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INTERACTING PARTICLE SYSTEMS *

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Abstract. We present in this paper a number of recent and various works of statistical physics, which involve an interacting particle system. In *kinetically constrained models*, the particles are placed on $\mathbb{Z}^d$, with local constraints on the dynamics, that can slow down the evolution and reproduce the behaviour of glassy systems. The *contact process* (or Suspected-Infected-Susceptible) is a simple model for the spread of an infection on a (general) graph. In the *Kuramoto model*, the motion of the particles is determined by a diffusion system with mean-field interaction. Finally, in the modeling of the *vertex reinforced jump process* (a particle that interacts with itself via its path), we show an unexpected link with an interacting particle system.

Résumé. Nous présentons dans ce papier plusieurs travaux récents et divers de physique statistique, qui mettent en jeu un système de particules en interaction. Dans les *modèles cinétiquement contraints*, les particules sont placées sur $\mathbb{Z}^d$ avec des contraintes locales sur la dynamique qui peuvent ralentir celle-ci et reproduire le comportement des systèmes vitreux. Le *processus de contact* (ou Susceptible-Infected-Susceptible) modélise quant à lui la propagation d'une infection sur un graphe quelconque. Dans le *modèle de Kuramoto*, le mouvement des particules est déterminé par un système de diffusions, avec interaction de type champ moyen. Enfin, pour le *processus de sauts renforcé par site*, la particule interagit avec elle-même (avec sa trajectoire), la modélisation présentant un lien inattendu avec un système de particules en interaction.

Introduction

Many complex systems can be modeled by stochastic particles that interact with each other. The term "Interacting Particle Systems" is widely used for a large family of models that take into account these interactions and their effect on the model behaviour. For example, the particles themselves can be driven by some stochastic partial differential equations, with interaction terms. One can consider mean-field interactions or some more specific graphs of interaction. Or, the particles are entirely determined by assigning a value in the finite set $\{0, 1\}$ on the vertices of a graph (for example, the value 1 typically can model an occupied vertex or an infected

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vertex), and the interactions are modeled by the edges, or more generally by the local structure of the graph. The evolution of the system is then driven by some statistical physics model on this graph.

The family of statistical physics models that are studied, as well as the family of SPDE and interaction terms that are considered, is growing fast. In this paper we try to give an overview of some recent developments concerning completely different models that come from various motivations and use specific methods.

In Section 1, Cristina Toninelli studies the Kinetically constrained models.

In section 2, Laurent Ménard studies the contact process and the cumulatively merged partitions.

In section 3, Eric Luçon studies the quenched Kuramoto model.

In section 4, Xiaolin Zeng studies the vertex reinforced jump process.

1. Kinetically constrained models and bootstrap percolation: critical time and length scales

1.1. Bootstrap percolation

In recent years, a great deal of progress has been made in understanding the behaviour of a class of monotone cellular automata, known as *bootstrap percolation* [7]. Fix a finite collection of finite subsets of $\mathbb{Z}^d \setminus 0$, $\mathcal{U} = \{X_1, \dots, X_m\}$. $\mathcal{U}$ is the *update family* of the process and each $X \in \mathcal{U}$ is an *update rule*. The $\mathcal{U}$-bootstrap percolation process on the d dimensional torus of linear size n, $\mathbb{Z}_n^d$, is then defined as follows. Given a set $A \subset \mathbb{Z}_n^d$ of initially infected sites, set $A_0 = A$, and define recursively for each $t \in \mathbb{N}$

$$A_{t+1} = A_t \cup \{x \in \mathbb{Z}_n^d : \ x + X_k \subset A_t \ \text{ for some } k \in (1, \dots m)\}$$

In words, site x is infected at time $t+1$ if the translate by x of at least one of the update rules is already entirely infected at time t, and infected sites remain infected forever. The set of sites that are eventually infected, $\cup_{t=0}^{\infty} A_t$, is called the $\mathcal{U}$-update closure of A and denoted by $[A]_{\mathcal{U}}$. This general class includes as specific examples the classical r-neighbour bootstrap percolation models (see [3] and references therein). In this case a site gets infected if at least r of its nearest neighbours are infected, namely the update family is formed by all the r-subset of nearest neighbours of the origin.

The key issue is the global behavior starting from q-random initial conditions, namely when each site of $\mathbb{Z}_n^d$ belongs independently with rate q to the initial set of infected sites. In particular one would like to know how large q should be in order that the closure typically covers the whole lattice. In [2, 6, 7] universality results for general $\mathcal{U}$-bootstrap percolation processes in dimension $d = 2$ have been established, yielding the behavior of the critical percolation threshold defined as

$$q_c(n;\mathcal{U}) = \inf\{q : \ \mathbb{P}_q([A]_{\mathcal{U}} = \mathbb{Z}_n^d) \geq 1/2\}$$

One can equivalently express these results in term of the critical length $L_c(q,\mathcal{U}) = \min\{n : q_c(n,\mathcal{U}) = q\}$. In turn, this length is naturally related to the infection time of the origin, $\tau(A,\mathcal{U}) := \min(t \geq 0 : 0 \in A_t)$. In particular, if A is q-random, and $\liminf_{n\to\infty} q_c(n,\mathcal{U}) = 0$, with high probability as $q \to 0$ it holds $\tau(A,\mathcal{U}) = L_c(q)$ [7].

1.2. Kinetically constrained models

A natural stochastic counterpart of bootstrap percolation models are interacting particle systems known as *kinetically constrained models* (KCM). Given a $\mathcal{U}$-bootstrap model, the associated kinetically constrained model (KCM) is a continuous time stochastic dynamics on $\mathbb{Z}^d$ in which each vertex is resampled (independently) at rate one by tossing a q-coin if it could be infected in the next step by the $\mathcal{U}$-bootstrap model and it is not updated otherwise [9]. More precisely a KCM is a stochastic interacting particle system with configurations defined by assigning to each vertex an occupation variable, either 0 or 1. In the former case we say that the site is empty (or infected in the bootstrap jargon), in the latter case it is filled (or not infected). The evolution is

a continuous time Markov process of Glauber type. Each site has a Poisson clock of mean time one, clocks on different sites are independent. When the clock for site x rings, if at least for one update rule X_i , $i \in (1, \dots, m)$ the set $X_i + x$ is completely empty, the configuration at x is updated by setting it to empty with probability q and occupied with probability $1 - q$. If this constraint is not satisfied the configuration is not updated and we wait for the next clock ring. Since the constraint does not depend on the to-be-updated site, detailed balance holds w.r.t. the product measure μ which gives weight q to empty sites and $1 - q$ to occupied sites. Therefore μ is an invariant reversible measure for the process.

The basic issues concerning the long time behavior KCM are in general not trivial. In particular, due to the presence of constraints, there exist configurations which do not evolve under the dynamics, and relaxation to μ is not uniform on the initial configuration. Also, at variance with the cellular automata, these stochastic dynamics are not monotone: the presence of more zeros facilitates motion and can therefore also allow killing more zeros. Thus coupling and censoring arguments which have been developed for attractive dynamics (e.g. Glauber dynamics for Ising model) cannot be applied. The main interest of KCM is that for $q \to 0$ they reproduce some of the most striking features of the liquid/glass transition, a major and still largely open problem in condensed matter physics. In particular, they display an heterogeneous dynamics and anomalously long mixing times.

1.3. **Results**

In [9] we have established that the ergodic regime for any KCM coincides with the regime in which blocked clusters do not percolate for the corresponding cellular automata. Furthermore we prove exponential mixing in the whole ergodic regime: the relaxation time T_{rel} (i.e. the inverse of the spectral gap of the Markov generator) is finite for $q > q_c$. Another natural time scale is the mean (over the process and the distribution μ for the initial configuration) of the random first time τ_0 at which the occupation variable at the origin is updated, $\mathbb{E}_\mu(\tau_0)$. In [9] using a Feynman-Kac formula approach we proved that $\mathbb{E}_\mu(\tau_0) \le T_{rel}/\min(q, 1-q)$. In particular $\mathbb{E}_\mu(\tau_0)$ is also finite for $q > \liminf_{n\to\infty} q_c(n, \mathcal{U})$.

A key issue both from the physical and mathematical point of view is to determine the divergence of the time scales T_{rel} and $\mathbb{E}_\mu(\tau_0)$ when we approach the critical density at which blocked clusters percolate, namely for $q \downarrow q_c = \liminf_{n\to\infty} q_c(n, \mathcal{U})$. Via a general argument based on the finite-speed of propagation, it is easy to establish (see [9]) that $\mathbb{E}_\mu(\tau_0)$ is lower bounded by the critical length of the corresponding cellular automata, L_c. Instead, there is not a direct connection with the cellular automata which allows to compute an upper bound on the time scales, and the best general upper bound is $\mathbb{E}_\mu(\tau_0) \le \exp(cL_c^d)$ ([9]). Though this bound has been refined for certain models leading in some cases the sharp behavior [9–11], the techniques are always ad hoc and valid only for very special choices of the constraints.

The main result that we present is a new toolbox which allows to pin down the dominant relaxation mechanism and obtain a much tighter upper bound for $\mathbb{E}_\mu(\tau_0)$ and T_{rel} [27]. In particular we apply our technique to the KCM with update rule corresponding to r-neighbour bootstrap percolation. This is a very popular KCM, known in physics literature as Friedrickson Andersen k-facilitated model. In this case the sharp scaling of L_c has been determined in a series of works (see [3] and references therein), leading to

$$L_c(q) = \exp_{(k-1)}\Big(\frac{\lambda(d,k) + o(1)}{q^{1/(d-k+1)}}\Big),$$

with $\lambda(d, k)$ an explicit constant and $\exp_{(r)}$ the r-times iterated exponential, $\exp_{(r+1)}(x) = \exp(\exp_{(r)}(x))$. In [27] we prove that for $k = 2$ there exists $\alpha > 0$ such that

$$\mathbb{E}_\mu(\tau_0) = O\Big(L_c(q)^{\log(1/q)^\alpha}\Big)$$

and for $3 \le k \le d$ there exists $c > \lambda(d, k)$ such that

$$\mathbb{E}_\mu(\tau_0) \le \exp_{(k-1)}\big(c/q^{1/(d-k+1)}\big)$$

This, together with the lower bound $\mathbb{E}_\mu(\tau_0) \geq L_c$, establishes a much tighter connection between $\mathbb{E}_\mu(\tau_0)$ and L_c than previous results.

2. The contact process and cumulatively merged partitions

This is based on the joint work by L.M. and Arvind Singh [28].

2.1. **Motivation: the contact process**

The contact process is a classical model of interacting particle system introduced by Harris in [21]. It is commonly seen as a model for the spread of an infection inside a network. Roughly speaking, given a locally finite connected graph G with vertex set V, the contact process on G is a continuous time Markov process taking value in $\{0,1\}^V$ (sites having value 1 at a given time are said to be *infected*) and with the following dynamics:

- Each infected site heals at rate 1.
- Each healthy site becomes infected at rate λN where $\lambda > 0$ is the infection parameter of the model and N is the number of infected neighbours.

We refer the reader to the books of Liggett [23, 24] for a comprehensive survey on interacting particle systems, including the contact process. Durett's book [15] also provides a nice survey on these models in the setting of random graphs.

An important feature of the model is the existence of a critical infection rate $\lambda_c \in [0, +\infty]$ such that the process starting from a finite number of infected sites dies out almost surely when $\lambda < \lambda_c$ but has a positive probability to survive for all times as soon as $\lambda > \lambda_c$. If the underlying graph is infinite, there is always a super-critical phase *i.e* $\lambda_c < +\infty$ (on a finite graph, the process necessarily dies out since it takes values in a finite space with zero being the unique absorbing state). This follows, for instance, from comparison with an oriented percolation process, see [23]. On the other hand, if the graph has bounded degrees, then there also exists a non trivial sub-critical phase, *i.e.* $\lambda_c > 0$. This can be seen by coupling the contact process with a continuous time branching random walk with reproduction rate λ.

However, without the boundedness assumption on the degrees of G, the situation is much more complicated and the existence of a sub-critical phase is not guaranteed. For example, Pemantle [31] proved that, on a Galton-Watson tree with reproduction law B such that, asymptotically, $\mathbb{P}\{B \geq x\} \geq \exp(-x^\beta)$ for some $\beta < 1$, then $\lambda_c = 0$. Thus, the degree distribution of a random tree can have moments of all orders and yet the contact process on it will still survive with positive probability even for arbitrarily small infection rates. This is a very different behavior from the one observed on regular trees with similar average degree and it indicates that the survival of the contact process depends on finer geometric aspects of the underlying graph than just its growth rate. In fact, to the best of our knowledge, there was no (non trivial) example of a graph with unbounded degrees for which it has been shown that λ_c is non-zero before the work [28] by L.M. and Arvind Singh that will be presented here.

Let us first briefly and informally explain how vertices with large degree help maintaining infections in the contact process and how they contribute to diminish the critical parameter λ_c. This heuristic will motivate the introduction of a partition of the vertex set of the graph called the *cumulatively merged partition* in the next section.

Heuristic. It is well known that the contact process on a star graph of degree d (*i.e* a vertex joined to d leaves) has a survival time of exponential order (say, to simplify $\exp(d)$) when the infection parameter λ is larger than some value $\lambda_c(d) > 0$. Now, consider the contact process on an infinite graph G with unbounded degrees, and fix a very small infection rate λ so that there are only very few sites in the graph where the contact process is locally super-critical (those with degree larger than say, d_0).

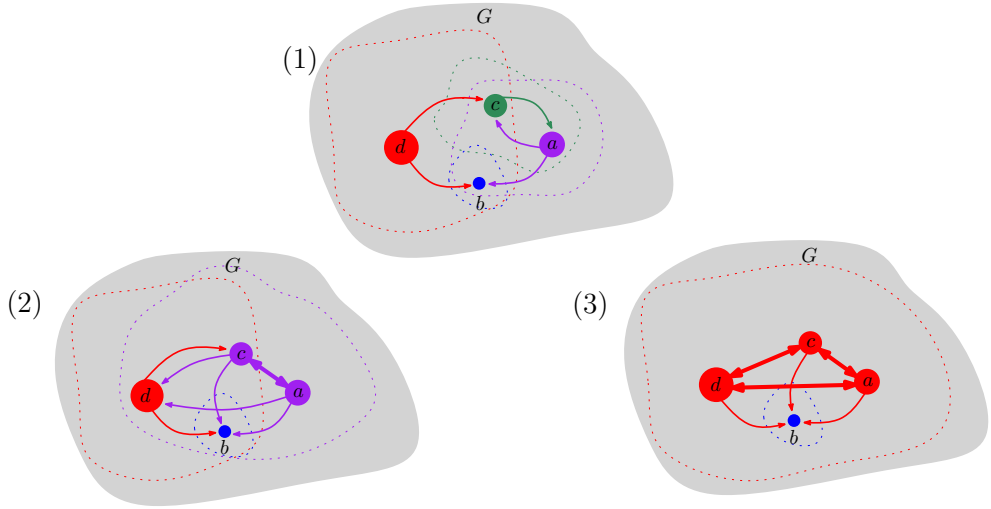

FIGURE 1. Illustration of the heuristic. In dotted lines are the maximal distances attained by infections started from each of the vertices a, b, c and d. Arrows represent infection fluxes and double arrows symbolize the grouping of sites, a group acting as a single site. In this example, a and c first merge together in (2), then the resulting cluster merges with site d in (3). Note that site b receives infections from the other sites but since it cannot reciprocate, it stays isolated during the merging procedure.

To see the influence of these vertices with anomalously large degree, imagine that we start the process with a single infected site a having degree $d_a > d_0$. In addition, suppose that in a neighbourhood of a, every vertex has degree smaller than d_0. Now run the process while forcing a to stay infected for a time of order $\exp(d_a)$ after which the whole star around site a recovers. By that time, roughly $\exp(d_a)$ infections will have been generated by the star around a. But, inside the neighbourhood of a, vertices have small degrees so the process is sub-critical and each infection emitted from a propagates only up to a distance with finite expectation and exponential tail. This tells us that the maximal distance reached by the infections generated from a should be roughly of order d_a.

Now imagine that within distance smaller than d_a from a, there is some other vertex b with degree $d_b > d_0$. Suppose also that d_b is much smaller than the distance between a and b. The previous heuristic applied to b tells us that, in that case, the contact process started from site b has little chance to ever infect a. Thus, infections generated by a will propagate to b but the converse is false. This means that, while a is infected, infections regularly reach site b but this flux stops when a recovers, then b survives for an additional time $\exp(b)$ without reinfecting a. So, the whole process survives for a time of order $\exp(d_a) + \exp(d_b) \approx \exp(d_a)$.

Consider now the case where there is a vertex c, again at distance less than d_a from a, but this time with degree d_c also larger than the distance between a and c. In that case, infections generated by a can reach c and *vice-versa*. Consequently, when either site a or c recovers, the other site has a high probability to reinfect it before its own recovery. This reinforcement effect means that, in order for the process to die out, both vertices a and c must recover almost simultaneously. This will happen after a time of order $\exp(d_a) \times \exp(d_c) = \exp(d_a + d_c)$. Thus, for the purpose of studying the extinction time, we can see both vertices a and c acting like a single vertex of degree $d_a + d_c$ (see Figure 1 part (1) and (2)).

But now, our combined pair of vertices (a, c) will send infections to a larger distance $d_a + d_b$ and will possibly find other vertices to interact with (for example, vertex d in Figure 1). Iterating this procedure, we recursively group vertices together, with the condition that two groups merge whenever the sum of the degrees inside each group is larger than the distance between them. Assuming that this procedure is well defined and converges,

the limiting partition should satisfy the condition that, for any two equivalence classes A and B,

$$d(A,B) > \min\{r(A); r(B)\}$$

where d is the graph distance and $r(A)$ is the sum of the degrees of the vertices of A.

It turns out that the limiting partition exists and does not depend on the order the merging procedure is performed. Its study is the purpose of Section 2.2 where we rigorously define it for a general weighted graph. Then, we examine some of its properties in 2.3 and we finally give a statement that makes the previous heuristic rigorous in Section 2.4.

2.2. **Cumulatively merged partition**

Let $G = (V, E)$ denote a locally finite connected graph. We use the notation $d(\cdot,\cdot)$ for the usual graph distance on G. For any pair $A, B \subset V$, we set $d(A,B) = \inf\{d(x,y) \ : \ (x,y) \in A \times B\}$. The graph G is equipped with a sequence of non-negative weights defined on the vertices:

$$(r(x),\ x \in V) \in [0,\infty)^V.$$

It is convenient to see r as a measure on V so the total weight of a set $A \subset V$ is given by $r(A) = \sum_{x\in A} r(x)$. We also fix a parameter

$$1 \leq \alpha < +\infty$$

to which we refer as the *expansion exponent*. For reasons that will become clear later, we call the quantity $r(A)^\alpha$ the *influence radius* of the set A. The goal of this section is to study the partitions of V which satisfy the following property:

Definition 2.1. *A partition $\mathcal{C}$ of the vertex set V of G is said to be* admissible *if it is such that*

$$\forall C, C' \in \mathcal{C}, \qquad C \neq C' \quad \Longrightarrow \quad d(C,C') > \min\{r(C), r(C')\}^\alpha. \tag{1}$$

The trivial partition $\{V\}$ is admissible and intersections of admissible partitions remain admissible. Therefore, we can consider the finest admissible partition, called the *cumulatively merged partition* (CMP) defined by

$$\mathscr{C} := \bigcap_{\substack{\text{admissible} \\ \text{partitions } \mathcal{C} \text{ of } G}} \mathcal{C}.$$

Elements of this partition are called *clusters* and the radius of influence of a cluster C is the quantity $r(C)^\alpha$. The partition $\mathscr{C}$ can be constructed by an algorithm justifying its name:

Proposition 2.2. *Consider the following algorithm:*

1. *Start with the finest partition $\mathcal{C} = \{\{v\}, v \in V\}$.*
2. *If there exists $C, C' \in \mathcal{C}$ such that $d(C,C') \leq \min\{r(C), r(C')\}^\alpha$, then merge C and C' into a single cluster. Repeat this step as long as some mergings are possible.*

This algorithm constructs a sequence of coarsening partitions that converges towards $\mathscr{C}$, independently of the order of the mergings.

Figure 2 gives an example of CMP on $\mathbb{Z}$. Two important obervations are:

- If $r(v) < 1$, then v is isolated in the CMP, meaning $\{v\} \in \mathscr{C}$.
- A cluster is infinite if and only if its radius of influence is infinite. Therefore, $\mathscr{C}$ always has at most one infinite cluster.

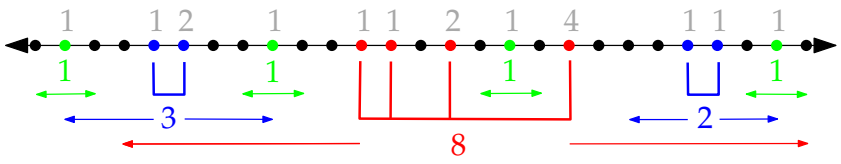

FIGURE 2. Example of CMP on $\mathbb{Z}$ with $\alpha = 1$. The weight of each vertex is in gray above it if it is not 0. Three clusters of size larger than 1 are created: one of weight 2 and one of weight 3, each containing two vertices; the red cluster contains 4 vertices and has weight 8. Arrows indicate the radius of influence of the clusters.

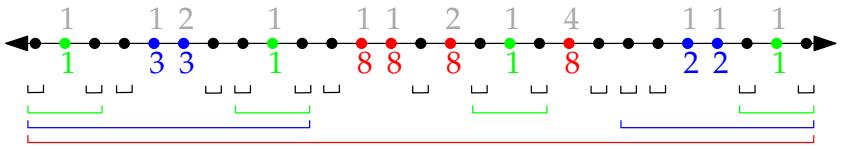

FIGURE 3. Stabilisers of the clusters of Figure 2. Observe that the whole represented interval is stable. The nested structure of the stabilisers is a feature of their multiscale structure.

It is quite easy to construct examples of graphs for which changing the weight of a single vertex creates an infinite cluster. Therefore local modifications of the graph can have global consequences for the CMP. However, the symmetry between clusters in the definition of admissible partitions force the CMP to have some local properties captured by the notion of *stable sets*.

Definition 2.3. *For $v \in \mathscr{C}$, we denote by $\mathscr{C}_v$ the cluster of $\mathscr{C}$ cointaining v. A set $H \subset V$ is* stable *if and only if*

$$\bigcup_{x \in H} B(x, r(\mathscr{C}_x(G))^\alpha) = H.$$

Informally, stable sets are the subsets H such that no radius of influence reach outside of H: sites inside H cannot merge with sites outside of H. The fact that a set is stable or not can be verified by a local algorithm (namely by doing the CMP procedure on the subgraph induced by H).

Intersections and unions of stable sets are still stable. This allows to define the *stabiliser* $\mathcal{S}_A$ of any set of vertices A as the smallest stable set containing A (see Figure 3 for an illustration). Stabilisers are instrumental in studying the existence of an infinite cluster in the CMP as can be seen in the following Theorem.

Theorem 2.4. *[**Criterion for the existence of an infinite cluster**] Suppose that G is an infinite graph. For every $v \in V$, the following statements are equivalent:*

$$|\mathscr{C}_v| = \infty \quad \Leftrightarrow \quad |\mathcal{S}_v| = \infty \quad \Leftrightarrow \quad \mathcal{S}_v = V.$$

Equivalently, the partition $\mathscr{C}$ has no infinite cluster if and only if there exists an increasing sequence (S_n) of stable subsets such that $\lim \uparrow S_n = V$.

2.3. Phase transition for percolation on the CMP

In this section, we consider the CMP on several random weighted graphs. We investigate whether or not the partition $\mathscr{C}$ contains an infinite cluster. At first look, one might fear that this will always be the case due to the amplification phenomenon resulting from the additive nature of cluster merging. Or, on the contrary, the cumulative effect might be quite weak and percolation by cumulative merging could be very similar to classical site percolation. It turns out that both worries are unfounded and that, for a wide variety of random weighted graphs, there is a non-trivial phase transition differing from that of classical site percolation.

Model 1 (**Bernoulli CMP**). *The underlying graph G is a deterministic infinite graph (e.g. $\mathbb{Z}^d$, a tree ...) and the weights $(r(x), x \in V)$ are independent identically distributed Bernoulli random variables with parameter $p \in [0,1]$. We denote by $\mathbb{P}_p^G$ the law of (G,r).*

Model 2 (**Continuum CMP**). *The underlying graph G is a deterministic infinite graph and the weights $(r(x), x \in V)$ are independent identically distributed random variables with law λZ, where $\lambda \geq 0$ and Z is a random variable taking value in $[0,\infty)$. We denote by $\mathbb{P}_\lambda^{G,Z}$ the law of (G,r).*

Model 3 (**Degree-weighted CMP**). *The underlying graph G is a random infinite graph (e.g. a Galton-Watson tree, a random Delaunay triangulation, a random planar map, ...) and the weights are defined by $r(x) := \deg(x)\mathbf{1}_{\{\deg(x)\geq\Delta\}}$, with $\Delta \geq 0$. We denote by $\mathbb{P}_\Delta^G$ the law of (G,r).*

The first two models are the counterparts in the context of cumulative merging of classical (site) percolation and boolean models. The third model may seem artificial at first. However, as we already explained in the introduction, it appears naturally in the connection between cumulative merging and the contact process. We will investigate this relationship in the next section.

Each model has a free parameter (p for Bernoulli, λ for continuum, and Δ for degree-weighted CMP) so we ask, the expansion exponent α being fixed, whether of not $\mathscr{C}$ contains an infinite cluster depending on the value of this parameter. By monotonicity of the CMP with respect to α and the weight sequence r, the probability of having an infinite cluster is monotone in both α and the free parameter of the model.

Definition 2.5. *For Bernoulli CMP we define*

$$p_c(\alpha) := \inf\left\{p \in [0;1] \,:\, \mathbb{P}_p^G\left\{\mathscr{C}(G,r,\alpha) \text{ has an infinite cluster}\right\} > 0\right\} \in [0;1].$$

Similarly, for continuous CMP we define

$$\lambda_c(\alpha) := \inf\left\{\lambda \geq 0 \,:\, \mathbb{P}_\lambda^{G,Z}\left\{\mathscr{C}(G,r,\alpha) \text{ has an infinite cluster}\right\} > 0\right\} \in [0;+\infty],$$

and for degree biased CMP, we set

$$\Delta_c(\alpha) := \sup\left\{\Delta \geq 1 \,:\, \mathbb{P}_\Delta^G\left\{\mathscr{C}(G,r,\alpha) \text{ has an infinite cluster}\right\} > 0\right\} \in \mathbb{N}^* \cup \{+\infty\}.$$

Under fairly general assumptions on G, it is easy to check that the existence of an infinite cluster is an event of either null or full probability.

Proposition 2.6. *Suppose that G is a vertex transitive graph, then for Bernoulli CMP (model 1) or continuum CMP (model 2), we have*

$$\mathbb{P}\left\{\mathscr{C} \text{ has an infinite cluster}\right\} \in \{0,1\}.$$

In the case of Model 3, one needs, of course, to make some assumptions on the random graph G in order to get a 0-1 law. However, for a large class of graphs, the existence of an infinite cluster is still a trivial event thank again to general ergodicity properties. This is in particular the case for random geometric graphs, Delaunay triangulations as well as Galton-Watson trees or unimodular random graphs.

Theorem 2.7. *Consider CMP for model 1 or 2 with an expansion exponent $\alpha \geq 1$. If the underlying graph G is infinite, then*

$$p_c < 1 \quad \text{and} \quad \lambda_c < +\infty.$$

It is enough to prove this result for $G = \mathbb{N}$ since every infinite graph contains a copy of it. The main idea is to use the multiscale structure of the clusters of the CMP to construct inductively bigger and bigger intervals with macroscopic clusters with increasing probabilities.

Knowing that CMP always has a supercritical phase, it remains to prove that a subcritical phase is also present:

Theorem 2.8. *Consider CMP on $\mathbb{Z}^d$, $d \geq 1$ with expansion exponent $\alpha \geq 1$.*

- Model 1. *One has $p_c > 0$.*
- Model 2. *If $\mathbb{E}\left[Z^\beta\right] < +\infty$ for $\beta = 4\alpha d^2$, then $\lambda_c > 0$.*

The proof is fairly technical and uses once again renormalization argument particularly suited to the recursive and multiscale structure of the CMP. The main idea is to construct bigger and bigger boxes containing stable sets that will eventually exhaust the graph.

For Model 3. we studied two models where the graph G is either a random geometric graph or a Delaunay triangulation constructed from a Poisson point process $\mathcal{P}$ in $\mathbb{R}^d$ with Lebesgue intensity. First, we quickly recall the definition of these graphs:

• *Geometric graph with parameter $R > 0$.* The vertex set is composed of the atoms of the point process $\mathcal{P}$ and, for any pair of points $x, y \in \mathcal{P}$, there is an edge between x and y if and only if $\|x - y\| < R$, where $\|\cdot\|$ denotes the Euclidian norm in $\mathbb{R}^d$. If R is above the critical parameter for continuum percolation, then this graph has a unique infinite connected component. We assume this is the case and denote this graph $\mathcal{G}(R, \mathcal{P})$.

• *Delaunay Triangulation.* For any $x \in \mathcal{P}$, we define the Voronoï cell of x as the set of points of $\mathbb{R}^d$ which are closer to x than to any other point of the Poisson point process:

$$\mathrm{Vor}_{\mathcal{P}}(x) := q\left\{z \in \mathbb{R}^d : \|x - z\| < \|y - z\| \, \forall y \in \mathcal{P}\right\}.$$

The Delaunay triangulation of $\mathcal{P}$ is the dual of the Voronoï tessellation: its vertex set is again the set of atoms of $\mathcal{P}$ and two vertices share an edge if and only if their corresponding cells are adjacent (i.e. they share a $d-1$ dimensional face). We denote this graph by $\mathcal{D}(\mathcal{P})$.

Both these graphs have unbounded degrees but typical sites have degrees with exponential moments. We are able to adapt the renormalization arguments of Theorem 2.8 to prove that they also exhibit a non trivial phase transition for CMP percolation:

Theorem 2.9. *Consider the CMP of Model 3. on $\mathcal{G}(R, \mathcal{P})$ or on $\mathcal{D}(\mathcal{P})$ with expansion exponent $\alpha \geq 1$. Then, we have*

$$\Delta_c(\alpha) < \infty.$$

2.4. **Back to the contact process**

The following result partially makes the heuristic given in Section 2.1 rigorous.

Theorem 2.10. *Let $G = (V, E)$ be a locally finite connected graph. Consider the CMP of Model 3. on G (*i.e. *with weights given by $r(x) = deg(x)\mathbf{1}_{\{deg(x) \geq \Delta\}}$). Suppose that for some expansion exponent $\alpha \geq \frac{5}{2}$ and some $\Delta \geq 0$, the partition $\mathscr{C}$ has no infinite cluster. Then, the contact process on G has a sub-critical phase: there exists $\lambda_0 > 0$ such that, for any infection parameter $\lambda < \lambda_0$, the process starting from a finite configuration of infected sites dies out almost surely.*

Let us make a few comments on Theorem 2.10:

- First, we find it remarkable that, in a way, all the geometry of the graph needed to prove the existence of a sub-critical phase is encoded in the merging procedure: the radii $r(x)$ give the degrees sites but provide no information on the local shape or growth of the graph around a site. In particular, the theorem requires no assumption on the growth rate of G.
- The exponent $5/2$ is not optimal. However, the proof we describe cannot yield an exponent smaller than 2 so we did not find it worth the effort to clutter the proof with additional technical details for very little gain. In order to get an exponent close to 1, we believe that one needs a better understanding of the inner structure of clusters. The real challenge is to prove (or disprove) the theorem for $\alpha = 1$.

The proof of Theorem 2.10 is delicate as it requires to get rid of the randomness of the contact process by controlling precisely how infections propagate in the graph. The main idea is to show that when a cluster

$C \in \mathscr{C}$ is infected, even if it stays infected for a duration exponential in $r(C)$, the number of infections it manages to send outside its stabiliser is less than 1 in average. Once this is established, we can couple the contact process with a subcritical branching random walk on the graph of the clusters of the CMP. The main technical point is thus to prove that the stabiliser of each cluster is subcritical enough to dissipate infections sent by the corresponding cluster. Here, the recursive nature of the cluster comes yet again in play in a proof by induction on the possible cluster sizes. The assumption $\alpha \geq 5/2$ is needed here to have uniform estimates.

Combining Theorem 2.10 and Theorem 2.9 finally give examples of graphs with unbounded degree for which the contact process has a non trivial phase transition:

Corollary 2.11. *The contact process on a random geometric graph or on a Delaunay triangulation admits a sub-critical phase.*

3. Long-time behavior of the disordered stochastic Kuramoto model

This review is based on [18], in collaboration with Giambattista Giacomin and Christophe Poquet and [26] with Christophe Poquet.

3.1. The mean-field Kuramoto model for synchronization

The purpose of this work is to address the long time dynamics of the Kuramoto model with noise and disorder [22], which describes the evolution of a population of N interacting rotators ($N \geq 1$), each of them represented by a phase φ_i, $i = 1, \dots, N$, on the circle $\mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$. The evolution of each clock φ_i obeys to two contradictory influences: an intrinsic dynamics which consists here in a constant rotation on the circle $\mathbb{S}$ with speed ω_i (that may differ from one rotator to another) and a mean-field interaction with the other rotators, through a sine coupling. We consider here the stochastic version of the Kuramoto model, where thermal noise is added to the dynamics. Putting all these ingredients together, the dynamics of the Kuramoto model is defined by the following system of interacting diffusions

$$\mathrm{d}\varphi_{i,t} = \delta\omega_i \mathrm{d}t - \frac{K}{N}\sum_{j=1}^{N} \sin(\varphi_{i,t} - \varphi_{j,t})\mathrm{d}t + \sigma \mathrm{d}B_{i,t}, \ i = 1, \dots, N, \ t \in [0, T], \tag{2}$$

where $T > 0$ is a fixed but arbitrary time horizon, $\{B_i\}_{i=1,\dots,N}$ is a sequence of i.i.d. standard Brownian motions, $K \geq 0$ is the intensity of interaction between rotators and $\delta > 0$ is a scaling parameter. By the time change $t \to t/\sigma^2$ (up to the obvious modification $\delta \to \delta/\sigma^2$ and $K \to K/\sigma^2$), one can assume with no loss of generality that $\sigma = 1$ in the following.

The Kuramoto model was originally introduced to describe synchronization [1]. Synchronization is a universal phenomenon, widely studied in physics and biology, which occurs in various situations (metronomes, neuronal activity, collective behavior of social insects, etc.). Understanding precisely the dynamics of all these examples would require possibly high-dimensional systems with a complex graph of interaction and a large numbers of parameters. In this sense, one can think of the Kuramoto model as an idealized, one-dimensional version of those models, which, despite its simplicity, captures a nontrivial transition from incoherence to synchrony.

The Kuramoto model and its numerous variants have attracted a considerable interest in recent years, mostly from the physics community (it would be hopeless to review the vast literature on the subject, see e.g. [37] and references therein) and more recently from a mathematical point of view [4, 18, 19]). In particular, a striking extension of (2) concerns the *active rotator model* (in which the local dynamics is more complex than a simple constant rotation [36]) where the emergence of periodic behaviors for large systems of excitable units has been observed [20]).

Remark 3.1. *A crucial property of the Kuramoto model which is at the basis of the dynamical features addressed here is its* invariance by rotation*: if $\{\varphi_j\}_{j=1,\dots,N}$ solves* (2)*, so does $\{\varphi_j + \psi_0\}_{j=1,\dots,N}$ for any arbitrary constant $\psi_0 \in \mathbb{S}$. This rotational invariance has a continuous counterpart in terms of the mean-field limit of* (2)*, which is*

the existence of a circle $\mathscr{C}$ of stationary solutions for the mean-field PDE associated with (2) *(see Section 3.3.1 below). Understanding the dynamics of the empirical measure of system* (2) *in the neighborhood of this circle is the cornerstone of the present work.*

3.1.1. *Perturbations of mean field particle systems*

We stress the fact that there are two sources of randomness in (2): the randomness coming from the initial choice of the frequencies $\{\omega_i\}_{i=1,\dots,N}$ (chosen to be i.i.d. according to a law λ) and the randomness coming from the thermal noise $\{B_i\}_{i=1,\dots,N}$ and the initial condition $\{\varphi_{i,0}\}_{i=1,\dots,N}$. Seeing $\{\omega_i\}_{i=1,\dots,N}$ as a random environment for (2), we adopt here the quenched point of view: we want to understand the behavior of (2) for a fixed choice of the environment $\{\omega_i\}_{i=1,\dots,N}$. In this context, a natural perspective is to see (2) as a particular instance of disordered perturbations of mean-field models of interacting diffusions, that is systems of the type

$$\mathrm{d}\varphi_{i,t} = c(\varphi_{i,t},\omega_i)\mathrm{d}t + \frac{1}{N}\sum_{j=1}^{N}\Gamma(\varphi_{i,t},\varphi_{j,t},\omega_i,\omega_j)\mathrm{d}t + \sigma\mathrm{d}B_{i,t},\ i=1,\dots,N,\ t\in[0,T], \tag{3}$$

for regular (say bounded Lipschitz continuous for simplicity) dynamics $c(\cdot)$ and $\Gamma(\cdot,\cdot)$. In the absence of disorder (i.e. when $c(\varphi,\omega)=c(\varphi)$ and $\Gamma(\varphi,\varphi',\omega,\omega')=\Gamma(\varphi,\varphi')$), we retrieve the classical mean-field framework which has been addressed thoroughly in recent years (see e.g. [17, 29, 38] and references therein): under mild assumptions on c and Γ, the empirical measure $\mu_{N,t} := \frac{1}{N}\sum_{i=1}^{N}\delta_{\varphi_{i,t}}$ converges on each bounded time interval $[0,T]$, as a measure-valued process, to the deterministic solution $q_t(\mathrm{d}\varphi)$ of a nonlinear PDE of McKean-Vlasov type (see e.g. [17] for a precise statement). The relevance of the approximation of (3) by its mean-field PDE has been studied by many authors, through the notion of propagation of chaos [38]: at the level of generality of (3), this essentially boils down to proving Gronwall-type estimates of the form $\sup_{i=1,\dots,N}\mathbf{E}\left(\sup_{s\le t}|\varphi_{i,s}-\bar\varphi_{i,s}|\right)\le e^{Ct}N^{-1/2}$, for an appropriate coupling of *nonlinear processes* $\{\bar\varphi_i\}_{i=1,\dots,N}$ whose law solves the McKean-Vlasov PDE (see [38] for precise definitions). The exponential constant in the previous estimate tells us that (without any further assumptions on the system) the mean-field approximation is only meaningful up to a time of order $c\log N$ for some $c>0$. This restriction is in fact not technical: when we look at (3) on a time scale longer than $\log N$, it may happen that the microscopic dynamics differs significantly from its macroscopic limit, due to the influence of the thermal noise and random environment. Various examples in this direction have already been considered in the literature: in case the mean-field PDE admits an isolated stable fixed point, the finite-size system exits from any neighborhood at exponential times in N [13, 30] whereas the system escapes at a time or order $\log N$ in case of an unstable fixed point [32]. The Kuramoto model (2) corresponds to a situation where the mean-field PDE admits a whole stable manifold of stationary solutions. A previous work [5] has addressed the Brownian fluctuations of (2) on a time scale of order N, in the case with no disorder (that $\omega_i=0$ for all i).

The purpose of the present work is to address the same issue in the presence of the random medium $\{\omega_i\}_{i\ge1}$. Adding a random environment in mean-field systems such as (3) is actually a major complication to the analysis, for two main reasons: first, in the absence of disorder, a crucial property of mean-field systems such as (3) is *exchangeability*: provided the law of $(\varphi_{1,0},\dots,\varphi_{N,0})$ is invariant under permutation at $t=0$, the same property holds for any positive time $t>0$. This property is obviously lost in the presence of a nontrivial quenched random environment. Secondly, in the case of the Kuramoto model without disorder (that is $\omega_i=0$ for all $i=1,\dots,N$ in (2)), it is easy to see that the dynamics is reversible, with unique invariant measure

$$\mu_{N,\infty}(\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_N)\propto\exp\left(\frac{K}{N}\sum_{i,j=1}^{N}\cos(\varphi_i-\varphi_j)\right)\mathrm{d}\varphi_1\dots\mathrm{d}\varphi_N. \tag{4}$$

The measure $\mu_{N,\infty}$ is a classical object in statistical physics, i.e. the Gibbs measure of the mean-field XY model with spin state $\mathbb{S}$. Adding a nontrivial disorder to the system makes the dynamics irreversible [4] and we fall into the domain of non equilibrium statistical mechanics. This observation motivates in particular the introduction

of the scaling parameter δ in (2): the main results will be stated for small δ, as they rely on perturbation arguments from the reversible case $\delta = 0$, studied in the seminal work [4].

3.1.2. *Intuition on synchronization and long time behavior*

The main issue is to understand the influence of a quenched disorder on the long time behavior of the system (2). Let us remain first at an intuitive level. As it will be explained below rigorously, the Kuramoto model is known to exhibit a phase transition: there exists $K_c = K_c(\lambda) > 0$ (depending only on the law λ of the disorder) such that if the strength of interaction between particles is sufficiently large (that is $K > K_c$), the finite size system $(\varphi_{1,t}, \dots, \varphi_{N,t})$ concentrates around a randomly chosen center of synchronization along a deterministic density. Note that, without disorder, $K_c(\delta_0) = 1$. Once the system is synchronized, the rotators feel the mutual influence of the thermal noise and their own frequencies. The main intuition here is that any asymmetry in the chosen quenched frequencies $\{\omega_i\}_{i=1,\dots,N}$ will generate an asymmetry in the system which will induce macroscopic traveling-waves. This intuition is particularly obvious when the law of the frequencies is not centered: $\mathbb{E}_\lambda(\omega) \neq 0$. In this case, the simple change of variables $\varphi_{i,t} \to \varphi_{i,t} - t\mathbb{E}_\lambda(\omega)$ maps (2) to the same system with centered frequencies. This means that one can always assume that $\mathbb{E}_\lambda(\omega) = 0$ and if it is not the case, one observes at first order a traveling wave of the center of synchronization with speed $\mathbb{E}_\lambda(\omega)$. This intuition is no longer obvious when $\mathbb{E}_\lambda(\omega) = 0$. The main questions we would like to address are: can we characterize the existence of disorder-induced traveling waves for (2)? If so, do the direction and speed of these traveling waves depend only on the law λ of the frequencies or also on a typical realization of $(\omega_1, \dots, \omega_N)$? At which time scale do we observe these traveling waves? Do we see them immediately (that is on bounded time interval $[0, T]$) or do we need to consider larger time intervals w.r.t. to the size N of the population?

These questions find actually different answers depending on the nature of the asymmetry we are dealing with. The most simple situation corresponds to a *macroscopic asymmetry*, that is when the law λ itself is asymmetric: assuming for instance that $\lambda((0, +\infty)) > \frac{1}{2}$, this means that, as $N \to \infty$, a majority of rotators will be associated to a positive frequency w.r.t. a minority with negative frequencies. We first show in Section 3.2.2 that, in the limit of an infinite population, this asymmetry makes the whole system rotate at a constant speed that depends only on the law λ and this rotation is noticeable at the scale of the law of large numbers associated to (2). This part is based on [18].

In the case where λ is symmetric, the previous argument cannot be applied since in the limit $N = \infty$, the population is perfectly balanced between positive and negative frequencies. What is relevant here for the long time dynamics of (2) is the *microscopic asymmetry* of the disorder: even if λ is symmetric, finite-size fluctuations of a sample $\{\omega_1, \dots, \omega_N\}$ make it not symmetric so that the fluctuations of the disorder compete with the fluctuations of the Brownian motions $\{B_i\}_{i=1,\dots,N}$ and make the whole system rotate with speed and direction depending on the realization of the disorder $\{\omega_i\}_{i=1,\dots,N}$ (and not only on the law λ itself!). These fluctuations are typically small (of order $N^{-1/2}$) so that we need to wait a time of order $N^{1/2}$ to observe macroscopic traveling waves. We detail this intuition in Section 3.3. This part is based on [26], which uses critical ideas from a previous article [5] where the Kuramoto model without disorder is addressed. Similar techniques are already present in [8, 16].

3.2. **Mean-field limit on bounded time intervals**

3.2.1. *The McKean-Vlasov equation*

The object of interest for the analysis of (2) is the double-layer empirical measure on both rotators and frequencies

$$\mu_{N,t} := \frac{1}{N} \sum_{i=1}^{N} \delta_{(\varphi_{i,t}, \omega_j)} \tag{5}$$

Assuming that $\mathbb{E}_\lambda(|\omega|) < +\infty$, it is possible to show (see [25] for details) that, for almost every choice of the disorder $\{\omega_i\}_{i \geq 1}$, for every $T > 0$, the empirical measure μ_N of the system (2) converges as $N \to \infty$, as a measure-valued process on $[0, T]$ to the deterministic process $t \mapsto q_t$ such that q_t is a probability measure on

$\mathbb{S} \times \mathrm{Supp}(\lambda)$ for all $t \geq 0$ solving the nonlinear Fokker-Planck (or McKean-Vlasov) equation

$$\partial_t q_t(\theta,\omega) \,=\, \frac{1}{2}\partial^2_\theta q_t(\theta,\omega) - \partial_\theta \left(q_t(\theta,\omega) \left(-K \int_{\mathbb{R}} \int_{\mathbb{S}} \sin(\psi) q_t(\theta-\psi,\tilde{\omega}) \mathrm{d}\psi \lambda(\mathrm{d}\tilde{\omega}) + \delta\omega \right) \right),\ \omega \in \mathrm{Supp}(\lambda),\ \theta \in \mathbb{S},\ t \geq 0. \tag{6}$$

Here, we voluntarily identified q_t with its density $q_t(\theta,\omega)$ w.r.t. $\ell(\mathrm{d}\theta) \otimes \lambda(\mathrm{d}\omega)$, where ℓ is the Lebesgue measure on $\mathbb{S}$ (using the regularizing properties of the heat kernel, it is not difficult to show that such a density exists and is a regular function in θ; we refer to [18] for details).

3.2.2. *The asymmetric case*

We suppose in this paragraph that the disorder is with bounded support: $\mathrm{Supp}(\lambda) \subset [-M, M]$ for some $M > 0$. In view of Remark 3.3 below, we restrict ourselves in this paragraph to the case where the law of the disorder is asymmetric: in this case, the traveling waves occur on a finite time scale $[0, T]$, at the level of the Fokker-Planck PDE (6), and the direction and speed of convergence only depend on the law of the disorder λ, not on a particular realization of the disorder (at least at first order, since it is natural to expect finite-size corrections to this deterministic rotation).

Theorem 3.2 ([18])**.** *For every $K > 1$, there exists $\delta_0 = \delta_0(K, M) > 0$ such that for $|\delta| \leq \delta_0$, there exist a value $c_\lambda(\delta) \in \mathbb{R}$ and solution to* (6) *of the form*

$$q_t^{(\psi)}(\theta,\omega) \,:=\, \widetilde{q}_\delta(\theta - c_\lambda(\delta)t - \psi),\ \psi \in \mathbb{S}. \tag{7}$$

Moreover, the family $\{q_t^{(\psi)}\}_\psi$ is stable.

We refrain to give to many details on the notion of stability at stake here and refer to [18] for precise definitions. The proof of Theorem 3.2 relies on strong stability estimates of the manifold of stationary solutions to (6) in the case without disorder (especially the fact that this manifold is a stable normally hyperbolic manifold and that this kind of structure is stable under small perturbations).

Remark 3.3. *Note that it is possible to have an arbitrary precise development of the speed of rotation c_λ in terms of the scaling parameter δ. A crucial property is that this speed of rotation is equally zero when the law of the disorder λ is symmetric, so that Theorem 3.2 is only relevant for asymmetric disorder.*

3.3. **Traveling waves in the symmetric disorder case**

We now turn to the main result of this review, that is the behavior of (2) when λ is symmetric.

3.3.1. *Manifold of invariant solutions and its stability*

A remarkable feature of the Kuramoto model with symmetric disorder is that it is possible to compute in a semi-explicit way the stationary solutions of (6): each stationary solution of (6) is a rotation of a profile q (i.e. given by $(\theta,\omega) \mapsto q(\theta+\psi_0,\omega)$ for some $\psi_0 \in \mathbb{S}$) of the form

$$q(\theta,\omega) := \frac{\int_{\theta-2\pi}^{\theta} e^{-2Kr(\cos(u)-\cos(\theta))-2\omega(u-\theta)}\mathrm{d}u}{Z(\omega,2Kr)}, \tag{8}$$

where $Z(\omega,2Kr)$ is a normalizing constant ensuring that $q(\cdot,\omega)$ is indeed a probability measure on $\mathbb{S}$ for all $\omega \in \mathrm{Supp}(\lambda)$. In (8), $r \in [0,1]$ is a parameter that needs to verify a fixed point relation

$$r = \Psi(2Kr) := \int \frac{\int_{\mathbb{S}} \cos(\theta) \int_{\theta-2\pi}^{\theta} e^{-2Kr(\cos(u)-\cos(\theta))-2\omega(u-\theta)}\mathrm{d}u\mathrm{d}\theta}{Z(\omega,2Kr)} \lambda(\mathrm{d}\omega). \tag{9}$$

Solving this fixed point relation enables to discriminate between two cases: there exists some $K_c(\lambda) > 0$ such that if $K \leq K_c$, $r = 0$ is the only solution to (9) and this corresponds to the incoherent solution $q \equiv \frac{1}{2\pi}$, whereas

if $K > K_c$, $r = 0$ coexists with nontrivial solutions $r > 0$. Such positive solutions may not be unique in general but it is the case if the disorder is sufficiently small: for all $K > 1$, there exists $\delta_1(K) > 0$ such that for all $|\delta| \leq \delta_1(K)$, there exists a unique positive solution to (9) (see [26] for more details). This $r > 0$ gives rise to a nontrivial (synchronized) stationary solution q to (6). In view of the rotational invariance of the problem (recall Remark 3.1), such a solution actually generates a whole circle of stationary solutions

$$\mathscr{C} := \{q(\cdot + \psi_0),\ \psi_0 \in \mathbb{S}\}. \tag{10}$$

A starting point of the analysis is to prove that this manifold is actually stable for the dynamics induced by (6). In a few words, considering for all $K > 1$ the linearization of the dynamics around one element $q \in \mathscr{C}$

$$L_q(u) := \frac{1}{2}\partial_\theta^2 u - \partial_\theta \left(u \left(-K \int_{\mathbb{R}} \int_{\mathbb{S}} \sin(\psi) q(\cdot - \psi, \tilde{\omega}) \mathrm{d}\psi \lambda(\mathrm{d}\tilde{\omega}) + \delta\omega \right) - Kq \int_{\mathbb{R}} \int_{\mathbb{S}} \sin(\psi) u(\cdot - \psi, \tilde{\omega}) \mathrm{d}\psi \lambda(\mathrm{d}\tilde{\omega}) \right), \tag{11}$$

it is possible to find an Hilbert space of distributions $\mathbf{H}$ and $\delta_2(K) > 0$ such that if $\delta \leq \delta_2(K)$, the operator L_q is closable in $\mathbf{H}$, sectorial, has 0 for eigenvalue, with eigenvector $\partial_\theta q$ (this reflects the invariance by rotation of the manifold $\mathscr{C}$) and the rest of the spectrum is negative, separated from 0 by a spectral gap $\gamma_L > 0$. This means in particular that the dynamics in any neighborhood of $\mathscr{C}$ can be decomposed into an orthogonal direction along which there is exponential attraction to $\mathscr{C}$ and a tangential direction that is left invariant by the dynamics. We refer to [26], § 2.3 for more details.

3.3.2. *Disorder-induced traveling waves*

We address here the main issue of this review, that is the appearance of traveling waves in the case of a symmetric law of disorder. For the simplicity of exposition, we restrict ourselves to the case where the disorder $\{\omega_i\}_{i\geq 1}$ is an i.i.d. sequence with law $\lambda = \frac{1}{2}(\delta_{+1} + \delta_{-1})$ (for possible extensions, see Remark 3.5 below). In this framework, it is easy to reformulate (6) by splitting $(\varphi_1, \ldots, \varphi_N)$ into two subpopulations $(\varphi_1, \ldots, \varphi_{N^+})$ and $(\varphi_{N^+ + 1}, \ldots, \varphi_N)$ of respective sizes N^+ and N^- such that the first (resp. second) subpopulation has $+1$ (resp. -1) as local frequency. Obviously, $N_+ + N_- = N$. In this framework, the randomness of the disorder is captured by the random asymmetry between the sizes N^+ and N^-.

In this case, Theorem 3.2 is essentially empty: there is no macroscopic traveling waves for at the level of the Fokker-Plack PDE (6). What really happens is a finite-size effect of the system (2) around its mean-limit: one can imagine that fluctuations in the finite sample $\{\omega_1, \ldots, \omega_N\} \in \{\pm 1\}^N$ may lead, for example, to a majority of $+1$ with respect to -1, so that the rotators with positive frequency induce a global rotation of the whole system in the direction of the majority. When N is large, this asymmetry is small, typically of order $N^{-1/2}$ and is not sufficient to make the empirical measure drift away from the attracting manifold $\mathscr{C}$, but induces a small drift that becomes macroscopic at times of order $N^{1/2}$.

To make this intuition rigorous, define the empirical proportions of frequencies in the N-sample $(\omega_1, \ldots, \omega_N)$

$$\lambda_N^+ := \frac{N^+}{N},\ \lambda_N^- := \frac{N^-}{N}, \tag{12}$$

and the associated fluctuation process

$$\xi_N^+ := N^{1/2}\left(\lambda_N^+ - \frac{1}{2}\right),\ \xi_N^- := N^{1/2}\left(\lambda_N^- - \frac{1}{2}\right), \tag{13}$$

A usual Borel-Cantelli argument shows that almost-every sequence of i.i.d. r.v. $\{\omega_i\}_{i\geq 1}$ with law $\lambda = \frac{1}{2}(\delta_{+1} + \delta_{-1})$ satisfies the two following properties:

(1) *Law of large numbers*: for all $k \in \{\pm 1\}$, λ_N^k converges to $\frac{1}{2}$, as $N \to \infty$.

(2) *Central limit behavior*: for all $\zeta > 0$, there exists N_0 (possibly depending on the sequence $\{\omega_i\}_{i\geq 1}$) such that for all $N \geq N_0$,

$$\max_{k\in\{\pm 1\}} \left|\xi_N^k\right| \leq N^{\zeta}\,.$$

The main result of the paper is the following. The space $\mathbf{H}$ and parameters $\delta_1(K), \delta_2(K)$ below are defined in Section 3.3.1. The notation $\mathbf{P}$ stands for the law of the noise $\{B_i\}_{i\geq 1}$ and the initial condition $\{\varphi_{i,0}\}_{i\geq 1}$.

Theorem 3.4 ([26])**.** *For all $K > 1$, for all $\delta \leq \delta(K) := \min(\delta_1(K), \delta_2(K))$, there exists a linear form $b : \mathbb{R}^2 \to \mathbb{R}$ (depending on K, δ) and a real number $\varepsilon_0 > 0$ such that the following holds: for almost every i.i.d. sequence $\{\omega_i\}_{i\geq 1}$, any probability measure p_0 satisfying $dist_{\mathbf{H}}(p_0, \mathscr{C}) \leq \varepsilon_0$ such that for all $\varepsilon > 0$,*

$$\mathbf{P}\left(\|\mu_{N,0} - p_0\|_{\mathbf{H}} \geq \varepsilon\right) \to 0, \text{ as } N \to \infty\,, \tag{14}$$

there exists $\theta_0 \in \mathbb{S}$ (depending on p_0) and a constant c such that for each finite time $t_f > 0$ and all $\varepsilon > 0$, denoting $t_0^N = cN^{-1/2}\log N$, we have

$$\mathbf{P}\left(\sup_{t\in[t_0^N, t_f]} \left\|\mu_{N,N^{1/2}t} - q_{\theta_0 + b(\xi_N)t}\right\|_{\mathbf{H}} \geq \varepsilon\right) \to 0, \text{ as } N \to \infty\,. \tag{15}$$

Moreover, $\xi \mapsto b(\xi)$ has the following expansion in δ: for all ξ such that $\xi^+ + \xi^- = 0$, we have

$$b(\xi) \;=\; 2\delta\xi^+ + O(\delta^2)\,. \tag{16}$$

Theorem 3.4 is simply saying that, on a time scale of order $N^{1/2}$, the empirical measure (5) is asymptotically close to a synchronized profile $q \in \mathscr{C}$, traveling at speed $b(\xi_N)$ along $\mathscr{C}$. This drift depends on the asymmetry ξ_N of the quenched disorder $\{\omega_i\}_{i\geq 1}$. In (15), t_0^N represents the time necessary for the system to get sufficiently close to the manifold $\mathscr{C}$.

Remark 3.5. *We stress the fact that this random traveling waves mechanism is essentially a* quenched *phenomenon, that is, true for a fixed realization of the disorder $\{\omega_i\}_{i\geq 1}$. In particular, the result does not really depend on the underlying mechanism that produced the sequence $\{\omega_i\}_{i\geq 1}$, it only depends on the asymmetry of this sequence. Theorem 3.4 remains indeed valid for any sequence satisfying the law of large numbers and central limit behavior stated before Theorem 3.4. Extensions to asymmetries different from $N^{1/2}$ are also possible. We refer to [26], § 2, for details.*

3.3.3. *Idea of proof*

Let us give a quick sketch of proof of Theorem 3.4: writing a semimartingale decomposition of the empirical measure μ_N, one realizes that the dynamics of μ_N in a neighborhood of $\mathscr{C}$ is, at first order, given by the sum of several contributions: the deterministic linear dynamics induced by the operator L_q defined in (11), a drift term involving the asymmetry of the sequence of the disorder and a noise term. Hence, we divide the dynamics on $[0, tN^{1/2}]$ into subintervals of length of order 1 and proceed recursively on each of these subintervals. On each subinterval, the local orthogonal stability of the manifold $\mathscr{C}$ forces the system to stay close to $\mathscr{C}$ and to move essentially along the tangential direction, with a small drift of order $N^{-1/2}$. One difficulty is to prove that this small drift is mainly induced by the asymmetry of the disorder whereas the contribution of the noise is negligible. Projecting back the drifted empirical measure on the manifold $\mathscr{C}$, we proceed recursively on the next subinterval, so that the overall drift becomes of order 1 on a time of order $N^{1/2}$.

4. A characterization of the mixing measure of the vertex reinforced jump process via interacting Brownian motions

4.1. **Introduction**

In this manuscript we point out an intriguing relation between Brownian hitting time and the mixing measure of the Vertex reinforced jump processes. Let us start by recalling some well known fact on Brownian motion and its first hitting time to 0. Denote $(B_t)_{t\geq 0}$ the standard Brownian motion, first consider Brownian motion starting at 1:

$$X_t = 1 + B_t,$$

and define the stopping time

$$\tau = \inf\{t \geq 0,\ X_t = 0\}.$$

Classically, τ equals in law to the inverse of a Gamma random variable of parameter Gamma$(\frac{1}{2}, \frac{1}{2})$, and conditioned on the value of τ, X_t is a 3-dimensional Bessel bridge from 1 to 0 during $[0, \tau]$. Now for any $v > 0$, consider the drifted Brownian motion

$$X_t = 1 + B_t - vt$$

instead, the first hitting time to 0, τ is Inverse Gaussian distributed with parameter $IG(\frac{1}{v}, 1)$[1], again, conditioned on τ, X_t is 3-dimensional Bessel bridge from 1 to 0 during $[0, \tau]$.

Turning to the vertex reinforced jump process (abbreviate to VRJP in the sequel and is defined in 4.2), a breakthrough in the study of this self interacting processes with linearly reinforcement is the discovery of its mixing measure, with explicit density function, in a series of papers [12, 33, 34]. This mixing measure also plays a role in the supersymmetric hyperbolic sigma model introduced by Zirnbauer [39], and studied in e.g. [14].

In the sequel we present a generalization of the facts on 1-dimensional Brownian motion and its hitting time to higher dimension, that is, we construct an n dimensional drifted Brownian martingale, living on a weighted graph (with n being the number of vertices), in such a way that, its first hitting time to 0, as a random vector, is distributed according to the mixing measure of the VRJP on the same graph, and also, not surprisingly, conditioned on the value of this hitting time, this Brownian martingale becomes independent 3-dimensional Bessel bridges. This work has been done with C.Sabot and we refer to [35] for more details.

4.2. **The mixing measure of Vertex reinforced jump processes**

Consider a finite, non-directed and connected graph $\mathcal{G} = (V, E)$, endowed with some positive conductances $(W_e)_{e\in E}$. We use the identification $V = \{1, \dots, n\}$, and for any edge $e = \{i, j\}$, we denote indifferently $W_e = W_{i,j} = W_{j,i}$. The VRJP $(Y_t)_{t\geq 0}$ on $\mathcal{G}$ starting at $i_0 \in V$ with conductance $(W_e)_{e\in E}$ is a V valued continuous time process, such that $Y_0 = i_0$ a.s. and at time t, the process jumps from $Y_t = i$ to a neighbor j at rate

$$W_{i,j} L_j(t) \text{ where } L_j(t) = 1 + \int_0^t \mathbb{1}_{Y_s = j} ds.$$

Due to self interaction of its dynamic, the VRJP is not Markov, nevertheless, according to [33], the VRJP is a mixture of Markov jump processes, that is

$$\mathbb{P}_{i_0}^{\text{VRJP}}(\cdot) \overset{\text{up to time change}}{=} \int \mathbb{P}_{i_0}^{\text{Markov of parameter } \beta}(\cdot) d\nu_V^W(\beta)$$

where $\nu_V^W(d\beta)$ is called the mixing measure. Hence the study of this process is largely reduced to the study of its mixing measure, which we define now. Denote by P the matrix

[1] the Inverse Gaussian distribution, $IG(\lambda, \mu)$ is with density : $\left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right) \mathbb{1}_{x>0} dx$

$$P_{i,j} = \begin{cases} W_{i,j} & \text{if } i \neq j \text{ and } \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

If $(\beta_i)_{i\in V}$ is a function on the vertices, let $H_\beta = 2\beta - P$ where 2β represents the operator of multiplication by the diagonal matrix $\mathrm{Diag}(2\beta_i)$, the measure

$$d\nu_V^W(\beta) = \mathbb{1}_{H_\beta>0} \left(\frac{2}{\pi}\right)^{|V|/2} \exp\left(-\frac{1}{2}\langle 1, H_\beta 1\rangle\right) \frac{1}{\sqrt{\det H_\beta}} \prod_{i\in V} d\beta_i \tag{17}$$

is a probability distribution on $\mathbb{R}^V$, where $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^V$ and 1 in the inner product is the vector $\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$, $\mathbb{1}_{H_\beta>0}$ is the indicator that H_β is positive definite. Moreover, the Laplace transform of (17) is explicitly given by

$$\int e^{-\langle\lambda,\beta\rangle} \nu_V^W(d\beta) = \exp\left(-\sum_E W_{i,j}\left(\sqrt{(1+\lambda_i)(1+\lambda_j)}-1\right)\right) \prod_V \frac{1}{\sqrt{1+\lambda_i}}.$$

From this expression, it follows that for all $i \in V$, $\frac{1}{2\beta_i}$ follows an Inverse Gaussian law with parameters $(\frac{1}{\sum_{j:j\sim i} W_{i,j}}, 1)$. More details on this particular law are given in the next section, in particular a characterization that is to be discussed.

A generalization of (17) by Letac[2] is the following:

$$\nu_V^{W,\eta}(d\beta) = e^{-\frac{1}{2}\langle \eta, H_\beta^{-1}\eta\rangle + \langle 1,\eta\rangle} \nu_V^W(d\beta) \tag{18}$$

is a probability measure for any $\eta = (\eta_1, \ldots, \eta_n)$ such that $\eta_i \geq 0$ for all i. Moreover, let $V_1 \subset V$, then the marginal law on V_1 of ν_V^W is $\nu_{V_1}^{W_{V_1},\eta}$ with W_{V_1} the restriction of W to $V_1 \times V_1$ and $\eta = P_1 1$ where P_1 is the $V_1 \times (V \setminus V_1)$ block of P.

4.3. **Hitting times of Brownian motion**

In this section we provide the main guide lines of the proof of the fact on drifted Brownian motion and its hitting time discussed in 4.1, the proof of our main theorem (in 4.4) use same kind of ideas.

Let us recall this fact in a slightly more general form, let $X_t = a + B_t - vt$ be a drifted $\mathbb{P}$-Brownian motion with negative drift $-v < 0$ starting from $a > 0$, the first hitting time of 0

$$\tau = \inf\{t \geq 0,\ X_t = 0\},$$

is Inverse Gaussian distributed with parameters $(a/v, a^2)$. Conditionally on τ, $(X_t)_{0\leq t\leq\tau}$ has the law of a 3-dimensional Bessel bridge from a to 0 in time τ.

We provide one possible proof of this well known fact, the distribution of the hitting time can be deduced by explicit computation of its Laplace transform, by classical application of exponential martingale and optional stopping theorem. For the conditioned law, we first use Girsanov's theorem to kill the drift, that is, let $\mathbb{Q}_{|\mathcal{F}_t} = e^{vB_t - \frac{1}{2}v^2 t}\mathbb{P}_{|\mathcal{F}_t}$, then X_t is a $\mathbb{Q}$-Brownian motion starting from a. Under the law $\mathbb{Q}$, the distribution of τ is the law of the stopping time a Brownian motion starting from a first hit 0, its density is

$$f(a,t) = \mathbb{1}_{t>0} \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}.$$

[2] personal communication

Let $\hat{\mathbb{Q}}^T$ be the conditioned law $\mathbb{Q}(\cdot \mid \tau = T)$. By Doob's h-transform, we have

$$M_t := \left(\frac{d\hat{\mathbb{Q}}^T}{d\mathbb{Q}}\right)_{|\mathcal{F}_t} = \frac{X_t}{a}\frac{1}{\sqrt{(1-t/T)^3}}\exp\left(-\frac{X_t^2}{2(T-t)} + \frac{a^2}{2T}\right)$$

Now Ito's formula shows that

$$d\log(M_t) = \left(\frac{1}{X_t} - \frac{X_t}{T-t}\right)dX_t,$$

this means that, by Girsanov's theorem, under $\hat{\mathbb{Q}}^T$, X_t is the unique solution of the stochastic differential equation

$$X_t = B_t + \int_0^t \frac{1}{X_s} - \frac{X_s}{T-s}ds,$$

which is the SDE of a 3 dimensional Bessel bridge.

4.4. **Interacting drifted Brownian motion on a conductance network**

Let V_1 be a subset of V, for any vector $T = (T_i)_{i\in V_1} \in (\mathbb{R}_+ \cup \{\infty\})^{V_1}$, we denote $\mathrm{Diag}_V^T(t)$ the diagonal matrix with entries

$$\left(\mathrm{Diag}_V^T(t)\right)_{i,i} = \begin{cases} t \wedge T_i & \text{if } i \in V_1 \\ 0 & \text{if } i \in V \setminus V_1 \end{cases}$$

and

$$K_{V_1}^T(t) = \mathrm{Id} - \mathrm{Diag}_{V_1}^T(t)P.$$

Theorem 1. *Let $(B_i(t))_{1\le i\le n}$ be n dimensional standard Brownian motion, the following stochastic differential equations are well defined for all $t \ge 0$, and has unique pathwise solution:*

$$\begin{cases} X_i(t) = 1 + \int_0^t \mathbb{1}_{s<T_i} dB_i(s) - \int_0^t \mathbb{1}_{s<T_i}\left(P\psi(s)\right)_i ds & i \in V_1 \\ X_i(t) = 1, \ \forall t \ge 0 & i \in V \setminus V_1 \end{cases} \tag{19}$$

where $\psi(t) = (K_{V_1}^T(t))^{-1}X(t)$ and $T = (T_i)_{i\in V_1}$ is the random vector of stopping times defined by

$$T_k = \inf\{t \ge 0, \quad X_t(k) = 0\}, \ \forall k \in V_1.$$

Moreover, $T_k < \infty$ a.s. for all $k \in V_1$.

Theorem 2. *With the notations of the above theorem, $(\frac{1}{2T_i})_{i\in V_1}$ has law $\nu^{W_{V_1},\eta}$ with $\eta = P_1 1$. Conditionally on $(T_i)_{i\in V_1}$, $(X_t(i))_{i\in V_1}$ are independent 3-dimensional Bessel bridges from 1 to 0 during $[0, T_i]$ for each i.*

As examples, when $V = V_1 = \{1\}$, $X(t)$ is a Brownian motion with initial value 1, the hitting time T_1 is an inverse of a Gamma distribution. When $V_1 = \{i_1\} \subset V$, $X_{i_1}(t)$ is a Brownian motion with drift $(P1)_{i_1}$, we thus recover the two facts discussed in 4.1.

Note that the drift term in (19), $\mathbb{1}_{s<T_i}(P\psi(s))_i ds$ only involves local interactions, that is, it only depends on the values of the X_j for j neighbors of i, since the martingale term is just $\mathbb{1}_{s<T_i}dB_i(s)$, we can say that $(X_i(t))_{i\in V}$ are Brownian motions on the vertices with drift depending only on its neighbors, in particular, any marginal of the process $(X_i(t))_{i\in V}$ is a Brownian motion with constant drift, this is coherent with the fact that the marginal of $(\beta_i)_{i\in V}$ is Inverse Gaussian distribution.

For the proof of Theorem 2, as in 4.3, we replace the density of Gamma $f(x,t)$ by the density of $\nu_V^W(d\beta)$, then construct the corresponding martingale, surprising simplification will occur when we perform Ito differentiation on $\log M_t$, and it will result to independent Bessel bridges. For details, c.f. [35]

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