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ON HYPERSPACES OF MAX-PLUS AND MAX-MIN CONVEX SETS

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Abstract. The paper is devoted to some new results concerning the topology of hyperspaces of max-plus convex subsets in Euclidean spaces and some other spaces.

Résumé. L'article est dévoué à de nouveaux résultats concernant la topologie de hyperespaces de sous-ensembles max-plus convexes dans des espaces euclidiens et d'autres espaces.

Introduction

Nadler, Quinn, and Stavrakas [14] proved that the hyperspace of compact convex subsets of the Euclidean space $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to the space $Q \setminus \{*\}$, where Q is the Hilbert cube. This result as well as another results from [14] opened the door to applications of methods of infinite-dimensional topology to convexity.

In the recent decade, there were widely investigated the so called max-plus and max-min convex sets. They are counterparts of the convex sets in the idempotent mathematics, i.e. a part of mathematics in which (some of) the usual operations in the set of reals is replaced by some idempotent operations; see, e.g., [13]. Motivations of consideration of max-plus convex sets as well as survey of results in max-plus convexity can be found in [10].

Some recent results are devoted to the hyperspaces of the max-plus and max-min convex sets. In particular, in [2,3], results analogous to the mentioned theorem of Nadler, Quinn, and Stavrakas were obtained. The main result of [4] stated that the hyperspace of max-plus convex subsets in the spaces $\mathbb{R}^\tau$ is homeomorphic to $\mathbb{R}^\tau$ if $\tau \in \{\omega, \omega_1\}$.

The present note is also devoted to the hyperspaces of max-plus convex sets. Some of the obtained results can be extended over the case of max-min convex sets.

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1. Preliminaries

All maps of topological spaces are assumed to be continuous. Recall that a map is open if the image of every open set is open. A map is proper if the preimages of compact subsets are compact.

By I we denote the segment $[0, 1]$. By $\bar{A}$ we denote the closure of a set A in a topological space.

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1.1. **Max-plus and max-min convexity**

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ and let τ be a cardinal number. Given $x, y \in \mathbb{R}^\tau$ and $\lambda \in \mathbb{R}_{\max}$, we denote by $x \oplus y$ the coordinatewise maximum of x and y and by $\lambda \odot x$ the vector obtained from x by adding λ to every its coordinate. Given $\lambda \in \mathbb{R}_{\max} \cup \{\infty\}$ and $x = (x_\alpha) \in \mathbb{R}^\tau$, we define $\lambda \otimes x = (\min\{\lambda, x_\alpha\})$.

A subset A in $\mathbb{R}^\tau$ is said to be *max-plus convex* if $\alpha \odot a \oplus \beta \odot b \in A$ for all $a, b \in A$ and $\alpha, \beta \in \mathbb{R}_{\max}$ with $\alpha \oplus \beta = 0$. A subset A in $\mathbb{R}^\tau$ is said to be *max-min convex* if $\alpha \otimes a \oplus b \in A$ for all $a, b \in A$ and $\alpha \in \mathbb{R}_{\max}$.

The minimal max-plus (respectively max-min) convex set containing a subset A in $\mathbb{R}^n$ is called the *max-plus convex hull* (respectively *max-min convex hull*)of A and is denoted by $h_{\mathrm{mp}}(A)$ (respectively $h_{\mathrm{mm}}(A)$).

1.2. **Hyperspaces**

Given a topological space X, by $\exp X$ we denote the space of all nonempty compact subsets in X. The sets of the form

$$\langle U_1, \dots, U_n \rangle = \{A \in \exp X \mid A \subset \cup_{i=1}^n U_i,\ A \cap U_i \neq \emptyset \text{ for all } i\},$$

where $U_1, \dots, U_n$ are open sets in X, comprise a base of the Vietoris topology in $\exp X$. If (X, d) is a metric space, then the Vietoris topology on $\exp X$ is generated by the Hausdorff metric d_H:

$$d_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\},$$

where $A, B \in \exp X$.

Let $\exp_n X$ denote the subspace of $\exp X$ consisting of all sets of cardinality $\leq n$.

By $\mathrm{mpcc}(X)$ we denote the hyperspace of nonempty max-plus convex compact subsets in $X \subset \mathbb{R}^n$. Given $n \in \mathbb{N}$, let $\mathrm{mpcc}_n(X)$ denote the subspace of $\mathrm{mpcc}(X)$ consisting of convex hulls of the sets of cardinality $\leq n$.

1.3. **Absolute retracts and some infinite-dimensional spaces**

A metrizable space X is an absolute (neighborhood) retract (briefly: AR (ANR)) if it is a retract of (an open subset of) any metrizable space containing X as a closed subset. The convex subsets of normed spaces are examples of AR-spaces. See, e.g., [5] for the backgrounds of the theory of retracts.

The notion of A(N)R can be also defined in another classes of topological spaces, in particular, in the class of compact Hausdorff spaces (see [15]).

In the sequel, we will need the following notion, which was introduced in [11]: a *c*-structure on a topological space X is an assignment, to every nonempty finite subset A of X, of a contractible subspace $F(A)$ of X, such that $F(A) \subset F(A')$ whenever $A \subset A'$. A pair (X, F), where F is a *c*-structure on X, is called a *c*-space. A subset E of X is called an F-set if $F(A) \subset E$ for any finite $A \subset E$. A metric space (X, d) is said to be a metric *l.c.*-space if all the open balls in it are F-sets and all open r-neighborhoods of F-sets are also F-sets.

A compact metric space X is said to satisfy the Disjoint Disc Property (DDP) if every map $f\colon I^n \to X$ can be approximated in the sense of uniform convergence by maps with disjoint images. The following characterization theorem for the Hilbert cube $Q = I^\omega$ is proved by H. Toruńczyk [16]: a compact metrizable AR-space X is homeomorphic to Q if and only if X satisfies DDP.

A metrizable space X is said to be a Hilbert manifold if X is locally homeomorphic to the Hilbert space ℓ^2.

Let $\mathcal{U}$ be an open cover of a topological space X. We say that a map $f\colon Y \to X$ is a $\mathcal{U}$-domination if f is a proper map and there is a map $g\colon X \to Y$ such that fg is $\mathcal{U}$-homotopic to 1_X (the latter means that there exists a homotopy $H\colon X \times [0,1] \to X$ of the maps fg and 1_X such that, for every $x \in X$, there is $U \in \mathcal{U}$ with $H(\{x\} \times [0,1]) \subset U$). It is known that every proper retraction onto X is a $\mathcal{U}$-domination, for every open cover $\mathcal{U}$ of X.

The following result is proved by H. Toruńczyk [17, Theorem 4.1].

Theorem 1.1. *Let X be a complete ANR. If, for every open cover $\mathcal{U}$ of X, the space X is $\mathcal{U}$-dominated by a Hilbert manifold, then X is a Hilbert manifold itself.*

By $\mathbb{R}^\infty$ we denote the direct limit of the sequence

$$\mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \dots$$

By Q^∞ we denote the direct limit of the sequence

$$Q \to Q \times \{0\} \hookrightarrow Q \times Q \to Q \times Q \times \{0\} \hookrightarrow Q \times Q \times Q \dots$$

A closed subset A of a space X is called a Z-set in X, if the identity map of X can be approximated (in the topology of uniform convergence) by maps whose image misses A.

2. Hyperspaces of max-plus convex polyhedra

The cone cone(X) of a topological space X is the quotient space $(X \times [0,1])/(X \times \{0\})$. As usual, $\mathbb{R}P^n$ stands for the n-dimensional real projective space.

Proposition 2.1. *The set* $\mathrm{mpcc}_2(I^n)$ *is homeomorphic to* $\mathrm{cone}(\mathbb{R}P^n) \times I^{n-1}$.

Proof. Let $\alpha\colon \exp_2 I^n \to \mathrm{mpcc}_2(I^n)$ denote the map that assigns to every A its max-plus convex hull. Then, clearly, α is a homeomorphism. The result follows from [18]. □

Proposition 2.2. *Let X be a compact max-plus convex subspace in a Euclidean space $\mathbb{R}^k$. Then the space $\mathrm{mpcc}_n(X)$ is an AR-space, for any $n \in \mathbb{N}$.*

Proof. We proceed by induction. Clearly, $\mathrm{mpcc}_n(X)$ is homeomorphic to X and therefore is an AR-space. Assume that we have already shown that $\mathrm{mpcc}_{n-1}(X)$ is an AR-space. Without loss of generality, one may assume that X is a max-plus convex subset in a cube J^k, where J is a closed segment in $\mathbb{R}$. Let $\alpha\colon (J^k)^n \to \mathrm{mpcc}_n(J^k)$ be the map defined by the formula

$$\alpha(x_1, \dots, x_n) = h_{\mathrm{mp}}(\{x_1, \dots, x_n\}).$$

Note that the subspace $\alpha^{-1}(\mathrm{mpcc}_{n-1}(X))$ of $(J^k)^n$ is determined by finitely many linear inequalities and therefore is an ANR-space. By [5], the adjunction space

$$Y = (J^k)^n \cup_{\alpha|\alpha^{-1}(\mathrm{mpcc}_{n-1}(X))} \alpha^{-1}(\mathrm{mpcc}_{n-1}(X))$$

is an ANR-space.

The nth symmetric group S_n acts naturally on the space Y and the orbit space of this action is exactly the space $\mathrm{mpcc}_n(X)$. The result the follows from Floyd's theorem on the orbit spaces of ANR-spaces [9] (see also [1]). □

Let $p\colon I^m \to I^n$ denote the projection map, where $m \geq n$. This map induces the map of the hyperspaces $A \mapsto p(A)\colon \mathrm{mpcc}_i(I^m) \to \mathrm{mpcc}_i(I^n)$; we denote it by p_i^*.

Proposition 2.3. *Let $m > n \geq 2$. Then the map p_i^* is open if and only if $i \in \{1, 2\}$.*

Proof. If $i = 1$, then there is nothing to prove.

The map p_2^* is open being homeomorphic to the map $A \mapsto p(A)\colon \exp_2 I^m \to \exp_2 I^n$ (see the proof of Proposition 2.1). The latter map is known to be open (see, e.g., [15]).

Now suppose that $i \geq 3$. Let $A \in \mathrm{mpcc}_{i-1}(I^n) \setminus A \in \mathrm{mpcc}_{i-2}(I^n)$. Suppose that A be the max-plus convex hull of a set $\{a_1, \dots, a_{i-1}\}$. Without loss of generality, one may assume that there exists $B \in \mathrm{mpcc}_i(I^m) \setminus \mathrm{mpcc}_{i-1}(I^m)$ with the property that $B \cap f^{-1}(a_1)$ is not a singleton.

One can also suppose that there is a sequence (A_j) in $\mathrm{mpcc}_i(I^n) \setminus \mathrm{mpcc}_{i-1}(I^n)$ such that A_j is the max-plus convex hull of the set $A \cup \{b_j\}$, where (b_j) is a sequence in I^n converging to a_2.

Then, clearly, there is no sequence (B_j) in $\mathrm{mpcc}_i(I^m)$ such that $\lim_{j\to\infty} B_j = A_j$ and $p_i^*(B_j) = A_j$ for every j. In turn, this implies that the map p_i^* is not open.

□

Counterparts of the results of this section can be proved also for the max-min convexity.

3. Max-plus convex cone

A map $f\colon X \to Y$ of topological spaces is called soft [15] if, for every paracompact space Z, every its closed subset A, and every maps $\varphi\colon A \to X$, $\psi\colon Z \to Y$ with $\psi|A = \varphi$, there exists a map $\Phi\colon Z \to Y$ such that $f\Phi = \psi$ and $\Phi|A = \varphi$.

We define the max-convex cone of the cube I^τ as follows:

$$\mathrm{cone}_{\mathrm{mp}}(I^\tau) = \{((x_\alpha), t) \mid t \in [0,1],\ x_\alpha \geq t \text{ for all } \alpha\} \subset I^\tau \times I = I^{\tau+1}.$$

Clearly, $\mathrm{cone}_{\mathrm{mp}}(I^\tau)$ is a max-plus convex subset in $I^{\tau+1}$.

Proposition 3.1. *The hyperspace* $\mathrm{mpcc}(\mathrm{cone}_{\mathrm{mp}}(I^\omega))$ *is homeomorphic to* I^ω.

Proof. We represent $\mathrm{cone}_{\mathrm{mp}}(I^\omega)$ as inverse limit $\varprojlim(\mathrm{cone}_{\mathrm{mp}}(I^m), \pi_m)$, where the bonding maps π_m are defined by the formula

$$\pi_m((x_1, \dots, x_{m+1}), t) = ((x_1, \dots, x_m), t).$$

Then

$$\mathrm{mpcc}(\mathrm{cone}_{\mathrm{mp}}(I^\omega)) = \varprojlim(\mathrm{mpcc}(\mathrm{cone}_{\mathrm{mp}}(I^m)), \pi_m^*),$$

where π_m^* is the map defined as $A \mapsto \pi_m(A)$.

Similarly as in [2] one can show that the maps π_m^* are open. Then, by applying selection theorem from [19] we conclude that these maps are soft and therefore the limit space $\mathrm{mpcc}(\mathrm{cone}_{\mathrm{mp}}(I^\omega))$ is an absolute retract (see [15]).

Now, we are going to establish the DDP for the space $\mathrm{mpcc}(\mathrm{cone}_{\mathrm{mp}}(I^\omega))$. Since this space is an AR-space, it is enough to prove that its identity map cam be approximated in the sense of uniform convergence by maps with disjoint images.

Let $\xi \in (0,1)$. Denote by $f_{m,\xi}, g_{m,\xi}\colon \mathrm{cone}_{\mathrm{mp}}(I^\omega) \to \mathrm{cone}_{\mathrm{mp}}(I^\omega)$ the maps acting by

$$f_{m,\xi}((x_i)_{i=1}^\infty, t) = (x_1, \dots, x_m, 0, x_{m=2}, x_{m+3}, \dots), \xi t),\ \ g_{m,\xi}((x_i)_{i=1}^\infty, t) = (x_1, \dots, x_m, \xi, x_{m=2}, x_{m+3}, \dots), \xi t).$$

The maps $f_{m,\xi}$ and $g_{m,\xi}$ induce the maps $f_{m,\xi}^*$ and $g_{m,\xi}^*$ of the hyperspaces of the max-plus convex sets. Clearly, the images of the maps $f_{m,\xi}^*$ and $g_{m,\xi}^*$ are disjoint and, if ξ is close enough to 1 and m is large enough then $f_{m,\xi}^*$ and $g_{m,\xi}^*$ are close enough to the identity map of $\mathrm{mpcc}(\mathrm{cone}_{\mathrm{mp}}(I^\omega))$.

The result now follows from Toruńczyk's characterization theorem for the Hilbert cube Q.

□

Remark that the notion of the max-plus convex cone can be defined also for all compact max-plus convex sets.

4. Function spaces

The notion of max-plus convex sets can be naturally extended to some functional spaces. We consider here a very special case of spaces of continuous functions on compact metric spaces.

Let X be a compact metric space. By $C(X)$ we denote the Banach space of continuous functions endowed with the sup-norm.

A subset $A \subset C(X)$ is said to be *max-plus convex* if $t \odot x \oplus y \in A$, for every $x, y \in A$ and $t \in [-\infty, 0]$. By $\mathrm{mpcc}(C(X))$, we denote the hyperspace of the max-plus convex sets in $C(X)$.

Given a subset $A \subset C(X)$, we denote by $h_{\mathrm{mp}}(A)$ the max-plus convex hull of A, i.e. the minimal max-plus convex set containing A.

Lemma 4.1. *The max-plus convex hull of a finite set in $C(X)$ is compact.*

Proof. Let $A \subset C(X)$ be finite and let $c = -\max\{\|x\| \mid x \in A\}$. It is easy to see that

$$h_{\mathrm{mp}}(A) = \oplus\{t_x \odot x \mid x \in A,\ t_x \in [c, 0]\}.$$

Therefore, $h_{\mathrm{mp}}(A)$ is the image of the compact set $[c, 0]^A$ under the map

$$(t_x)_{x\in A} \mapsto \oplus\{t_x \odot x \mid x \in A\},$$

i.e. is compact. □

The proof of the following statement is immediate.

Lemma 4.2. *Let $x_i, y_i \in A$, $i = 1, \ldots, n$. If $\|x_i - y_i\| < \varepsilon$, for every $i = 1, \ldots, n$, then*

$$\|(\oplus_{i=1}^n t_i \odot x_i) - (\oplus_{i=1}^n t_i \odot y_i)\| < \varepsilon,$$

for any $t_1, \ldots, t_n \in \mathbb{R}$.

Lemma 4.3. *For every precompact set $A \subset C(X)$, the set $h_{\mathrm{mp}}(A)$ is also precompact.*

Proof. Given $\varepsilon > 0$, find a finite $(\varepsilon/2)$-net B in A. Then $h_{\mathrm{mp}}(B)$ is compact and there is a finite $(\varepsilon/2)$-net B in $h_{\mathrm{mp}}(B)$.

Now, let $x \in h_{\mathrm{mp}}(A)$. Write $x = \oplus_{i=1}^k \alpha_i \odot a_i$, where $\alpha_i \leq 0$, $a_i \in A_i$, $i = 1, \ldots, k$, $\oplus_{i=1}^k \alpha_i = 0$. For every i, one can find $b_i \in B$ such that $\|a_i - b_i\| < \varepsilon/2$. Let $y \oplus_{i=1}^k \alpha_i \odot b_i$, then by Lemma 4.2, $\|x - y\| < \varepsilon/2$. There exists $c \in C$ such that $\|c - y\| < \varepsilon/2$. Finally, $\|x - c\| < \varepsilon$. □

Lemma 4.4. *The map*

$$A \mapsto \overline{h_{\mathrm{mp}}(A)} \colon \exp(C(X)) \to \mathrm{mpcc}(C(X))$$

is nonexpanding.

Proof. Suppose that $d_H(A, B) < r$, where $A, B \in \exp(C(X))$ and $r > 0$.

Let $\varepsilon > 0$. Let $x \in \overline{h_{\mathrm{mp}}(A)}$, then there exists $y \in h_{\mathrm{mp}}(A)$ such that $\|x - y\| < \varepsilon$. Write $y = \oplus_{i=1}^k \alpha_i \odot a_i$, where $\alpha_i \leq 0$, $a_i \in A_i$, $i = 1, \ldots, k$, $\oplus_{i=1}^k \alpha_i = 0$. For every i, find $b_i \in B$ such that $\|a_i - b_i\| < r$. Then $z = \oplus_{i=1}^k \alpha_i \odot b_i \in \overline{h_{\mathrm{mp}}(B)}$ and $\|x - z\| < r + \varepsilon$.

One can similarly prove that, for any $x \in \overline{h_{\mathrm{mp}}(B)}$, there exists $z \in \overline{h_{\mathrm{mp}}(A)}$ such that $\|x - z\| < r + \varepsilon$. Therefore, $d_H(\overline{h_{\mathrm{mp}}(A)}, \overline{h_{\mathrm{mp}}(B)}) < r + \varepsilon$. Since $\varepsilon > 0$, we are done. □

Proposition 4.5. *Let X be a compact metrizable space. Then the space $\mathrm{mpcc}(C(X))$ is an AR-space.*

Proof. One can proceed similarly as in the proof of Lemma 3.1 from [4]. Define a c-structure on the space $\mathrm{mpcc}(C(X))$ as follows: given any $A_1, \ldots, A_n \in \mathrm{mpcc}(C(X))$, let

$$F(\{A_1, \ldots, A_n\}) = \left\{ \bigoplus_{i=1}^n \alpha_i \odot A_i \mid \alpha_1, \ldots, \alpha_n \in [-\infty, 0], \bigoplus_{i=1}^n \alpha_i = 0 \right\}.$$

The verification of the conditions from the definition of c-structure is straightforward.

We are going to show that every set of the form $F(\{A_1, \ldots, A_n\})$ is contractible. Let $A = \oplus_{i=1}^n A_i$. Then $A \in F(\{A_1, \ldots, A_n\})$. Define a map

$$H \colon F(\{A_1, \ldots, A_n\}) \times [0, 1] \to F(\{A_1, \ldots, A_n\})$$

by the formula:

$$H(C,t) = C \oplus (\ln t) \odot A.$$

Note that H is well-defined, $H(C,0) = C \oplus \{-\infty\} = C$ and $H(C,1) = C \oplus 0 \odot A = A$, for every $C \in F(\{A_1,\dots,A_n\})$. Thus, H contracts the set $F(\{A_1,\dots,A_n\})$ to A.

The rest of the proof follows that of [4, Lemma 3.1].

□

Theorem 4.6. *Let X be an infinite compact metrizable space. Then the space* $\mathrm{mpcc}(C(X))$ *is homeomorphic to the separable Hilbert space ℓ^2.*

Proof. By [8, Theorem E], the space $\exp(C(X))$ is homeomorphic to ℓ^2. Clearly, the map

$$A \mapsto \overline{h_{\mathrm{mp}}(A)}\colon\ \exp(C(X)) \to \mathrm{mpcc}(C(X))$$

is a proper retraction. Now, we use Theorem 1.1 to conclude that $\mathrm{mpcc}(C(X))$ is an ℓ^2-manifold.

Since the space $\mathrm{mpcc}(C(X))$ is contractible, we are done. □

4.1. Non-separable case

By $\ell^2(\kappa)$ we denote the Hilbert space of density κ. K. Koshino [12, Main Theorem] proved, in particular, that if a space X is connected, locally connected, topologically complete, nowhere locally compact, and for each point $x \in X$, any neighborhood of x in X is of density κ, then $\exp X$ is homeomorphic to $\ell^2(\kappa)$.

Theorem 4.7. *Let X be a compact Hausdorff space of weight κ. Then the hyperspace* $\mathrm{mpcc}(C(X))$ *is homeomorphic to $\ell^2(\kappa)$.*

Proof. The Banach space $C(X))$ is known to be of density κ. By Koshino's theorem stated above, $\exp C(X)$ is homeomorphic to $\ell^2(\kappa)$. Now, we remark that the methods applied in Section 4 work also in the non-separable case. Therefore, the hyperspace $\mathrm{mpcc}(C(X))$ is also homeomorphic to $\ell^2(\kappa)$ as an AR-space which is dominated by $\ell^2(\kappa)$. □

The notion of max-plus convex set can be naturally defined for the space $\mathbb{R}^\infty$.

Theorem 4.8. *The hyperspace* $\mathrm{mpcc}(\mathbb{R}^\infty)$ *is homeomorphic to Q^∞.*

Proof. Since

$$\mathrm{mpcc}(\mathbb{R}^\infty) \simeq \mathrm{mpcc}(\varinjlim [-n,n]^n) \simeq \varinjlim \mathrm{mpcc}([-n,n]^n),$$

it suffices to prove that, for every natural n, the set $\mathrm{mpcc}([-n,n]^n)$ is a Z-set in the space $\mathrm{mpcc}([-n-1,n+1]^{n+1})$.

Indeed, for every $r > 0$, the map

$$A \mapsto \bar{O}_r(A)\colon \mathrm{mpcc}([-n-1,n+1]^{n+1}) \to \mathrm{mpcc}([-n-1,n+1]^{n+1})$$

is a map which is r-close to the identity and misses $\mathrm{mpcc}([-n,n]^n)$. It follows from the Z-set unknotting theorem (see [7]) that the pair

$$(\mathrm{mpcc}([-n-1,n+1]^{n+1}), \mathrm{mpcc}([-n,n]^n))$$

is homeomorphic to the pair $(Q \times Q, Q \times \{0\})$, which implies the result.

□

Counterparts of the results of this section can be proved also for the max-min convexity.

5. $\mathbb{B}$-convexity

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \geq 0\}$. A subset $C \subset \mathbb{R}^n$ is said to be $\mathbb{B}$-convex (see, e.g., [6]) if, for any $x, y \in C$ and $t \in [0,1]$, $tx \oplus y \in C$. The map $j\colon \mathbb{R}^n \to \mathbb{R}^n_+$, $j(x_1, \dots, x_n) = (e^{x_1}, \dots, e^{x_n})$, is an embedding which sends the max-plus convex sets in $\mathbb{R}^n$ to $\mathbb{B}$-convex sets in $\mathbb{R}^n_{++} \subset \mathbb{R}^n_+$ and such that the preimage of every $\mathbb{B}$-convex set is a max-plus convex set. Indeed, let $A \subset \mathbb{R}^n$ be a max-plus convex set and $j(x), j(y) \in j(A)$, $t \in [0,1]$ (without loss of generality, we assume that $t \in (0,1]$). Then $tj(x) \oplus j(y) = j(\ln t \odot x \oplus y) \in j(A)$.

This remark allows us to prove for the hyperspace $\mathbb{B}$-cc$(\mathbb{R}^n)$, $n \geq 2$, of compact $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$ counterparts of results of [2].

One can extend the notion of $\mathbb{B}$-convex set over arbitrary vector lattice. To be specific, let ℓ^2_+ denote the positive cone of the separable Hilbert space ℓ^2. We say that a subset B of ℓ^2_+ is $\mathbb{B}$-convex if for all $x, y \in B$ and all $t \in [0,1]$ one has $tx \oplus y \in B$.

Theorem 5.1. *The hyperspace* $\mathbb{B}$-*cc*(ℓ^2_+) *is homeomorphic to* ℓ^2.

Proof. Since ℓ^2_+ is complete separable, connected, locally connected, nowhere locally compact metric space, $\exp \ell^2_+$ is homeomorphic to ℓ^2, by [8, Theorem E]. The rest of the proof follows that of the main result of Section 4. □

6. Remarks and open questions

The following questions remain open.

1) Let U be an open subset of $\mathbb{R}^\infty$. Is the hyperspace mpcc(U) homeomorphic to $U \times Q^\infty$?

2) The notion of $\mathbb{B}$-convex set can be formulated also for the space ℓ^2 endowed with the bounded-weak* topology (denoted by bw*). Is the hyperspace mpcc(ℓ^2, bw^*) homeomorphic to (ℓ^2, bw^*)?

3) Is the hyperspace of all max-plus (max-min) polyhedra in $\mathbb{R}^n$, $n \geq 2$, homeomorphic to the pre-Hilbert space $\ell^2_{\mathrm{f}} = \{(x_i) \in \ell^2 \mid x_i = 0$ for all but finitely many $i\}$?

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