing:} & \frac{\partial}{\partial z_i} c_i(y_i, z_i) = -p. \end{array}\right\} \tag{13}$$

While still *out-of-equilibrium,* the first two conditions in (13) hold, but factor pricing isn't uniform. Therefore, at any stage, two agents i, j, for which

$$\Delta := \frac{\partial}{\partial z_j} c_j(y_j, z_j) - \frac{\partial}{\partial z_i} c_i(y_i, z_i) \neq 0,$$

undertake an exchange (6) with step-size $s \geq 0$ and transfer direction

$$d := \begin{cases} \Delta & if\ z_i, z_j > 0 \\ \min\{\Delta, 0\} & if\ z_i > 0,\ z_j = 0 \\ \max\{\Delta, 0\} & if\ z_i = 0,\ z_j > 0 \\ 0 & otherwise. \end{cases}$$

A suitable step-size takes the form $s := \min\{s_k, \hat{s}\}$ with

$$\hat{s} := \begin{cases} z_i/d & if\ d < 0 \\ z_j/d & if\ d > 0 \end{cases}$$

In line with the arguments above, for any feasible factor allocation $z = (z_i)$, players easily find Cournot equilibrium (10).

Some numerical simulations. Let Cournot oligopolist $i \in I = \{1, 2, 3\}$ put out $y_i \in \mathbb{R}_+$, using input $z_i \in \mathbb{R}_+$, at cost

$$c_i(y_i, z_i) = \frac{\alpha_i y_i}{1 + z_i}.$$

The inverse demand curve for products is linear: $P(y_I) = 1 - y_I$ where $y_I = \sum_{i \in I} y_i$. To ensure interior solutions, let each $\alpha_i \leq 1/3$. Firms come immediately up with Cournot output levels

$$y_i(z_i, z_{-i}) = \frac{1}{4}\left(1 - 3\frac{\alpha_i}{1 + z_i} + \sum_{j \neq i} \frac{\alpha_j}{1 + z_j}\right).$$

Example 1 (*Identical costs*). Choose cost parameters $\alpha_i = 1/4$, endowments $(e_i) = (3, 0, 0)$, and step-sizes $s_k = 200/k$, to get

i	z_i	y_i	p
1	1.000	0.2187	0.0205
2	1.000	0.2187	0.0205
3	1.000	0.2187	0.0205

Table 1 Results for Example 1.

Example 2 (*Asymmetric costs*). Choose cost parameters $(\alpha_i) = (1/3, 1/4, 1/5)$, endowments $e_i = 1$, and step-sizes $s_k = 200/k$ to get

i	z_i	y_i	p
1	1.1446	0.1916	0.0208
2	0.9990	0.2220	0.0208
3	0.8565	0.2393	0.0208

Table 2 Results for Example 2.

Examples 1 and 2 confirm the emergence of equilibria with symmetric and asymmetric agents.

7. Links to Literature

This paper ties oligopolies for final products to exchange of production factors. A particular feature is that experienced producers adapt their outputs swiftly, at the spur of the moment, but adjust their factor use slowly and stepwise. Nonetheless, granted monotone margins (4), overall equilibrium may eventually obtain. Such monotonicity holds in some workhorse models of industrial organization, notably in Cournot oligopolies.

The paper relates to a large literature. Regarding production strategies, there is room for more generality than indicated above. Partial equilibrium (10) could come à la Cournot, Bertrand, Stackelberg, Arrow-Debreu (1954), Gabszewicz-Vial (1972), or Shapley-Shubik (1977). It imports that the production component of equilibrium be stable.

While adapting their factor use, agents move slowly, driven by differences in personal valuations. Bilateral exchange is the only vehicle. It's agent-based, decentralized, and non-coordinated. This optic has several consequences, briefly mentioned next:

• The Shapley-Shubik (1977) market game, in which players put payments and quantities on diverse tables, is a one-shot, strategic-form episode. Hence it doesn't fit our emphasis on repeated, small-size trades.

• Hahn (1984) and Westskog (1996) consider an oligopoly, surrounded by a price-taking fringe which generates a price curve. Identifying that curve might require more knowledge than the oligopolists have [23]. Further, it's intricate to discern or predict who comes forward as oligopolist [34].

• The Cournot-Walras setting - as modelled by Codognato and Gabszewicz (1991) - presumes that some scarce goods are owned by just a few agents. Whence only the latter act as oligopolists. Such is not our setting. To good approximation, endowments are widely held. For that reason, factor owners are already "in the game". Entry or exit becomes less of an issue, hence not modelled here [28].
• Admittedly, owners of scarce rights may game the situation - say, by destructing, withholding, or misrepresenting endowments [2], [33], [31]. Whatever be the moves which precede the game proper, including maybe reallocation of endowments [24], property rights are here construed as reliable and well defined.
• The factor market, modelled above, is essentially Walrasian, like in Montgomery (1972).[8] However, during trade, no player is a price-taker. There are no prices to take; they are all personal - and not necessarily posted. Moreover, quantities are bounded. Consequently, trade may evolve as in order markets [4]. Despite decentralization, equilibrium is apt to emerge all the same.
• Attainment of equilibrium in exchange economies has long been studied; see [7], [8], [9], [14], [16] and references therein. A novelty here is that exchange is superimposed on non-cooperative production. There are some similarities to [12], but two major differences: first, the aggregate endowment is fixed; second, out-of-equilibrium behavior is explicitly modelled.
• Since the product market is imperfect, Pareto efficiency is rare, and overall payoff is apt to evolve in non-monotone (non-potential) manner. These features are at variance with transferable-utility pure exchange models [7], [8], [9], [14], [16]. Also noteworthy is that individual constraints can be general and criteria non-smooth.

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