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SOME GENERAL PRINCIPLES IN TROPICAL CONVEXITIES

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Abstract. We present what the title says with an emphasis on separation and topological properties as they relate to some of the fundamental principles of nonlinear analysis.

1. Introduction

The text is structured along the lines of a short talk given at ETAMM 2016. Some details are given in the first part, which is mainly algebraic or set theoretical; few details are given in the second part where algebra and topology intertwine. The first part revolves maily around the Kakutani Separation Property, also known as the Stone-Kakutani Property or the algebraic Hahn-Banach; going into the analytic or the geometric Hahn-Banach would have taken us too far. The second part revolves around a basic intersection theorem, the theorem of Knaster-Kuratowski-Mazurkiewicz - which is an avatar of Brouwer's Fixed Point Theorem - and some of its consequences, mainly fixed point theorems for single valued or multivalued maps. We refrained from giving the most general statement or the most general proof, when we gave one at all. With the last part we come back to the first part of the paper; the result stated their shows that the Kakutani Separation Prop erty, along with some natural topological properties, like connectedness of the convex sets, are at the heart of the fixed points results - and others too - that were presented in the preceding section. We have not tried to be exhaustive - duality and variational inequalities in tropical convexities are not to be found here - we tried to show on a few basic results why and how things work.
Applications are made mainly to the standard tropical convexities, maxplus in $\mathbb{R}^n$ and maxtimes, also called $\mathbb{B}$-convexity, in $\mathbb{R}^n_+$. Some infinite dimensional tropical convexities are listed among the examples but they are not studied here. The topological structures of some infinite dimensional spaces naturally related to tropical convexity are studied in [3] and [4].

2. Convexities and convex spaces

2.1. The general framework

Definition 2.1.1. *A convexity on a set X is a family $\mathcal{C}$ of subsets of X such that:*

(Conv1) $\emptyset \in \mathcal{C}$ *and* $X \in \mathcal{C}$ *and,* $\forall x \in X$, $\{x\} \in \mathcal{C}$*;*
(Conv2) *if $\mathcal{A}$ is a subfamily of $\mathcal{C}$ then $\bigcap \mathcal{A}$ belongs to $\mathcal{C}$;*
(Conv3) *if $\mathcal{A}$ is an updirected[1] subfamily of $\mathcal{C}$ then $\bigcup \mathcal{A}$ belongs to $\mathcal{C}$.*

Elements of $\mathcal{C}$ are called convex sets and $(X, \mathcal{C})$ is a convex space.

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[1] $\forall C, C' \in \mathcal{C} \quad \exists C'' \in \mathcal{C}$ such that $C \cup C' \subset C''$.

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Given $A \subset X$ let $[A]_{\mathcal{C}} = \bigcap\{C \in \mathcal{C} : A \subset C\}$; It is the $\mathcal{C}$**-hull** of A.

By **Conv1** and **Conv2**, $\forall A \subset X \quad [A]_{\mathcal{C}} \in \mathcal{C}$ and $[A]_{\mathcal{C}} = A \Leftrightarrow A \in \mathcal{C}$. Furthermore,

$$\forall A, B \subset X \quad \begin{cases} (1) & A \subset [A]_{\mathcal{C}} \\ (2) & A \subset B \Rightarrow [A]_{\mathcal{C}} \subset [B]_{\mathcal{C}} \\ (3) & [\,[A]_{\mathcal{C}}\,]_{\mathcal{C}} = [A]_{\mathcal{C}} \end{cases} \tag{1}$$

For all non empty sets S let $\langle S \rangle$ be the family of nonempty finite subsets of S. Condition **(Conv3)** implies that

Lemma 2.1.2. *For all convex space $(X, \mathcal{C})$ and for all subsets A of X*

$$[A]_{\mathcal{C}} = \bigcup_{F \in \langle A \rangle} [F]_{\mathcal{C}} \tag{2}$$

Proof. We can assume that $A \neq \emptyset$. The family $\langle A \rangle$ of nonempty finite subsets of A is updirected and consequently so is the family of convex sets $\{[F]_{\mathcal{C}} : F \in \langle A \rangle\}$ and therefore $\bigcup_{F \in \langle A \rangle} [F]_{\mathcal{C}}$ is convex, call this set C. If $F \in \langle A \rangle$ then $F \subset [F]_{\mathcal{C}} \subset [A]_{\mathcal{C}}$ from which $C \subset [A]_{\mathcal{C}}$. From $A = \bigcup_{F \in \langle A \rangle} F$ we have $A \subset C$ and consequently, since $C \in \mathcal{C}$, $[A]_{\mathcal{C}} \subset C$. We have shown that $[A]_{\mathcal{C}} = C$. □

Lemma 2.1.3. *For all convex space $(X, \mathcal{C})$ and for all subset C of X, $C \in \mathcal{C}$ if and only if, for all finite subset F of C, $[F]_{\mathcal{C}} \subset C$.*

Proof. This is a restatement of Lemma 2. □

The $\mathcal{C}$-hull of a finite set is a $\mathcal{C}$**-polytope**. From 1 and 2 we have

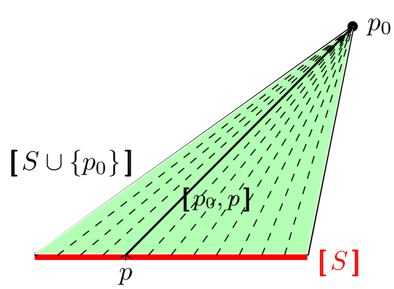
$$\forall C \in \mathcal{C} \quad C = \bigcup_{\substack{P \subset C \\ P \text{is a polytope}}} P \tag{3}$$

Conditions 1 say that the application $A \mapsto [A]_{\mathcal{C}}$ is a Moore closure operator while 2 says that it is an algebraic closure operator. One can easily check, that given a algebraic Moore closure operator $[\,\cdot\,] : \mathcal{P}(X) \to \mathcal{P}(X)$ on a set X such that $[\emptyset] = \emptyset$ and, forall $x \in X$, $[\{x\}] = \{x\}$, the set of fixed points of $[\,\cdot\,]$ is a convexity $\mathcal{C}$ on X for such that $[\,\cdot\,]_{\mathcal{C}} = [\,\cdot\,]$.

From here on, if no confusion is possible, the convex hull operator associated to a given convexity will mostly be written $[\,\cdot\,]$ and we will write $[p_1, \cdots, p_m]$ for $[\{p_1, \cdots, p_m\}]$.

Definition 2.1.4. *A convex space $(X, \mathcal{C})$ is* **recursive** *if, for all non empty finite subset S of X and for all point $p_0 \in X$*

$$[S \cup \{p_0\}] = \bigcup_{p \in [S]} [p, p_0]. \tag{4}$$



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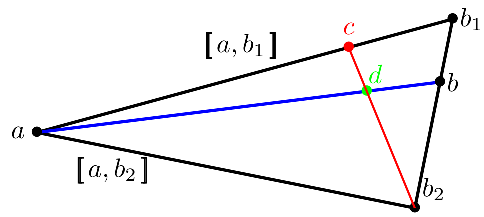
Let us say that a convexity $\mathcal{C}$ on a set X is an **interval convexity** if, for all $C \subset X$, the following two conditions are equivalent :

$$\begin{cases} (1) & C \in \mathcal{C} \\ (2) & \forall p, q \in C \quad [\![p, q]\!] \subset C \end{cases} \tag{6}$$

Lemma 2.1.5. *A recursive convexity $\mathcal{C}$ on a set X is an interval convexity.*

Proof. Let $S = \{p_0, \cdots, p_m\}$ be an arbitrary non empty subset of C. Assume that, for all $p, q \in C$, $[\![p, q]\!] \subset C$. We show, by induction on the cardinality of S, that $[\![S]\!] \subset C$. If $m \leq 1$ there is nothing to show. Assume that $m > 1$; from $D = [\![S \setminus \{p_m\}]\!] \subset C$ and from $[\![S]\!] = \cup_{q \in D} [\![p_m, q]\!]$ we have $[\![S]\!] \subset C$; by Lemma 2.1.3, C is convex; □

Definition 2.1.6. *A convex space $(X, \mathcal{C})$ has* **Property C** *if, for all triple (a, b_1, b_2) of points of X and all points $c \in [\![a, b_1]\!]$ and $d \in [\![c, b_2]\!]$ there exists a point $b \in [\![b_1, b_2]\!]$ such that $d \in [\![a, b]\!]$*



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Proposition 2.1.7. *A convexity $\mathcal{C}$ on a set X is recursive if and only if it is an interval convexity with Property C.*

Proof. Assume that $\mathcal{C}$ is recursive, by Lemma 2.1.5 it is an interval convexity. Let the points a, b_1, b_2, c and d be as in definition 2.1.6. From $[\![a, b_1, b_2]\!] = \bigcup_{b \in [\![b_1, b_2]\!]} [\![a, b]\!]$ and $d \in [\![c, b_2]\!] \subset [\![a, b_1, b_2]\!]$ there exists $b \in [\![b_1, b_2]\!]$ such that $d \in [\![a, b]\!]$.

Assume now that $\mathcal{C}$ is an interval convexity for which Property C holds and let us show that $\mathcal{C}$ is recursive. Since $S_0 \cup \{p_0\} \subset \bigcup_{p \in [\![S_0]\!]} [\![p, p_0]\!]$ it is enough to see that $\bigcup_{p \in [\![S_0]\!]} [\![p, p_0]\!]$ is convex since we then have $[\![S_0 \cup \{p_0\}]\!] \subset [\![\bigcup_{p \in [\![S_0]\!]} [\![p, p_0]\!]]\!] = \bigcup_{p \in [\![S_0]\!]} [\![p, p_0]\!]$.

Let p_1, p_2 be two points of $\bigcup_{p \in [\![S_0]\!]} [\![p, p_0]\!]$. We show that $[\![p_1, p_2]\!] \subset \bigcup_{p \in [\![S_0]\!]} [\![p, p_0]\!]$.

There exist $q_1, q_2 \in [\![S_0]\!]$ such that $p_i \in [\![p_0, q_i]\!]$; let $c \in [\![p_1, p_2]\!]$.

Applying Property C to the configuration $\begin{cases} a = p_0 \\ c_1 = p_1 \\ b_1 = q_1 \\ c_2 = b_2 = p_2 \end{cases}$

we find a point $b' \in [\![q_1, p_2]\!]$ such that $c \in [\![p_0, b']\!]$.

With the configuration $\begin{cases} a = p_0 \\ c_1 = b_1 = q_1 \\ b_2 = q_2 \\ c_2 = p_2 \end{cases}$ we find a point $b \in [\![q_1, q_2]\!]$ such that $b' \in [\![p_0, b]\!]$.

From $c \in [\![p_0, b']\!]$ and $b' \in [\![p_0, b]\!]$ we have $c \in [\![p_0, b]\!]$.

From $b \in [\![q_1, q_2]\!]$ and $q_1, q_2 \in [\![S_0]\!]$ we have $b \in [\![S_0]\!]$ and finally $c \in \bigcup_{p \in [\![S_0]\!]} [\![p, p_0]\!]$ and therefore $[\![p_1, p_2]\!] \subset \bigcup_{p \in [\![S_0]\!]} [\![p, p_0]\!]$. □

Given a set X let $X^{\{0,1\}}$ be the family of non empty finite subsets of X of cardinality at most 2 ; given a map $\mathcal{I} : X^{\{0,1\}} \to \mathcal{P}(X)\setminus\{\emptyset\}$ let

$$\mathcal{C}_{\mathcal{I}} = \big\{C \subset X : \forall\{x,y\} \subset C \quad \mathcal{I}(\{x,y\}) \subset C\big\}.$$

The proofs of Lemmas 2.1.8 and 2.1.9 below are left to the reader.

Lemma 2.1.8. $\big(X, \mathcal{C}_{\mathcal{I}}\big)$ *is a convex space if and only if the following condition holds:*
$\mathbf{L_0}$ *:* $\forall x \in X \ \mathcal{I}(\{x\}) = \{x\}$
in which case it is an interval convexity.

The convex interval $\big[x,y\big]_{\mathcal{C}_{\mathcal{I}}}$, which is the intersection of all the $\mathcal{C}_{\mathcal{I}}$ convex sets containing $\{x,y\}$, does not have to be $\mathcal{I}(\{x,y\})$.

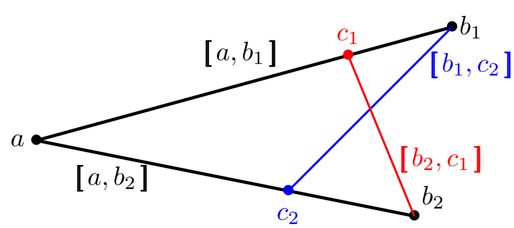
Lemma 2.1.9. *Assume that condition* $\mathbf{L}_0$ *holds. The following statements are equivalent:*
(1) *For all* $\{x,y\} \subset X$ $\big[x,y\big]_{\mathcal{C}_{\mathcal{I}}} = \mathcal{I}(\{x,y\})$.
(2) *For all* $\{x,y\} \subset X$ *and for all* $\{w,z\} \subset \mathcal{I}(\{x,y\})$, $\mathcal{I}(\{w,z\}) \subset \mathcal{I}(\{x,y\})$.

General convex spaces are more akin to general topological spaces or measure spaces, in that the structure is defined by a family of subsets, than to the usual linear convex sets where the convex structure is associated to some previously defined algebraic structure. This of course does not preclude a given convexity structure from being associated to an underlying algebraic structure as, for example, it is the case for tropical, or linear, convexity. In any case, in the absence of an underlying primary structure from which the convexity structure is derived, it seems natural to define a morphism of convex spaces as it is done for topological spaces (continuous maps) or measure spaces (measurable maps) ; morphisms of general convex spaces are called, following Van Mill and Van de Vell, **convexity preserving maps**. A convexity preserving map from a convex space $(X_1, \mathcal{C}_1)$ to a convex space $(X_2, \mathcal{C}_2)$ is a map $\varphi : X_1 \to X_2$ such that, for all $C \in \mathcal{C}_2$, $\varphi^{-1}\big(C\big) \in \mathcal{C}_1$.
Obviously, the class of convex spaces and convexity preserving maps constitute a category, whose study is not the subject matter of this paper. All we will need is the associated notion of isomorphism : two **convex spaces** $(X_1, \mathcal{C}_1)$ and $(X_2, \mathcal{C}_2)$ **are isomorphic** if there exists a bijective convexity preserving map $\varphi : X_1 \to X_2$ whose inverse is also convexity preserving; such a map is naturally called an isomorphism of convexity spaces. Clearly, if $\varphi : (X_1, \mathcal{C}_1) \to (X_2, \mathcal{C}_2)$ is an isomorphism then so is $\varphi^{-1} : (X_2, \mathcal{C}_2) \to (X_1, \mathcal{C}_1)$ and $\varphi : (X_1, \mathcal{C}_1) \to (X_2, \mathcal{C}_2)$ is an isomorphism if and only if it is bijective and for all $C_1 \in \mathcal{C}_1$ all $C_2 \in \mathcal{C}_2$, $\varphi\big(C_1\big) \in \mathcal{C}_2$ and $\varphi^{-1}\big(C_2\big) \in \mathcal{C}_1$.

2.2. Algebraic separation of convex sets

Definition 2.2.1. *A convex space* $(X, \mathcal{C})$ *has the* **Pash-Peano Property** *if, for all quintuple* (a, b_1, b_2, c_1, c_2) *of points of* X *such that* $c_i \in \big[a, b_i\big]$*, one has*

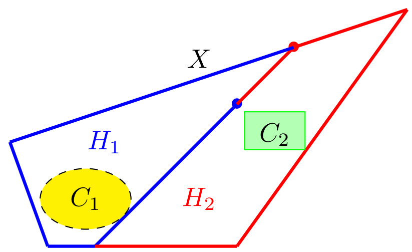
$$\big[b_1, c_2\big] \cap \big[b_2, c_1\big] \neq \emptyset.$$



(8)

Definition 2.2.2. *A half-space of a convex space* $(X, \mathcal{C})$ *is a convex set* $H \in \mathcal{C}$ *whose complement* $X \setminus H$ *is also in* $\mathcal{C}$.

Definition 2.2.3. *A convex set $(X, \mathcal{C})$ has the Kakutani Property if, for all pair (C_1, C_2) of disjoint non empty convex sets there exists a half-space H such that $C_1 \subset H$ and $C_2 \subset X \setminus H$.*



Kakutani Property

Theorem 2.2.4. *A recursive convex space $(X, \mathcal{C})$ has the Kakutani Property if and only if it has the Pash-Peano Property.*

Proof. A proof can be found in [20]. □

Corollary 2.2.5. *If an interval convexity $\mathcal{C}$ on a set X has Property C and the Pash-Peano Property then it has the Kakutani Property.*

Proof. By Proposition 2.1.7. □

Both Property C and the Pash-Peano Property are properties of polytopes spanned by at most three points; one could informally say that, for interval convexities, the Kakutani property is a "2-dimentional property". One can compare with Proposition 2.2.8 below.

Definition 2.2.6. *A convex space $(X, \mathcal{C})$ has the* **Generalized Pash-Peano Property** *if for all non empty finite subsets B_1 and B_2 of X and all triple of points (a, c_1, c_2) with $c_i \in [\![\{a\} \cup B_i]\!]$ one has $[\![\{c_1\} \cup B_2]\!] \cap [\![\{c_2\} \cup B_1]\!] \neq \emptyset$.*

A proof of the following proposition can be found in [20].

Proposition 2.2.7. *A convex space $(X, \mathcal{C})$ has the Kakutani Property if and only if it has the Generalized Pash-Peano Property.*

Given a convex space $(X, \mathcal{C})$ and a convex set $P \in \mathcal{C}$ the family $\{P \cap C : C \in \mathcal{C}\}$ is a convexity on P called the **induced convexity**. Proposition 2.2.8 below is due to Keimel and Wieczorek; their proof can be found [23].

Proposition 2.2.8. *A convex space $(X, \mathcal{C})$ has the Kakutani Property if and only if all polytopes have, with respect to the induced convexity, the Kakutani Property.*

Proof. Notice that the Generalized Pash-Peano Property holds if and only if it holds on polytopes. □

Property C is due to Coppel, [14] where some of the results stated here, albeit in a somewhat different form and for a different purpose can also be found. Convexities with the Kakutani Property are studied in [14] and [29]; Holmes in [17] explicitely uses the Pash-Peano Property - without giving it a name since in a real vector space it is obvious - to prove the algebraic separation theorem.

3. Tropical convexity

A **semiring** is a set A, with two operations $\oplus : A \times A \to A$ and $\odot : A \times A \to A$ and two distinguished elements $\mathtt{1}_A$ and $\mathtt{0}_A$ such that:
(1) $(A, \oplus, \mathtt{0}_A)$ is an abelian semigroup.
(2) for all $x, y, z \in A$, $z \odot (x \odot y) = (z \odot x) \odot y$.
(3) for all $x, y, z \in A$, $z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y)$.
(4) for all $x \in A$, $\mathtt{0}_A \odot x = \mathtt{0}_A$ and $\mathtt{1}_A \odot x = x$

For simplicity we will assume that $\odot$ is commutative.

An **idempotent semiring**, also called a **tropical semiring** or a **Maslov semiring** is a semiring A such that, for all $x \in A$, $x \oplus x = x$.
A **(tropical) semifield** is a (tropical) semiring for which $(A_\star, \odot, \mathtt{1}_A)$ is a group ($A_\star = A \setminus \{\mathtt{0}_A\}$). Since tropical multiplication can sometimes be the "usual addition" or the "usual multiplication" of scalars we will denote, for an arbitrary semifield $(A_\star, \odot, \mathtt{1}_A)$, by u^{inv} the inverse of $u \in A_\star$.

An idempotent abelian semigroup $(A, \oplus, \mathtt{0}_A)$ is a semilattice with smallest element $\mathtt{0}_A$, and reciprocally, where $x \oplus y$ is the least upper bound of $\{x, y\}$ with respect to the partial order $x \leqslant y$ if $x \oplus y = y$.

Some tropical semirings

(1)(maxplus semiring) $A = \mathbb{R} \cup \{-\infty\}$, $\mathtt{0}_{\max+} = -\infty$, $\mathtt{1}_{\max+} = 0$, $x \oplus y = \max\{x, y\}$, $x \odot y = x + y$.
On could could also take A to be $\mathbb{Z} \cup \{-\infty\}$ or $\mathbb{Q} \cup \{-\infty\}$.
(2)(maxtimes semiring) $A = \mathbb{R}_+$, $\mathtt{0}_{\max\times} = 0$, $\mathtt{1}_{\max\times} = 1$, $x \oplus y = \max\{x, y\}$, $x \odot y = xy$. One could also take for A $\mathbb{N}$ or $\mathbb{Q}_+$.
(3)(distributive lattice) $\left(L, \vee, \wedge, \mathtt{0}_L, \mathtt{1}_L\right)$ is a disributive lattice with a smallest and a largest element; let $x \oplus y = x \vee y$ and $x \odot y = x \wedge y$.

The maxplus and the maxtimes semirings are semifields ; if $u \in \mathbb{R}$ the multiplicative inverse of u in the maxplus semifield is $-u$; if $u \in \mathbb{R}_+ \setminus \{0\} = \mathbb{R}_{++}$ the multiplicative inverse of u in the maxtimes structure is u^{-1}.

A **(tropical) semimodule over a (tropical) semiring** $(A, \oplus_A, \odot_A, \mathtt{0}_A, \mathtt{1}_A)$ is an abelian (idempotent) semigroup $\left(M, \oplus, \mathtt{0}_M\right)$ endowed with an operation
$\odot : A \times M \to M$ such that :

$$\forall a, b \in A \quad \forall x, y \in M \begin{cases} a \odot (x \oplus y) = (a \odot x) \oplus (b \odot y) \\ (a \oplus_A b) \odot x = (a \odot x) \oplus (b \odot x) \\ (a \odot_A b) \odot x = a \odot (b \odot x) \\ \mathtt{1}_A \odot x = x \text{ and } \mathtt{0}_A \odot x = \mathtt{0}_M \end{cases}$$

>From here on we will mostly drop the subscripts on the operations $\oplus$, $\odot$. Tropical semirings are also called idempotent semirings or Maslov semirings.

A tropical semiring A is a **totally ordered tropical semiring** if for all $x, y \in A$, $x \oplus y \in \{x, y\}$.

The maxplus semiring $\mathbb{A} \cup \{-\infty\}$ and the maxtimes times semiring $\mathbb{A}_+$ where $\mathbb{A} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \cdots\}$ are totally ordered.

Some tropical semimodules

(1) Let A be an arbitrary semiring and S a non empty set. On $M = A^S$, the set of all functions from S to A, let $\oplus$ and $\odot$ be the pointwise operations: $(f \oplus g)(s) = f(s) \oplus g(s)$ and $(a \odot f)(s) = a \odot f(s)$; and take $\mathtt{0}_M$ be the function which is identically $\mathtt{0}_A$.

(1.1) If A is the maxplus semiring $\mathbb{R} \cup \{-\infty\}$ and if $S = \{1, \cdots, n\}$ we have the maxplus semimodule $\big(\mathbb{R} \cup \{-\infty\}\big)^n$;
(1.2) If A is the maxtimes semiring $\mathbb{R}_+$ and S is as above we have the maxtimes semimodule $\big(\mathbb{R}_+\big)^n$.
(2) Let $(L, \vee)$ be a real vector space semilattice, for example the space of real valued bounded functions on a given set X or the set of real continuous functions on a compact set X where $\vee = \max$; on $L_+ = \{x \in L : 0 \leqslant x\}$ let $x \odot y = x \vee y$ and, for $t \in \mathbb{R}_+$, $t \odot x = tx$. With these operations L_+ is a semimodule over the maxtimes semiring $\mathbb{R}_+$.

A subset C of a tropical semimodule M over a tropical semiring A is **tropically convex** if :

$$\forall (x,y) \in C^2 \quad \forall (a,b) \in A^2 \text{ such that } a \oplus b = 1 \quad (a \odot x) \oplus (b \odot y) \in C$$

Proposition 3.0.1. *The set of convex sets of a given semimodule M over an arbitrary tropical semiring A is an interval convexity on M for which, for all $x, y \in M$, $[\![x,y]\!] = \big\{(a \odot x) \oplus (b \odot y) : (a,b) \in A^2$ and $a \oplus b = 1\big\}$. Furthermore, for an arbitrary finite and non empty subset $S = \{x_1, \cdots, x_m\}$ of M, $[\![S]\!] = \big\{ \bigoplus_{i=1}^m a_i \odot x_i : (a_1, \cdots, a_m) \in A^n$ s.t. $\bigoplus_{i=1}^m a_i = 1\big\}$*

Proof. Let $\mathcal{I}(\{x,y\}) = \big\{(a \odot x) \oplus (b \odot y) : (a,b) \in A^2$ and $a \oplus b = 1\big\}$. $\mathbf{L_0}$ holds since, with $a \oplus b = 1$, we have $(a \odot x) \oplus (b \odot x) = (a \oplus b) \odot x = 1 \odot x = x$.

The following straightforward calculation shows that (2) of Lemma 2.1.9 holds.
Assume that $a_i \oplus b_i = 1$ and $a \oplus b = 1$.
Let $w = (a_1 \odot x) \oplus (b_1 \odot y)$ and $z = (a_2 \odot x) \oplus (b_2 \odot y)$ with $a_i \oplus b_i = 1$. Then $(a \odot w) \oplus (b \odot z) = \big\{[(a \odot a_1) \oplus (b \odot a_2)] \odot x\big\} \oplus \big\{[(a \odot b_1) \oplus (b \odot b_2)] \odot y\big\}$.
>From $\big\{[(a \odot a_1) \oplus (b \odot a_2)]\big\} \oplus \big\{[(a \odot b_1) \oplus (b \odot b_2)]\big\} = a \odot (a_1 \oplus b_1) \oplus b \odot (a_2 \oplus b_2) = a \odot 1 \oplus b \odot 1 = a \oplus b = 1$ we have $(a \odot w) \oplus (b \odot z) \in \mathcal{I}(\{x,y\})$.

A similar straightforward calculation yields the second part. □

Let $\mathcal{C}_{\text{trop}}(M)$ denote the family of tropically convex subsets of a tropical semimodule M over a tropical semiring A; the emptyset belongs to $\mathcal{C}_{\text{trop}}(M)$.

Proposition 3.0.2. *If M is a tropical semimodule over a tropical semifield A then tropical convexity on M is recursive.*

Proof. . We have seen that $\mathcal{C}_{\text{trop}}(M)$ is an interval convexity ; let us see that Property C holds. Let c and d be two points of M such that

$$\begin{cases} (1) \quad c = (s \odot a) \oplus (t \odot b_1) \text{ with } s \oplus t = 1 \\ \text{and} \\ (2) \quad d = (s' \odot c) \oplus (t' \odot b_2) \text{ with } s' \oplus t' = 1 \end{cases} \tag{9}$$

which gives

$$d = (s' \odot s) \odot a \oplus ((s' \odot t) \odot b_1) \oplus (t' \odot b_2)$$

If $(s' \odot t) \oplus t' \neq 0$ let $u = \big((s' \odot t) \oplus t'\big)^{\text{inv}}$.

Then $[(s' \odot t) \odot b_1 \oplus (t' \odot b_2)] = ((s' \odot t) \oplus t') \odot \big((t_1 \odot b_1) \oplus (t_2 \odot b_2)\big)$ with $t_1 = u \odot (s' \odot t)$ and $t_2 = u \odot t'$.
Furthermore, $t_1 \oplus t_2 = [u \odot (s' \odot t)] \oplus (u \odot t') = u \odot [(s' \odot t) \oplus t'] = 1$.
We have shown that $(t_1 \odot b_1) \oplus (t_2 \odot b_2) \in [\![b_1, b_2]\!]$. Let $b = (t_1 \odot b_1) \oplus (t_2 \odot b_2)$.

From $d = (s' \odot s) \odot a \oplus ((s' \odot t) \oplus t') \odot b$ and $(s' \odot s) \oplus (s' \odot t) \oplus t' = s' \odot (s \oplus t) \oplus t' = s' \oplus t' = 1$ we have $d \in [\![a, b]\!]$.

If $(s' \odot t) \oplus t' = 0$ then $(s' \odot t) = t' = 0$. Since A is a semifield, either $s' = 0$ or $t = 0$; from $s' \oplus t' = 1$ and $t' = 0$ we have $s' = 1$ and therefore $t = 0$ and, from $s \oplus t = 1$, $s = 1$. So $a = c = d$ and $d = a \oplus (0 \odot b)$ with $b = b_1$. □

Proposition 3.0.3. *If M is a tropical semimodule over a totally ordered tropical semifield A then tropical convexity on M has the Pash-Peano Property.*

Proof. Let $c_i = (s_i \odot a) \oplus (t_i \odot b_i)$ with $s_i \oplus t_i = 1$. We have to show that there exists (u_1, v_1) and (u_2, v_2) in $A \times A$ such that $\begin{cases} (1) & u_1 \oplus v_1 = u_2 \oplus v_2 = 1 \\ (2) & (u_1 \odot c_1) \oplus (v_1 \odot b_2) = (u_2 \odot c_2) \oplus (v_2 \odot b_1) \end{cases}$ Substituting for c_1 and c_2, respectively, in (2) yields

$$\begin{cases} (u_1 \odot c_1) \oplus (v_1 \odot b_2) = [(u_1 \odot s_1) \odot a] \oplus [(u_1 \odot t_1) \odot b_1] \oplus [v_1 \odot b_2] \\ (u_2 \odot c_2) \oplus (v_2 \odot b_1) = [(u_2 \odot s_2) \odot a] \oplus [(u_2 \odot t_2) \odot b_2] \oplus (v_2 \odot b_1) \end{cases}$$

To complete the proof it is sufficient to find (u_i, v_i) such that $\begin{cases} (a) & u_1 \odot s_1 = u_2 \odot s_2 \\ (b) & u_1 \odot t_1 = v_2 \\ (c) & u_2 \odot t_2 = v_1 \end{cases}$ and $u_1 \oplus v_1 = u_2 \oplus v_2 = 1$.

Assume that either $s_1 \neq 0$ or $s_2 \neq 0$; let us say $s_2 \neq 0$.

$$\text{Given an arbitrary } u_1 \in A \backslash \{0\} \text{ let } \begin{cases} u_2 = u_1 \odot s_1 \odot s_2^{\text{inv}} \\ v_1 = u_2 \odot t_2 \\ v_2 = u_1 \odot t_1 \end{cases} \tag{10}$$

Equations (a), (b) and (c) hold. We have to see that u_1 can be chosen such that $u_1 \oplus v_1 = u_2 \oplus v_2 = 1$.

$$\begin{cases} u_1 \oplus v_1 = u_1 \oplus (u_2 \odot t_2) = u_1 \oplus (u_1 \odot s_1 \odot s_2^{\text{inv}} \odot t_2) = u_1 \odot [1 \oplus (s_1 \odot s_2^{\text{inv}} \odot t_2)] \\ u_2 \oplus v_2 = [u_1 \odot s_1 \odot s_2^{\text{inv}}] \oplus [u_1 \odot t_1] = u_1 \oplus [(s_1 \odot s_2^{\text{inv}}) \oplus t_1)] \end{cases} \tag{11}$$

Since A is totally ordered and $s_i \oplus t_i = 1$ there are only four cases to consider: $\begin{cases} (1) & s_1 = s_2 = 1 \\ (2) & s_1 = t_2 = 1 \\ (3) & t_1 = s_2 = 1 \\ (4) & t_1 = t_2 = 1 \end{cases}$

$$(1) \quad \begin{cases} u_1 \oplus v_1 = u_1 \odot (1 \oplus t_2) = u_1 \odot 1 \\ u_2 \oplus v_2 = u_1 \oplus t_1 \end{cases}$$

Take $u_1 = 1$.

$$(2) \quad \begin{cases} u_1 \oplus v_1 = u_1 \odot (1 \oplus s_2^{\text{inv}}) \\ u_2 \oplus v_2 = u_1 \oplus (s_2^{\text{inv}} \oplus t_1) \end{cases}$$

Take $u_1 = s_2$.

$$(3) \quad \begin{cases} u_1 \oplus v_1 = u_1 \odot [1 \oplus (s_1 \odot t_2)] = u_1 \odot 1 = u_1 \\ u_2 \oplus v_2 = u_1 \end{cases}$$

Take $u_1 = 1$.

$$\begin{cases} u_1 \oplus v_1 = u_1 \odot [1 \oplus (s_1 \odot s_2^{\text{inv}})] \\ u_2 \oplus v_2 = u_1 \oplus (s_1 \odot s_2^{\text{inv}}) \end{cases}$$

If $s_1 \odot s_2^{\text{inv}} \leq 1$ take $u_1 = 1$ and if $1 \leq s_1 \odot s_2^{\text{inv}}$ take $u_1 = \left(s_2 \odot s_1^{\text{inv}}\right)^{\text{inv}}$. □

Proposition 3.0.4 and Corollary 3.0.5 below follow from the previous propositions of this section and propositions 2.2.4 and 2.2.7

Proposition 3.0.4. *If M is a tropical semimodule over a totally ordered tropical semifield A then tropical convexity on M has the Kakutani Property and the Generalized Pash-Peano Property.*

Corollary 3.0.5. *An arbitrary convex subset of a tropical semimodule over a totally ordered tropical semifield is the intersection of all the half spaces containing it.*

The structure of halfspaces and separation theorems, algebraic and analytic, in maxplus convexity are studied in [8], [9], [12], [12] and [26].

As is well known, maxplus convexity in $\big(\mathbb{R}\cup\{-\infty\}\big)^n$ and $\mathbb{B}$-convexity (that is maxtimes convexity in $\mathbb{R}^n_+$) are isomorphic: $\mathbf{ln}(x_1,\cdots,x_n)=(\ln(x_1),\cdots,\ln(x_n))$, with $\ln(0)=-\infty$, is a convexity preserving bijection from $\mathbb{R}^n_+$ to $\big(\mathbb{R}\cup\{-\infty\}\big)^n$ whose inverse is $\boldsymbol{E}(x_1,\cdots,x_n)=(e^{x_1},\cdots,e^{x_n})$, with $0=e^{-\infty}$, is also convexity preserving.
Since $\mathbb{R}^n$ and $\mathbb{R}^n_{++}$ are, respectively, convex subsets of the maxplus semimodule $\big(\mathbb{R}\cup\{-\infty\}\big)^n$ and the maxtimes semimodule $\mathbb{R}^n_+$ and, from $\boldsymbol{E}\big(\mathbb{R}^n\big)=\mathbb{R}^n_{++}$, one has that maxplus convexity on $\mathbb{R}^n$ and $\mathbb{B}$-convexity on $\mathbb{R}^n_{++}$ are also isomorphic.

More tropical semimodules and tropical convex sets

Apart from the standard finite dimensional maxplus convexity on $\big(\mathbb{R}\cup\{-\infty\}\big)^n$ or maxtimes convexity $\mathbb{R}^n_+$ there are natural (possibly) infinite dimensional tropical convexities, for example:
(1) The "*Hom functor*" yields tropical semimodule: given a two tropical semimodules M_1 and M_2 over a tropical semiring A one can consider the set $\mathsf{Hom}_A(M_1,M_2)$ of all maps $\psi:M_1\to M_2$ such that
$$\begin{cases}(1)\,\forall(a,x)\in A\times M_1 & \psi(a\odot x)=a\odot\psi(x)\\ (2)\,\forall(x_1,x_2)\in M_1\times M_1 & \psi(x_1\oplus x_2)=\psi(x_1)\oplus\psi(x_2)\end{cases}$$
There is on $\mathsf{Hom}_A(M_1,M_2)$ an obvious tropical A-semimodule structure :
$(a\odot\psi)(x)=a\odot\big(\psi(x)\big)$ and $(\psi_1\oplus\psi_2)(x)=\psi_1(x)\oplus\psi_2(x)$.
For all $\psi\in\mathsf{Hom}_A(M_1,M_2)$, $\psi(\mathsf{0})=\mathsf{0}$; let $\ker\psi=\{x\in M_1:\psi(x)=\mathsf{0}\}$, it is a sub-semimodule of M_1.
Elements of $\mathsf{Hom}_A(M_1,M_2)$ are convexity preserving maps (as previously defined) but also, the image $\psi(C)$ of an element $C\in\mathcal{C}_{\text{trop}}(M_1)$ by a map $\psi\in\mathsf{Hom}_A(M_1,M_2)$ belongs to $\mathcal{C}_{\text{trop}}(M_2)$.
A map $\theta:C_1\to M_2$ from $C_1\in\mathcal{C}_{\text{trop}}(M_1)$, $C_1\neq\emptyset$, to M_2 is **A-affine** if, for all $x_1,x_2\in C_1$ and for all $a_1,a_2\in A$ such that $a_1\oplus a_2=\mathsf{1}$
$$\theta\big((a_1\odot x_1)\oplus(a_2\odot x_2)\big)=a_1\theta(x_1)\oplus a_2\theta(x_2)$$
The set $\mathsf{Aff}_A(C_1,M_2)$ of A-affine maps from C_1 to M_2 is, with respect to the pointwise $\odot$ and $\oplus$ operations, an A-semimodule (recall that the $\odot$ operation on A is assumed to be commutative). The image of a convex subset of C_1 by $\theta\in\mathsf{Aff}_A(C_1,M_2)$ is convex and the inverse images of convex subset of M_2 is a convex subset of C_1.
(2) Let us say that a subset K of an A-semimodule M is a **cone in M** if:
$$\begin{cases}(1)\,\forall(x_1,x_2)\in K\times K & x_1\oplus x_2\in K\\ (2)\,\forall(a,x)\in A_\star\times K & a\odot x\in K\end{cases}$$
A cone in M is a convex subset of M (if $a_1\oplus a_2=\mathsf{1}$ then either $a_1\neq\mathsf{0}$ or $a_2\neq\mathsf{0}$). If K is a cone in M then $K\cup\{\mathsf{0}\}$ is a subsemimodule of M and if A is a semifield and L is a sub-semimodule of M then $L_\star=L\backslash\{\mathsf{0}\}$ is a cone in M. For example, given a nonempty set X, let, for all $a\in A$, $\mathsf{c}_a\in A^X$ be the constant map $\mathsf{c}_a(x)=a$; A^X, with the pointwise operations is an A-semimodule and *if A is a semifield* $\{\mathsf{c}_a : a\in A_\star\}$ is a cone in A^X (which we identify with $A_\star$) and $A^X_\star$ is also a cone in A^X which contains $A_\star$.
Given two cones $K_1\subset M_1$ and $K_2\subset M_2$ let $\mathsf{Hom}_A(K_1,K_2)$ be the set of maps $\psi:K_1\to K_2$ such that

$$\begin{cases}(1)\,\forall(x_1,x_2)\in K_1\times K_1 & \psi(x_1\oplus x_2)=\psi(x_1)\oplus\psi(x_2)\\(2)\,\forall(a,x)\in A_\star\times K_1 & \psi(a\odot x)=a\odot\psi(x)\end{cases}$$

If A is a semifield then the set $\mathsf{Hom}_A(K_1,K_2)$ is itself a cone in the A-semimodule A^X.
If K is a subcone of $A_\star^X$ such that $\mathsf{c}_1\in K$ let

$$\mathcal{M}_A(K)=\{\psi\in\mathsf{Hom}_A(K,A_\star):\psi(\mathsf{c}_1)=1\}$$

Clearly, if $\psi\in\mathcal{M}_A(K)$ then, for all $a\in A_\star$, $\psi(\mathsf{c}_a)=a$.
To each $x\in X$ one can associate the map "*evaluation at x*", $\psi\to\delta_x(\psi)=\psi(x)$; clearly, $\delta_x\in\mathsf{Hom}_A(K,A_\star)$ and, $\delta_x(\mathsf{c}_1)=1$; this shows that $\mathcal{M}_A(K)$ is not empty.
If ψ_1 and ψ_2 are in $\mathcal{M}_A(K)$ then $\psi_1\oplus\psi_2$ is in $\mathcal{M}_A(K)$ and, if A is a semifield then, for all $a\in A_\star$ and all $\psi\in\mathcal{M}_A(K)$, $a\odot\psi$ is in $\mathcal{M}_A(K)$.
In other words, if A is a semifield then $\mathcal{M}_A(K)$ is a cone in the A-semimodule A^K, as a matter of facts a subcone of $A_\star^K$.
If one takes $A=\mathbb{R}\cup\{-\infty\}$ and if X is a compact topological space then $\mathcal{C}(X)$, the space on real valued continuous functions on X, is a maxplus-cone in $\big(\mathbb{R}\cup\{-\infty\}\big)^X$ then $\mathcal{M}_A\big(\mathcal{C}(X)\big)$ is the cone of **Maslov's measures on** X or **idempotent measures on** X , for its topological structure when X is a compact metric space see [3].
(4) Let us denote by $\mathcal{C}^\bullet_{\rm trop}(M)$ the set of nonempty convex sets of a tropical semimodule M over a tropical (commutative) semiring A; define on $\mathcal{C}^\bullet_{\rm trop}(M)$ the operations $\odot$ and $\oplus$ as follows: $\begin{cases}a\odot C=\{a\odot x:x\in C\}\\C_1\oplus C_2=\{u\oplus v:(u,v)\in C_1\times C_2\}.\end{cases}$

For $a\in A$ and $C_1,C_2\in\mathcal{C}^\bullet_{\rm trop}(M)$, $\begin{cases}C_1\oplus C_2\in\mathcal{C}^\bullet_{\rm trop}(M)\text{ and}\\a\odot(C_1\oplus C_2)=a\odot C_1\oplus a\odot C_2\end{cases}$

and, for $a,b\in A$ and $C\in\mathcal{C}^\bullet_{\rm trop}(M)$, $\begin{cases}a\odot C\in\mathcal{C}^\bullet_{\rm trop}(M)\\(a\odot b)\odot C=a\odot(b\odot C)\\1\odot C=C\text{ and }0\odot C=\{0\}\end{cases}$

That $(C_1\oplus C_2)\oplus C_3=C_1\oplus(C_2\oplus C_3)$ is also clear as well as the inclusion $(a\oplus b)\odot C\subset(a\odot C)\oplus(b\odot C)$.
If A is a semifield then $(a\odot C)\oplus(b\odot C)\subset(a\oplus b)\odot C$.
Indeed if $z=a\odot x\oplus b\odot y$ with $x,y\in C$ either $a\oplus b=0$ or $a\oplus b\neq 0$. In the first case $a=b=0$ and $(a\oplus b)\odot C=0\odot C=\{0\}$ while $(a\odot C)\oplus(b\odot C)=\{0\}\oplus\{0\}=\{0\}$. If $a\oplus b\neq 0$ let $c=(a\oplus b)^{\rm inv}$; then $(c\odot a)\odot x\oplus(c\odot b)\odot y\in C$ and $z=(a\oplus b)\big((c\odot a)\odot x\oplus(c\odot b)\odot y\big)\in(a\oplus b)\odot C$.
In conclusion: *if A is a semifield then $\mathcal{C}^\bullet_{trop}(M)$, endowed with elementwise multiplication by elements of A and elementwise addition is a tropical semimodule over A.*
(5) Let $\mathcal{P}\mathrm{oly}^\bullet_{\rm trop}(M)$ be the set of nonempty tropical polytopes of the A-semimodule M; it is a subset of $\mathcal{C}^\bullet_{\rm trop}(M)$ to which $\{0\}$ belongs. If $P=[\![S]\!]$ where S is finite nonempty subset of M then, for all $a\in A$, $a\odot P=[\![\{a\odot x:x\in S\}]\!]$ an therefore $a\odot P$ is a polytope. What might not be at first so clear is that $P_1\oplus P_2$ is a polytope if P_1 and P_2 are polytopes. We will show that this is the case if A is a totally ordered semifield and therefore, *if A is a totally ordered semifield then $\mathcal{P}oly^\bullet_{trop}(M)$ is a tropical semimodule over A as a matter of fact, a tropical subsemimodule of $\mathcal{C}^\bullet_{trop}(M)$.*
The set $T\subset M$ being finite and nonempty let us write $\bigoplus T$ for $\bigoplus_{x\in T}x$. Given two nonempty finite subsets S_1 and S_2 of M let

$$[\![S_1\mid S_2]\!]=[\![\,\textstyle\bigoplus T_1\oplus\bigoplus T_2:T_1\times T_2\subset S_1\times S_2\quad T_1\times T_2\neq\emptyset\}]\!] .$$

We have $[\![S_1\mid S_2]\!]\in\mathcal{P}\mathrm{oly}^\bullet_{\rm trop}(M)$ and $[\![S_1\mid S_2]\!]\subset[\![S_1]\!]\oplus[\![S_2]\!]$. Let us show by induction on the cardinality of S_1 that $[\![S_1]\!]\oplus[\![S_2]\!]\subset[\![S_1\mid S_2]\!]$. Let $S_1=\{x_1,\cdots,x_k\}$ and $S_2=\{y_1,\cdots,y_m\}$, $u=\oplus_{i=1}^k a_ix_i$, $v=\oplus_{j=1}^m b_jy_j$ and $\oplus_{i=1}^k a_i=\oplus_{j=1}^m b_j=1$.
If $k=1$ then $u\oplus v=x_1\oplus(\oplus_{j=1}^m b_j\odot y_j)=\oplus_{j=1}^m b_j\odot(x_1\oplus y_j)\in[\![S_1\mid S_2]\!]$.
Let $k=n+1$. By hypothesis A is totally ordered and $\oplus_{i=1}^{n+1}a_i=1$, one of the a_i is therefore equal to 1; without loss of generality one can assume that $a_1=1$. Let $u_1=\oplus_{i=1}^n a_ix_i$; from $\oplus_{i=1}^n a_i=1$ and the induction

hypothesis we have $u_1 \oplus v \in \llbracket S_1 \backslash \{x_{n+1}\} \mid S_2 \rrbracket \subset \llbracket S_1 \mid S_2 \rrbracket$.
Let us write $u_1 \oplus v = \oplus_{j=1}^{l} d_j \big(\bigoplus T_{1_j} \oplus \bigoplus T_{2_j} \big)$ where $\oplus_{j=1}^{l} d_j = 1$ and, for all j, T_{1_j} is a nonempty subset of $S_1 \backslash \{x_{n+1}\}$ and T_{2_j} is a nonempty subset of S_2 .
Then $u_1 \oplus v \oplus x_{n+1} = \oplus_{j=1}^{l} d_j \big(\bigoplus [T_{1_j} \oplus x_{n+1}] \oplus \bigoplus T_{2_j} \big)$ and therefore $u_1 \oplus v \oplus x_{n+1} \in \llbracket S_1 \mid S_2 \rrbracket$.
>From $1 \oplus a_{m+1} = 1$ we have $(u_1 \oplus v) \oplus [a_{m+1} \odot (u_1 \oplus v)] = u_1 \oplus v$ and from the convexity of $\llbracket S_1 \mid S_2 \rrbracket$ we have $(u_1 \oplus v) \oplus a_{m+1}(u_1 \oplus v \oplus x_{n+1}) \in \llbracket S_1 \mid S_2 \rrbracket$. But $(u_1 \oplus v) \oplus a_{m+1}(u_1 \oplus v \oplus x_{n+1}) = (u_1 \oplus v) \oplus a_{m+1}(u_1 \oplus v) \oplus a_{m+1}x_{m+1} = (u_1 \oplus v) \oplus a_{m+1}x_{m+1} = u \oplus v$.

For the topological structure of the space of nonempty compact maxplus convex subsets of a power of the real line see [4] and the references cited therein.

4. Convexity and topology

Given a set X a convexity $\mathcal{C}$ and a topology τ what kind of meaningful relationship can one impose between $\mathcal{C}$ and τ ? Connectedness of polytopes seems natural but, in itself, might not be sufficient to yield the fixed point property or other properties related to convexity in normed or locally convex topological vector spaces. Nonetheless, we will recall at the end of this section that connectedness is still good enough to capture some of the fundamental results of classical nonlinear analysis.

The standard n dimensional simplex, that is $\{(t_0, \cdots, t_n) \in [0,1]^{n+1} : \sum_{i=0}^{n} t_i = 1\}$ will be denoted, as usual, by Δ_n : the boundary $\partial\Delta_n$ of Δ_n is $\{(t_0, \cdots, t_n) \in \Delta_n : \prod_{i=0}^{n} t_i = 0\}$. The family of nonempty subsets of $\{0, \cdots, n\}$ is denoted by $\langle n \rangle$ and, for $J \in \langle n \rangle$, $\Delta_{n,J} = \{(t_0, \cdots, t_n) \in \Delta_n : \forall i \notin J\, t_i = 0\}$ and $\Delta_n^{-j} = \{(t_0, \cdots, t_n) \in \Delta_n : t_j = 0\}$.
Let us recall that a topological space Z is **homothopically trivial** if, for all $n \in \mathbb{N}$ and all continuous map $g : \partial\Delta_n \to Z$ there exists a continuous map $G : \Delta_n \to Z$ whose restriction to $\partial\Delta_n$ is g. A topological space Z is **contractible** if there exists a continuous map $h : [0,1] \times Z \to Z$ such that $h(0, -) : Z \to Z$ is constant and $h(1, -) : Z \to Z$ is the identity map; contractible spaces are homothopically trivial.

A **convex topological space** is a triple $(X, \tau, \mathcal{C})$ where τ is a topology on X and $\mathcal{C}$ is a convexity on X. Two convex topological spaces $(X_1, \tau_1, \mathcal{C}_1)$ and $(X_2, \tau_2, \mathcal{C}_2)$ are **homeomorphic convex topological spaces** if there exists an homeomorphism $\varphi : (X_1, \tau_1) \to (X_2, \tau_2)$ which is also an isomorphism of the convex spaces $(X_1, \mathcal{C}_1)$ and $(X_2, \mathcal{C}_2)$. For example, $x \mapsto \mathbf{ln}(x)$ is an homeomorphism from the $\mathbb{B}$-convex space $\mathbb{R}^n_{++}$ to the maxplus convex space $\mathbb{R}^n$; one could extend this homeomorphism from $\mathbb{R}^n_+$ to $\big(\mathbb{R} \cup \{-\infty\}\big)^n$ by taking as a neighbourhood base for $-\infty$ in $\mathbb{R} \cup \{-\infty\}$ the family of complements of closed half rays $[r, +\infty[$, $r \in \mathbb{R}$; which makes $\big(\mathbb{R} \cup \{-\infty\}\big)^n$ homeomorphic to $\mathbb{R}_+$.

It is not hard to see that $\mathbb{B}$-polytopes in $\mathbb{R}^n_+$ are compact and contractible, see [10], and obviously metrizable ; the same is therefore true of maxplus-polytopes in $\mathbb{R}^n$.

A family of topological results has played a fundamental role in nonlinear analysis and mathematical economics; these results pertain mainly to the existence of a fixed point (for a single valued or a multivalued map), to the existence of a continuous selection (or an approximate continuous selection) for a multivalued map or to the nonvacuity of the intersection of a given family of sets. Most of these results can be extended to topological convex spaces whose polytopes are homothopically trivial. One of these results is as easily stated as it is fundamental, in its classical version it can be seen as an avatar of Brouwer's fixed point theorem: it is the Knaster-Kuratowski-Mazurkiewicz theorem, also known as the **KKM Lemma**, whose standard form can be found in [15], [1], [2], [7] or [28] along with its various forms and applications. The classical "KKM Lemma" reads as follows:

(KKM) *If $\{F_0, \cdots, F_n\}$ is a family of closed sets of Δ_n such that, for all $J \in \langle n \rangle$, $\Delta_{nJ} \subset \cup_{j \in J} F_j$ then $\cap_{j=0}^{n} F_j \neq \emptyset$.*

Lassonde noticed that the KKM Lemma holds if, instead of being closed, all the F_i are open [25]; but see also [16] and [27]. We will still refer to the KKM Lemma for either a closed or an open covering as the "classical KKM".

4.1. **Some basic intersection theorems in topological convex spaces**

Given a topological convex space $\big(X,\tau,\mathcal{C}\big)$ let us say that a subset A of X is **finitely closed (resp. open)** if, for all polytope $P\in\mathcal{C}$, $A\cap P$ is closed (resp. open) in P with respect to the induced topology.

Proposition 4.1.1 (KKM)**.** *In a topological convex space $\big(X,\tau,\mathcal{C}\big)$ with homothopically trivial polytopes let $\{F_0,\cdots,F_n\}$ be a family of subsets of X all of which are finitely closed or all of which are finitely open. Then $\cap_{i=0}^n F_i\neq\emptyset$ if and only if there exists a family of points $\{x_0,\cdots,x_n\}$ of X, not necessarily distincts, such that, for all nonempty subset $J\subset\{0,\cdots,n\}$, $[\![\{x_j:j\in J\}]\!]\subset\cup_{j\in J}F_j$.*

Proof. Let $P=[\![\{x_0,\cdots,x_n\}]\!]$ and, for all $J\in\langle n\rangle$, let $P_J=[\![\{x_i:i\in J\}]\!]$.
As in [18], Theorem 1, one shows that there exists a continuous map $\theta:\Delta_n\to P$ such that,

$$\forall J\in\langle n\rangle,\ \theta\big(\Delta_{n,J}\big)\subset[\![\{x_i:i\in J\}]\!].$$

Let $M_j=\theta^{-1}(P\cap F_j)$; from $P\subset\cup_{j=0}^n F_j$ we have $\Delta_n=\cup_{j=0}^n M_j$ and also $\Delta_{n,J}\subset\cup_{j\in J}M_j$ and finally, from the classical KKM Lemma, $\cap_{j=0}^n M_j\neq\emptyset$. □

Proposition 4.1.2 (Alexandrov-Pasynkov)**.** *In a topological convex space $\big(X,\tau,\mathcal{C}\big)$ with homothopically trivial polytopes let $\{F_0,\cdots,F_n\}$ be a family of subsets of X all of which are finitely closed or all of which are finitely open. If there exists a family of points $\{x_0,\cdots,x_n\}$ of X, not necessarily distincts, such that, $[\![\{x_0,\cdots,x_n\}]\!]\subset\cup_{j=0}^n F_j$ and, for all $j\in\{0,\cdots,n\}$, $[\![\{x_i:i\neq j\}]\!]\subset F_j$ then $\cap_{i=0}^n F_i\neq\emptyset$.*

Proof. Let $P^{-j}=[\![\{x_i:i\neq j\}]\!]$; all the other symbols retain the meaning they had in the previous proposition. Since $P\subset\cup_{j=0}^n F_j$ we have $\Delta_n=\cup_{j=0}^n M_j$ and we also have, for all $j\in\{0,\cdots,n\}$, $\Delta_n^{-j}\subset M_j$ and finally, from the classical Alexandrov-Pasynkov Theorem, $\cap_{j=0}^n M_j\neq\emptyset$. □

Proposition 4.1.3 (Klee's Theorem)**.** *In a topological convex space $\big(X,\tau,\mathcal{C}\big)$ with homothopically trivial polytopes let $\{C_0,\cdots,C_n\}$ be a family of convex sets of X of which are finitely closed or all of which are finitely open. If, $\cup_{i=0}^n C_i$ is convex and, for all $j\in\{0,\cdots,n\}$, $\cap_{i\neq j}C_i\neq\emptyset$, then $\cap_{i=0}^n C_i\neq\emptyset$.*

Proof. For each $i\in\{0,\cdots,n\}$ pick a point $x_i\in\cap_{i\neq j}C_j$; we then have
$[\![\{x_0,\cdots,x_n\}]\!]\subset\cup_{j=0}^n C_j$ and, for all $j\in\{0,\cdots,n\}$, $\{x_i:i\neq j\}\subset C_j$. From the Alexandrov-Pasynkov Theorem, $\cap_{i=0}^n M_i\neq\emptyset$ and consequently, $\cap_{i=0}^n C_i\neq\emptyset$. □

Klee's Theorem [24] for the usual linear convexity was proved independently by Berge in 1959, [5] or [6], using the Hahn-Banach Theorem.

Let us say that a subset S of a convex topological space $\big(X,\mathcal{C}\big)$ is **starshaped** if there exists a point $x_0\in S$ such that, for all $x\in S$, $[\![x_0,x]\!]\subset S$; in which case we say that S is starshaped at x_0. A convex set is starshaped. The next result is a topological version of Breen's Theorem [11].

Proposition 4.1.4 (Breen's Theorem)**.** *Let $\big(X,\tau,\mathcal{C}\big)$ be a convex topological space in which starshaped subsets of polytopes are homothopically trivial. Let $\{U_0,\cdots,U_n\}$ be a family of subsets of X all of which are finitely closed or all of which are finitely open. If, for all nonempty subsets $J\subset\{0,\cdots,n\}$ the set $\cup_{j\in J}U_j$ is starshaped then $\cap_{j=0}^n U_j\neq\emptyset$.*

Proof. For each $J\in\langle n\rangle$ let $F_J=\cup_{j\in J}U_j$; choose $x_J\in F_J$ such that F_J is starshaped at x_J. Let $P=\big[\!\big[\big\{x_J: J\in\langle n\rangle\big\}\big]\!\big]$ and $S_J=P\cap F_J$; notice that S_J is starshaped at x_J. Given $J_1,\cdots,J_l\in\langle n\rangle$ let $J=J_1\cup\cdots\cup J_l$ and notice that $S_{J_1}\cup\cdots\cup S_{J_1}=S_J$. The family $\big\{S_J:J\in\langle n\rangle\big\}$ is made of starshaped subsets of the polytopes P and either all of its members are closed in P or all of its members are open in P; furthermore, by hypothesis,

all of the sets S_J are homothopically trivial subsets of P. Enumerate $\langle n\rangle$ arbitrarily, $\langle n\rangle = \big\{J_0, \cdots, J_m\big\}$; since, for all $\{J_{i_1}, \cdots, J_{i_k}\}$, $\cup_{j=1}^k S_{J_{i_j}} = S_{J_l}$ with $J_l = \cup_{j=1}^k J_{i_j}$ we see that $\cup_{j=1}^k S_{J_{i_j}}$ is a starshaped subset of P.
For all $\lambda \in \{0, \cdots, m\}$, let $T_\lambda = S_{J_\lambda} = \cup_{j\in J_\lambda} S_j$; T_λ is an homothopically trivial subset of P and the family $\big\{T_\lambda : \lambda \in \{0, \cdots, m\}\big\}$ is union closed.
For all $\Lambda \in \langle m\rangle$ let $G_\Lambda = \cup_{\lambda\in\Lambda} T_\lambda$. If $\Lambda \subset \Lambda'$ then $G_\Lambda \subset G_{\Lambda'}$ and these sets are homothopically trivial subsets of P. There exists a continuous map $\theta : \Delta_m \to P$ such that, for all $\Lambda \in \langle m\rangle$, $\theta\big(\Delta_{m,\Lambda}\big) \subset G_\Lambda$. Let $M_\lambda = \theta^{-1}(T_\lambda)$; notice that $\Delta_{m,\Lambda} \subset \cup_{\lambda\in\Lambda} M_\lambda$ and that, either all the sets M_λ are closed in Δ_m or all of them are open in Δ_m. By the standard KKM Theorem, $\cap_{\lambda=0}^m M_\lambda \neq \emptyset$ and therefore $\cap_{\lambda=0}^m T_\lambda \neq \emptyset$ that is $\cap_{J\in\langle n\rangle} S_J \neq \emptyset$ and *a fortiori*, $\cap_{J\in\langle n\rangle} F_J \neq \emptyset$ from which $\cap_{i=0}^n U_i \neq \emptyset$. □

Breen's original result is on the one hand more precise than Proposition 4.1.4 and on the other hand less general. She considered a family $\mathcal{F}$ of nonempty compact starshaped sets in $\mathbb{R}^d$ and proved that $\cap\mathcal{F} \neq \emptyset$ if the union of every subfamily of $\mathcal{F}$ of $d+1$ or fewer members of $\mathcal{F}$ has starshaped union. On the other hand, Proposition 4.1.4 holds in arbitrary topological vector spaces. For a topological proof of Breen's original theorem see [22].

One can show that *tropical starshaped sets in $\mathbb{R}^d$, that is maxplus starshaped sets, are homothopically trivial, as well as $\mathbb{B}$-convex (maxtimes) starshaped sets in $\mathbb{R}_+^d$.* We leave aside for now the question of a completely analoguous "tropical Breen's Theorem" in $\mathbb{R}^d$ or in $\mathbb{R}_+^d$.

There is another Proposition whose proof is somewhat similar to that of Proposition 4.1.4; to keep this section at a reasonable length the proof will not be given here.

Proposition 4.1.5. *Let $\big(X, \tau, \mathcal{C}\big)$ be a convex topological space in which starshaped subsets of polytopes are homothopically trivial. Let $\{U_0, \cdots, U_n\}$ be a family of subsets of X all of which are finitely closed or all of which are finitely open. If, for all nonempty subsets $J \subset \{0, \cdots, n\}$ of cardinality at most n the set $\cap_{j\in J} U_j$ is starshaped and if $\cup_{j=0}^n U_j$ is starshaped then $\cap_{j=0}^n U_j \neq \emptyset$.*

4.2. Fixed points theorems

The results of this section, given without proofs, rely on the fact that maxplus polytopes in $\mathbb{R}^n$ and $\mathbb{B}$-polytopes in $\mathbb{R}_+^n$ are contractible, and therefore homothopically trivial.

The first result is a tropical version of the Fan-Browder Theorem, page 143 in [15].

Proposition 4.2.1. *Let C be either a compact maxplus convex set in $\mathbb{R}^n$ or a compact $\mathbb{B}$-convex set in $\mathbb{R}_+^n$ and let $T : C \to C$ be a multivalued map with closed nonempty maxplus (respectively, $\mathbb{B}$) convex values. If, for all $y \in C$, the set $T^{-1}y = \{x \in C : y \in Tx\}$ is open in C then there exists $\bar{x} \in C$ such that $\bar{x} \in T\bar{x}$.*

Proof. It is a consequence of Proposition 4.1.1; see [19] Theorem 3 for the details. □

An arbitrary product $\prod_{i=1}^n [a_i, b_i]$ of real intervals is a $\mathbb{B}$-convex set in $\mathbb{R}_+^n$ consequently, an arbitrary nonempty $\mathbb{B}$-convex set in $\mathbb{R}_+^n$ has a neighbourhood base consisting of $\mathbb{B}$-convex sets ; $\mathbb{B}$-convex set are obviously metrizable. By the corollary on page 259 of [19], arbitrary nonempty $\mathbb{B}$-convex set in $\mathbb{R}_+^n$ are absolute retracts. The same is true of arbitrary nonempty maxplus convex sets in $\mathbb{R}^n$. Since we are here in finite dimention we could also reach the same conclusion from the fact that $\mathbb{B}$-convex sets are contractible and locally contractible.

Proposition 4.2.2. *Let C be either a maxplus convex set in $\mathbb{R}^n$ or a $\mathbb{B}$-convex set in $\mathbb{R}_+^n$ and $f : C \to C$ a continuous map whose image is contained in a compact subset of C. Then f has a fixed point.*

Proof. From the Generalized Schauder Theorem, page 165 in [15]. □

We close this section with Kakutani's Fixed Theorem in maxplus or $\mathbb{B}$-convexity. First, in $\mathbb{R}_+^n$ open balls with respect to the $\|\cdot\|_\infty$ norm are $\mathbb{B}$-convex sets and, for all $\mathbb{B}$-convex subset C of $\mathbb{R}_+^n$ and all $r > 0$ the

r-neighbourhood of C with respect to $\|\cdot\|_\infty$ are $\mathbb{B}$-convex; in the language of [19], $\mathbb{B}$-convex sets are l.c. spaces. The same is true of maxplus convex sets in $\mathbb{R}^n$.

Proposition 4.2.3. *Let C be either a compact maxplus convex set in $\mathbb{R}^n$ or a compact $\mathbb{B}$-convex set in $\mathbb{R}^n_+$ and $T : C \to C$ an an upper semicontinuous multivalued map with nonempty closed maxplus (respectively, $\mathbb{B}$) convex values. Then, there exists $\bar{x} \in C$ such that $\bar{x} \in T\bar{x}$.*

Proof. Page 262 of [19]. □

4.3. **Back to the begining**

The two previous sections lead naturally to the following question: *when does a convex topological space have homothopically trivial polytopes ?* As we have already observed, maxplus polytopes in $\mathbb{R}^n$, or $\mathbb{B}$-polytopes in $\mathbb{R}^n_+$ are homothopically trivial, as well as usual linear polytopes, and all of this can easily be shown by hand without appealing to any general theory. Let us nonetheless reformulate our question as follows: *what kind of "compatibility " properties should link the convexity and the topology of a given convex topological space $\big(X, \tau, \mathcal{C}\big)$ in order to have homothopically trivial polytopes ?* Of course, we would like the standard tropical convexities, maxplus in $\mathbb{R}^n$ or maxtimes in $\mathbb{R}^n_+$, to have those properties. Two answers will be given, one geometric, in terms of the existence of a compatible geodesic structure, and one topological. For simplicity, we will restrict our attention to metric spaces eventhough it is not necessary, and even not recommanded, think of convex sets in locally convex topological vector spaces. In everything that follows (X, d) is a metric space and $\mathcal{C}$ is a convexity on X.

4.3.1. *Uniform geodesic convexities*

Let us say that $(X, d, \mathcal{C})$ is a **uniform geodesic space** if $\mathcal{C}$ is an interval convexity and if there exists a continuous function $\lambda : X \times X \times [0, 1] \to X$ such that $\forall (x_0, x_1, t, s)$
$$\begin{cases} (1) \quad \lambda(x_0, x_1, 0) = x_0 \text{ and } \lambda(x_0, x_1, 1) = x_1 \\ (2) \quad d\big(\lambda(x_0, x_1, t), \lambda(x_0, x_1, s)\big) = |t - s| d(x_0, x_1) \\ (3) \quad \lambda\Big(x_0, x_1, [0, 1]\Big) = [\![x_0, x_1]\!] \end{cases}$$
If $(X, d, \mathcal{C})$ is a uniform geodesic space then all nonempty convex subsets of X are, with respect to the induced convexity and the induced metric, uniform geodesic spaces. These are indeed very strong conditions, and stronger than they need to be, but as they stand they hold for maxplus convexity in $\mathbb{R}^n$ and $\mathbb{B}$-convexity in $\mathbb{R}^n_{++}$. In the first case, the metric is given $d(x, y) = \|x - x \vee y\|_\infty + \|y - x \vee y\|_\infty$, one can check that it is the so called *Hilbert affine distance* (Stephane Gaubert showed to the author that maxplus segments in $\mathbb{R}^n$ are geodesics for the Hilbert affine distance) that is, for $x, y \in \mathbb{R}^n$, $d(x, y) = \max_i\{0, x_i - y_i\} + \max_i\{0, y_i - x_i\}$.
Conditions (1) and (3) imply together that starshaped sets are homothopically trivial, as a matter of fact they are contractible.
In a uniform geodesic space one can naturally define the **midpoint function** $\mu : X \times X \to X$ by $\mu(x_0, x_1) = \lambda(x_0, x_1, 1/2)$. From (2) we have $d(x_0, \mu(x_0, x_1)) = d(x_1, \mu(x_0, x_1)) = 1/2 d(x_0, x_1)$.

Proposition 4.3.1. *In a uniform geodesic space $(X, d, \mathcal{C})$ a closed set $C \subset X$ is convex if and only if it contains the midpoint of any two of its points, that is $\mu\big(C \times C\big) \subset C$.*

Proof. The nontrivial part of the proof is done by induction on the set of dyadic numbers in the real interval $[0, 1]$. □

One can see without too much difficulty that, for all maxplus convex segments $[\![x, y]\!] \subset \mathbb{R}^n$, $[\![x, y]\!] = [\![x, x \oplus y]\!] \cup [\![x \oplus y, y]\!]$ and therefore, a subset C of $\mathbb{R}^n$ is maxplus convex if and only if
$$\begin{cases} (1) \quad \forall x, y \in C \ x \oplus y \in C \ (C \text{ is a subsemilattice of } \mathbb{R}^n) \\ (2) \quad \forall x, y \in C \ x \leq y \Rightarrow [\![x, y]\!] \subset C. \end{cases}$$

From this remark and Proposition 4.3.1 one gets the following caracterization of closed maxplus convex subsets of $\mathbb{R}^n$.

Proposition 4.3.2. *A closed subset C of $\mathbb{R}^n$ is maxplus convex if and only if* $\begin{cases} (1) & C \text{ is a subsemilattice of } \mathbb{R}^n \\ (2) & \forall x, y \in C\ x \le y \Rightarrow \mu_{max}(x,y) \in C. \end{cases}$ *where $(x,y) \mapsto \mu_{max}(x,y)$ is the midpoint function on $\mathbb{R}^n$ associated to the Hilbert affine metric.*

One can show that, if $x \leqslant y$ then $\mu_{max}(x,y)_i = \max\left\{x_i, y_i - \dfrac{\max_{1\le j\le n}\{y_j - x_j\}}{2}\right\}$.

4.3.2. *Uniform convexities*

We restrict the discussion to metric spaces, but uniform topological spaces would do as well. Let $\mathcal{C}$ be a convexity on a metric space (X,d) such that polytopes are compact. The convexity is completely specified by the convex hull operator restricted to the set of nonempty finite subsets of X; let $\langle X\rangle$ be that space and let $\mathcal{K}(X)$ be the space of nonempty compact subsets of X. Both $\langle X\rangle$ and $\mathcal{K}(X)$ are metric spaces with respect to the Hausdorff metric associated to the metric d on X. Let us say that $\big(X,d,\mathcal{C}\big)$ is a **uniform convex space** if the convex hull operator $\{x_0,\cdots,x_n\} \mapsto [x_0,\cdots,x_n]$ is uniformly continuous from $\langle X\rangle$ to $\mathcal{K}(X)$. One can show that $\mathbb{B}$-convexity on $\mathbb{R}^n_+$ is a uniform convexity; the same is true of maxplus convexity on $\mathbb{R}^n$.

Proposition 4.3.3. *Let $\big(X,d,\mathcal{C}\big)$ be a convex space with compact polytopes. If $\big(X,d,\mathcal{C}\big)$ is a uniform convex space, if polytopes are connected and if the Kakutani Separation Property holds then convex sets are homothopically trivial.*

The proof is somewhat too involved to be presented here; it can be found in [20], for uniform topological spaces, or in [30] for metric spaces. All of the classical selection, extension and fixed point theorems for locally convex and normed spaces, as well as some topological properties, like being an absolute retract, hold in the context of uniform convexities. It has been said at the begining of this discussion that a convexity has the Kakutani Separation Property if and only if arbitrary polytopes have it; therefore, all the results alluded to follow from properties of polytopes ("finite dimensional properties") and one global property: uniform continuity of the convex hull operator.

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