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DETERMINISTIC WALK ON POISSON POINT PROCESS

Simon Le Stum[1]

Abstract. A deterministic walk on a Poisson point process in $\mathbf{R}^d$ is an oriented graph where each point of the process is connected to only one other point following a deterministic and stationary rule of connection. In the paper we investigate the absence of percolation for such graphs and our main result is based on two assumptions. The Loop assumption ensures that any forward branch of the graph merges on a loop provided that the Poisson point process is augmented with a finite collection of well-chosen points. The Shield assumption ensures that the graph is locally determined with possible random horizons. Among the models which satisfy these general assumptions and inherit in consequence the finite cluster property, we focus on the deterministic walk to the k-th neighbour, with k any integer greater than one.

Résumé. Une marche déterministe sur un processus ponctuel de Poisson de $\mathbf{R}^d$ consiste à connecter chaque point du processus à un et un seul autre en suivant une règle déterministe invariante par translation. Dans ce papier, nous donnons un résultat général d'absence de composante connexe infinie pour de tels graphes dès que le modèle vérifie deux hypothèses. La première hypothèse, appelée hypothèse Loop, garantit que les branches descendantes du graphe finissent par boucler du moment qu'une collection bien choisie de points est ajoutée au processus initial. La seconde, appelée hypothèse Shield, assure que le graphe est déterminé localement avec éventuellement un horizon de détermination aléatoire. Parmi tous les modèles satisfaisant ces deux hypothèses nous nous intéressons tout particulièrement à la marche déterministe au k-ième plus proche voisin, avec k un entier plus grand que un.

Introduction

The classical nearest neighbour walk based on a planar homogeneous Poisson point process consists in connecting each point to its nearest neighbour. The absence of infinite cluster for this model is due to the fact that almost surely a homogeneous Poisson point process does not have a descending chain. By a descending chain, we mean an infinite sequence $x_1, x_2, ...$ of points of the process for which $|x_{i-1} - x_i| \geq |x_i - x_{i+1}|$ for all $i \geq 2$. Let k be an integer larger than 2, is there an infinite cluster if each vertex of a Poisson point process is connected to its k-th nearest neighbour? It is easy to notice that the descending chain argument is not available in this setting and that the graph is more complicated and strongly interlaced, especially when k is large. If, instead of connect each vertices to its k-th nearest neighbour, the vertex is connected randomly and uniformly among its first k neighbours, the absence of percolation could also be obtained with a descending chain argument. Indeed the uniform random choice ensures that, with positive probability, the son and the father of a vertex are the same. So there is an infinite number of opportunities for producing a loop and therefore the loop exists with probability one. Obviously each opportunity is possible provided that the father of the vertex belongs to its first k neighbours, which is related to a descending chain argument: roughly speaking, along each possible infinite branch, it appear long descending chains. When the choice among the k neighbours is deterministic (as,

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for instance, in the deterministic walk to the k-th neighbour) the problem is more complicated since the creation of a loop is more difficult. As far as we know, there does not exist a proof of the absence of percolation for the deterministic k-th nearest neighbour walk by using a descending chain argument.

In this paper, the absence of percolation for the k-th nearest neighbour walk in $\mathbf{R}^d$ is obtained as a consequence of our main result (Theorem 1.1). This result is a simplified version of Theorem 3.1 in [2] where the connection rule is possibly random. In our main theorem we consider general oriented outdegree-one graphs based on the vertices of a stationary Poisson point process. Since the graph is oriented, we can define the Forward and Backward sets of any given vertex x: see Section 1.2 for a precise definition. Then, the cluster containing x merely is the union of these both sets. The outdegree-one structure implies that any forward branch is finite if and only if it contains a loop. A loop is defined as a finite sequence $x_1, x_2, \ldots, x_n$ of different vertices such that x_i is connected to x_{i+1} for $1 \leq i \leq n-1$ and x_n is connected to x_1. A general argument for stationary outdregree-one graphs, called mass transport principle, and mentioned for example in [4] and [2], ensures that the absence of forward percolation implies the absence of backward percolation. On the oriented outdegree-one Poisson graph the aim of the work is to provide general assumptions ensuring that any forward branch of Poisson outdegree-one graphs merges on a loop.

The proof of our main theorem, given in [2] is based on a general statement for Poisson outdegree-one graphs which can be interpreted as a counterpart of the mass transport principle: roughly speaking, if there exists, with positive probability, an infinite forward branch then the expectation of the size of a typical backward branch is infinite. An important part of the work in [2] was to exhibit two assumptions which guarantee that such expectation is finite. Let us describe briefly these assumptions. The first one, called the Loop assumption, assumes that any forward branch merges to a loop if the process is augmented with a finite collection of well-chosen points (without reducing the size of the backward). This assumption ensures that a loop is possible along a forward branch provided that some points are added. In general this fact is obvious for all models for which the loops are possible. The extra condition on the size of the backward is directly related to the method we use. The second one, called the Shield assumption, is directly inspired from the ones done by Hirsch, see Section 3 of [4]). More or less, it assumes that with high probability, the graph contains no edge crossing large boxes. In this paper we show in particular that the k-th nearest neighbour walk in $\mathbf{R}^d$ satisfies these assumptions and therefore the absence of percolation occurs.

The paper is organized as follows. In Section 2, we provide a precise description of stationary deterministic walks built on a Poisson point process, illustrated by the example of the k-th nearest neighbour walk in $\mathbf{R}^d$. We achieve this section by describing our both assumptions and the main result (Theorem 1.1) ensuring the absence of percolation. Section 3 is devoted to check that the k-th nearest neighbour walk in $\mathbf{R}^d$ satisfies the assumptions of Theorem 1.1. Finally, in Section 4, we give a sketch of proof of Theorem 1.1.

1. Results

In this section, we give another version of Theorem 3.1 of [2] adapted to the deterministic walks on Poisson environment.

1.1. Notations

In this paper, all geometric models take place in the Euclidean space $\mathbf{R}^d$. The configuration space $\mathscr{C}$ on $\mathbf{R}^d$ is defined as

$$\mathscr{C} = \Big\{\varphi \subset \mathbf{R}^d \, ; \, N_\Lambda(\varphi) < \infty, \text{ for any bounded } \Lambda \subset \mathbf{R}^d\Big\}$$

where $N_\Lambda(\varphi) = \#\varphi_\Lambda$ denotes the number of points of φ which lie in Λ (for a given subset Λ of $\mathbf{R}^d$, and $\varphi \in \mathscr{C}$, φ_Λ denotes the set of points of φ included in Λ: $\varphi_\Lambda = \varphi \cap \Lambda$).

As usual, the configuration space $\mathscr{C}$ is equipped with the σ-algebra

$$\mathscr{S} = \sigma\Big(P_{(A,n)} \, ; \, A \subset \mathbf{R}^d, \, n \geq 0\Big) \, ,$$

generated by the counting events $P_{(A,n)} = \{\varphi \in \mathscr{C};\ N_\Lambda(\varphi) \leq n\}$. Similarly, for any subset $\Lambda \subset \mathbf{R}^d$, we define the σ-algebra of events in Λ by

$$\mathscr{S}_\Lambda = \sigma\Big(P_{(A,n)}\,;\, A \subset \Lambda,\, n \geq 0\Big) \ .$$

Let $v \in \mathbf{R}^d$. The translation operator τ_v acts on $\mathbf{R}^d$ and $\mathscr{C}$ as follows: for any $x \in \mathbf{R}^d$ and $\varphi \in \mathscr{C}$, we set $\tau_v(x) = v + x$ and $\tau_v(\varphi) = \cup_{x\in\varphi}\{\tau_v(x)\}$. Finally, a subset $\mathscr{C}' \subset \mathscr{C}$ is said translation invariant if $\tau_v(\mathscr{C}') = \mathscr{C}'$, for any vector $v \in \mathbf{R}^d$.

1.2. The outdegree-one model

In our setting, an *outdegree-one graph* is an oriented graph whose vertex set is given by a configuration $\varphi \in \mathscr{C}$ and has exactly one outgoing edge per vertex. Such graph can be described by a *graph function* which determines, for any vertex, its outgoing neighbour.

Definition 1.1. *Let $\mathscr{C}' \subset \mathscr{C}$ be a translation invariant set. A function h from $\mathscr{C}' \times \mathbf{R}^d$ to $\mathbf{R}^d$ is called a **graph function** if:*

(i) $\forall \varphi \in \mathscr{C}', \forall x \in \varphi,\ h(\varphi, x) \in \varphi\backslash\{x\}$;
(ii) $\forall v \in \mathbf{R}^d, \forall \varphi \in \mathscr{C}', \forall x \in \varphi,\ h(\tau_v(\varphi), \tau_v(x)) = \tau_v(h(\varphi, x))$.

*The couple $(\mathscr{C}', h)$ is then called an **outdegree-one model**.*

In the sequel, we consider an outdegree-one model $(\mathscr{C}', h)$ and a configuration $\varphi \in \mathscr{C}'$. The oriented graph is made up of edges $(x, h(\varphi, x))$, for all $x \in \varphi$. Like Figure 1 shows, this graph is not necessarily planar.

Let us describe the structure of the clusters. Let $x \in \varphi$. The *Forward set* $\mathrm{For}(x, \varphi)$ of x in φ is defined as the sequence of the outgoing neighbours starting at x:

$$\mathrm{For}(x, \varphi) = \{x, h(\varphi, x), h(\varphi, h(\varphi, x)), \ldots\} \ .$$

The outdegree-one property ensures that the Forward set is finite if and only if it contains a *loop*, i.e. a subset $\{y_1, \ldots, y_l\} \subset \mathrm{For}(x, \varphi)$, with $l \geq 2$, such that for any $1 \leq i \leq l$, $h(\varphi, y_i) = y_{i+1}$ (where the index $i + 1$ is taken modulo l). In this case, the integer l is called the *size* of the loop. The *Backward set* $\mathrm{Back}(x, \varphi)$ of x in φ contains all the vertices $y \in \varphi$ having x in their Forward set:

$$\mathrm{Back}(x, \varphi) = \{y \in \varphi\,;\, x \in \mathrm{For}(y, \varphi)\} \ .$$

The Backward set $\mathrm{Back}(x, \varphi)$ admits a tree structure whose x is the root. The Forward and Backward sets of x may overlap; they (at least) contain x. Their union forms the *Cluster* of x in φ:

$$C(x, \varphi) = \mathrm{For}(x, \varphi) \cup \mathrm{Back}(x, \varphi) \ .$$

Our main theorem (Theorem 1.1) states that for a large class of random models, all the clusters are a.s. finite. Furthermore, the outdegree-one property implies that there is at most one loop in a cluster. Hence, a finite cluster is made up of one loop with some finite trees rooted at vertices of the loop. Obviously, this notion of loops will be central in our study.

In the sequel, the configuration will be generated by a Poisson point process (PPP) $\mathbf{X}$ with intensity measure λ_d (the Lebesgue measure on $\mathbf{R}^d$). This means that the random variable $\#(\mathbf{X} \cap A)$ follows a Poisson distribution with parameter $\lambda_d(A)$, for any bounded set $A \subset \mathbf{R}^d$. By scaling, any other (stationary) intensity measure of the form $z\lambda_d$ with $z > 0$ could be considered.

Finally, let us denote by $(\Omega, \mathscr{F}, \mathbf{P})$ a probability space on which the PPP $\mathbf{X}$ is defined.

Definition 1.2. *Let $(\mathscr{C}', h)$ be an outdegree-one model. If $\mathbf{P}(\mathbf{X} \in \mathscr{C}') = 1$ then the triplet $(\mathscr{C}', h, \mathbf{X})$ is called a **random outdegree-one model**.*

The next section introduces a natural example of **random outdegree-one model** which generalizes the nearest neighbours random walk in a PPP.

1.3. The k-nearest neighbour walk in $\mathbf{R}^d$

Let $k \in \mathbf{N}^*$ be an integer which will be the range of the walk. Given $x \in \mathbf{R}^d$ and $r > 0$, $B(x, r)$ define the open Euclidiean ball of radius r and centred on x. Let $\varphi \in \mathscr{C}$ and $x \in \varphi$, we define the set of the k-nearest neighbour of x in φ:

$$N_k(\varphi, x) = \{y \in \varphi \ ; \ N_{B(x,\|x-y\|_2)}(\varphi) = k\}.$$

Let us define $\mathscr{C}'$ as well,

$$\mathscr{C}' = \{\varphi \in \mathscr{C} \ ; \ \forall x \in \varphi, \ \#N_k(\varphi, x) = 1\}.$$

For each $\varphi \in \mathscr{C}'$ and each $x \in \varphi$, $h(\varphi, x)$ corresponds to the only point of $N_k(\varphi, x)$. Using standard properties of the PPP, it is easy to show that $\mathbf{X}$ contains no isosceles triangles a.s, and therefore $(\mathscr{C}', h, \mathbf{X})$ is a random outdegree-one model.

As mentioned in the introduction, for $k \geq 2$, in a forward branch the length of the edges is not necessary decreasing .

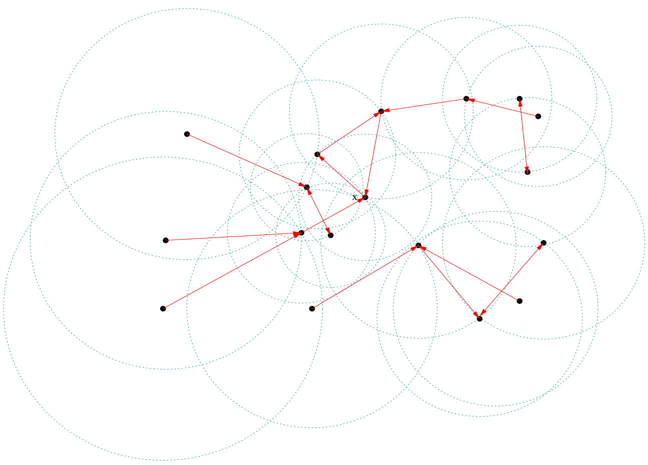

FIGURE 1. In this picture, the dimension d equals to 2 and $k = 3$. The oriented graph is drawn in red. We can see four clusters, three of them have a loop of size 2, the other one has a loop of size 3. The circles are drawn to help to recognize the 3-nearest neighbours of each point

Let us set others examples of deterministic walks on a Poisson point process. The hard sphere lilypond model built on a Poisson configuration introduced in [3] can be observed like a out-degree one graph. The authors prooved the absence of percolation for this model. We can also talk about the 2D-directed spanning forest, the authors of [1] prooved that this Poisson oriented graph is almost surely a tree (unicity of infinte cluster).

1.4. Assumptions and Theorems

We first establish in Theorem 1.1 the absence of percolation for all random outdegree-ones $(\mathscr{C}', h, \mathbf{X})$ satisfying two general assumptions, namely the *Loop and Shield assumptions*, which are described and commented below. Thus, Theorem 1.2 asserts that the k-nearest neighbour walk verifies these two assumptions and therefore does not percolate.

Loop assumption

The Loop assumption mainly expresses the possibility for any point $x \in \varphi$ to break its Forward set by adding a finite sequence of points $(x_1, \ldots, x_l)$ to the current configuration φ.

Let $\varphi \in \mathscr{C}'$ and l be a positive integer. The configuration φ is said *l-looping* if for any $x \in \varphi$, there exists an open set $A_x \subset \{(x_1, \dots, x_l) \in \mathbf{R}^{dl} \ ; \ \forall i \neq j, \ x_i \neq x_j\}$ such that, for all $(x_1, \dots, x_l) \in A_x$:

(i) $\text{For}(x, \varphi \cup \{x_1, \dots, x_l\}) \subset \{x, x_1, \dots, x_l\}$;
(ii) $\#\text{Back}(x, \varphi \cup \{x_1, \dots, x_l\}) \geq \#\text{Back}(x, \varphi)$.

Given x, conditions (i) and (ii) can be interpreted as a local modification of the configuration φ which breaks the Forward set of x without altering its Backward set– or at least without decreasing the cardinality of its Backward set. Whereas condition (i) is very natural to obtain a finite cluster, condition (ii) is more technical and will appear in the proof of Proposition 4.3 in [2]. The choice of the integer l will be specific to the random outdegree-one graph $(\mathscr{C}', h, \mathbf{X})$.

We will say that the random outdegree-one graph $(\mathscr{C}', h, \mathbf{X})$ satisfies the Loop assumption if there exists a positive integer l such that

$$\mathbf{P}(\mathbf{X} \text{ is } l\text{-looping }) = 1 \ .$$

Shield assumption

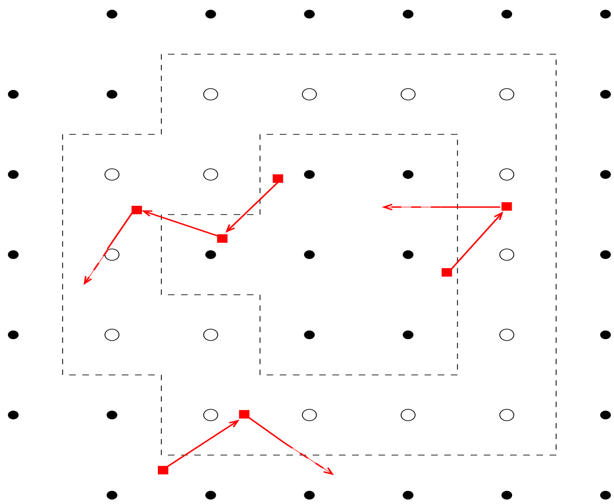

FIGURE 2. To simplify the picture, we have chosen $\alpha = \frac{1}{2}$ so that $\mathcal{A}_i = (A_i \oplus [\frac{-1}{2}; \frac{1}{2}]^d)$, $\forall i = 1, 2$. The white points are the elements of mV while the black ones are those of $m\mathcal{A}_1$ (inside mV) and $m\mathcal{A}_2$ (outside mV). Red squares are points of φ. If $\tau_{-mz}(\varphi) \in \mathscr{E}_m$ for all $z \in V$, then it is impossible for a (red) segment $[x; x']$, where $h(\varphi, x) = x'$, to cross the set $m(V \oplus [-\alpha; \alpha]^d)$ from $m\mathcal{A}_1$ to $m\mathcal{A}_2$– or from $m\mathcal{A}_2$ to $m\mathcal{A}_1$ by symmetry of Equation (2) w.r.t. indexes 1 and 2.

The Shield assumption is a kind of strong stabilizing property for the random outdegree-one graph $(\mathscr{C}', h, \mathbf{X})$ and has been first introduced in a slightly different way in [4].

We will say that the random outdegree-one graph $(\mathscr{C}', h, \mathbf{X})$ satisfies the Shield assumption if there exist a positive real α and a sequence of events $(\mathscr{E}_m)_{m \geq 1}$ such that:

(i) $\forall m \geq 1$, $\mathscr{E}_m \in \mathscr{S}_{[-\alpha m; \alpha m]^d}$;
(ii) $\mathbf{P}(\mathscr{E}_m) \underset{m \to \infty}{\longrightarrow} 1$;
(iii) Consider the lattice $\mathbf{Z}^d$ with edges given by $\{\{z, z'\}, |z - z'|_\infty = 1\}$ and any three disjoint subsets A_1, A_2, V of $\mathbf{Z}^d$ such that $\forall i = 1, 2$, the boundary $\partial A_i = \{z \in \mathbf{Z}^d \backslash A_i, \exists z' \in A_i, |z - z'|_\infty = 1\}$ is included in V. Let us set

$$\mathcal{A}_i = \left(A_i \oplus \left[\frac{-1}{2}; \frac{1}{2} \right]^d \right) \setminus (V \oplus [-\alpha; \alpha]^d) \ . \tag{1}$$

Then, for m sufficiently large and for any configurations $\varphi, \varphi' \in \mathscr{C}'$ such that $\tau_{-mz}(\varphi) \in \mathscr{E}_m$ for all $z \in V$, the following holds:

$$\forall x \in \varphi_{m\mathcal{A}_1},\ h(\varphi, x) = h(\varphi_{m\mathcal{A}_2^c} \cup \varphi'_{m\mathcal{A}_2}, x)\ . \tag{2}$$

In Condition (iii), the set mV acts as an uncrossable obstacle, i.e. a shield between sets $m\mathcal{A}_1$ and $m\mathcal{A}_2$. See Figure 2. Equation (2) says that the outgoing neighbour of any $x \in \varphi_{m\mathcal{A}_1}$ does not depend on the configuration on $m\mathcal{A}_2$. In particular, $h(\varphi, x) \in \varphi_{m\mathcal{A}_2^c}$.

Here are our main results.

Theorem 1.1. *Any random outdegree-one graph $(\mathscr{C}', h, \mathbf{X})$ satisfying the Loop and Shield assumptions does not percolate with probability* 1*:*

$$\mathbf{P}(\forall x \in \mathbf{X},\ \#C(x, \mathbf{X}) < \infty) = 1\ .$$

Theorem 1.1 is proved in [2] in a more general context and a sketch of its proof is given in Section 3 of this paper.

Checking that the model given in Section 1.3 satisfies the Loop and Shield assumptions, we get:

Theorem 1.2. *For each $k \in \mathbf{N}^*$, the k-nearest neighbour walk in $\mathbf{R}^d$ does not percolate with probability* 1*.*

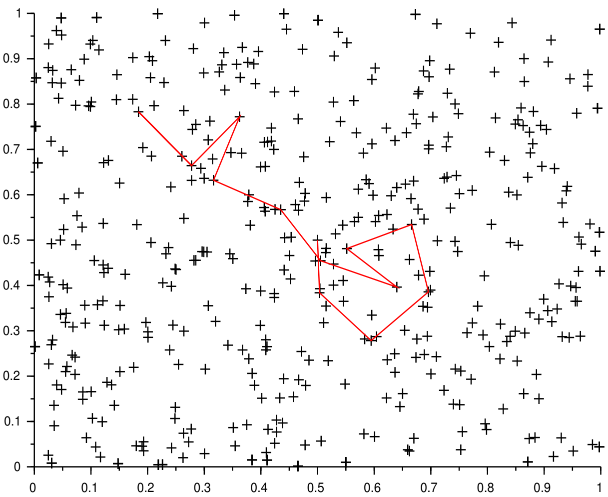

FIGURE 3. In this simulation, we draw, in a $[0,1]^2$, the 30-nearest neighbour walk starting to $(\frac{1}{2}, \frac{1}{2})$. The intensity value is 500. We observe that the forward component of $(\frac{1}{2}, \frac{1}{2})$ is finite.

Hence, we will see in Proposition 2.1 that l points suffice to make a loop for the l-nearest neighbour walk.

2. Proof of Theorem 1.2

This section is devoted to the verifications of Loop and Shield assumptions for the k**-nearest neighbour walk**. A positive integer k is fixed in all this section.

Let us introduce some usual notations. For a given vertex $x \in \varphi$, we split the Euclidean ball $B(x, \|x - h(\varphi,x)\|_2)$ into k disjoint regions. By induction, for $i \in \mathbf{N}$ we define $v_i(\varphi, x)$ as follows; $v_0(\varphi, x) = x$, and, for $i \geq 1$ $v_i(\varphi, x)$ is the unique nearest neighbour of x in the configuration $\varphi \setminus \{v_0(\varphi,x), \ldots, v_{i-1}(\varphi,x)\}$. Precisely, for $i \geq 1$, $v_i(\varphi, x)$ is the unique i^{th} nearest neighbour of x in φ, in particular, $h(\varphi, x) = v_k(\varphi, x)$. Thus we set, for $i \in \{1, \ldots, k\}$,

$$\begin{array}{rcl} C_1(\varphi,x) & = & B(x, \|x - v_1(\varphi,x)\|_2), \\ \forall i \in \{2,\ldots,k\},\ C_i(\varphi,x) & = & B(x, \|x - v_i(\varphi,x)\|_2) \setminus B(x, \|x - v_{i-1}(\varphi,x)\|_2). \end{array}$$

Then, we obtain

$$B(x, \|x - h(\varphi,x)\|_2) = \bigcup_{1 \leq i \leq k} C_i(\varphi, x).$$

2.1. Loop assumption

The **k-nearest neighbour walk** satisfies the Loop assumption.

Proposition 2.1. *Almost all configurations of $\mathscr{C}'$ is k-looping.*

Proof: We prove that there exists $\mathscr{C}'' \subset \mathscr{C}'$ such that any configuration of $\mathscr{C}''$ is k-looping and $\mathbf{P}[\mathscr{C}''] = 1$.

For a given point $x \in \varphi$, create a loop in the forward set of x is relatively easy, it is sufficient to reduce the radius of the open ball $C_1(\varphi, x)$ and to put exactly k points inside. In the rest of the proof, we note by $\mathbf{B}(\varphi, x)$ the ball $B(x, \frac{\|x - v_1(\varphi,x)\|_2}{2})$. The difficulty is that nothing ensures that the size of the backward set of x is not reduced. Precisely, it could exist a point $y \in \text{Back}(\varphi, x)$ such that $B(y, \|y - h(\varphi,y)\|_2)$ contains at least one point among the k points added. In this new configuration, the k^{th}-nearest neighbour of y would be changed. Let us consider the set of points

$$\mathscr{E}(x,\varphi) = \{y \in \varphi \setminus \{x\}\ ;\ B(y, \|y - h(\varphi,y)\|_2) \cap \mathbf{B}(\varphi, x) \neq \emptyset\}.$$

The main part of the proof consists in checking that if $\mathscr{E}(x,\varphi)$ is finite, then there exists an open set A_x which satisfies the two items of the Loop assumption. Then, the following subset of configurations is introduced

$$\mathscr{C}'' = \{\varphi \in \mathscr{C}'\ ;\ \forall x \in \varphi,\ \#\mathscr{E}(x,\varphi) < \infty\}$$

First, we check that $\mathbf{E}[\#\mathscr{E}(0, \mathbf{X} \cup \{0\})] < \infty$ following the strategy described by the proof of Lemma 5.1 in [2].

Now we have to show that all configurations in $\mathscr{C}''$ are k-looping. We fix $\varphi \in \mathscr{C}''$ and $x \in \varphi$. Since $\mathscr{E}(x,\varphi)$ is finite, then there exists an open ball Γ such that, for all $y \in \mathscr{E}(x,\varphi)$, and for all $i \in \{1,\ldots,k\}$,

$$C_i(\varphi, y) \cap \Gamma \neq \emptyset \implies \Gamma \subset C_i(\varphi, y). \tag{3}$$

This implication is illustrated in Figure 4.

The diameter of Γ is chosen smaller than the distance between x and Γ to ensure that the k^{th}-nearest neighbour of each point added is x. In other words, for all distinct points $x_1, \ldots, x_k \in \Gamma$ and for all $i \in \{1,\ldots,k\}$, $h(\varphi \cup \{x_1,\ldots,x_k\}, x_i) = x$.

Let us define the open set

$$A_x = \{(x_1, x_2, \ldots, x_k) \in \Gamma^k\ ;\ \forall i \neq j,\ x_i \neq x_j\}.$$

Then, we can verify that for all $(x_1, \ldots, x_k) \in A_x$,

(i) $\text{For}(x, \varphi \cup \{x_1,\ldots,x_k\}) \subset \{x, x_1, \ldots, x_k\}$;
(ii) $\#\text{Back}(x, \varphi \cup \{x_1,\ldots,x_k\}) \geq \#\text{Back}(x,\varphi)$.

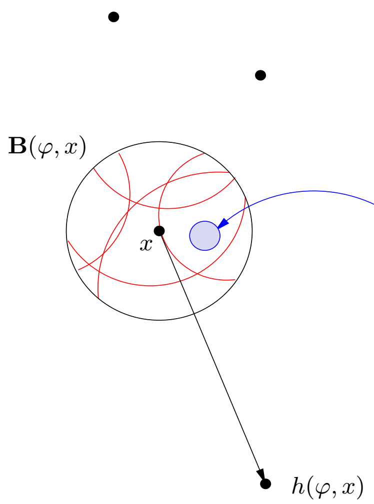

FIGURE 4. For each $y \in \mathscr{E}(x, \varphi)$, the boundary of $C_i(\varphi, y)$ is drawn in red. There is a finite number of red arc which overlap $\mathbf{B}(\varphi, x)$, that is why it is possible to insert a ball which is not overlapped by any arc.

The first item is obtained as a consequence of the two following facts; for $(x_1, \ldots, x_k) \in A_x$, $h(\varphi \cup \{x_1, \ldots, x_k\}, x) \in \{x_1, \ldots, x_k\}$ (because $x_i \in C_1(\varphi, x)$ for all $i \in \{x_1, \ldots, x_k\}$) and, $h(\varphi \cup \{x_1, \ldots, x_k\}, x_i) = x$ for any $i \in \{1, \ldots, k\}$.

The second item is obtained as a consequence of (3): Let $(x_1, \ldots, x_k) \in A_x$ and $y \in \mathrm{Back}(\varphi, x) \setminus \{x\}$. There exist $n \in \mathbf{N}^*$ and $y_0, \ldots, y_n \in \varphi$ such that; $y_0 = y$, $y_n = x$ and $y_{j+1} = h(\varphi, y_j)$ for all $j \in \{0, \ldots, n-1\}$. Two situations may occur:

- If, for all $j \in \{0, \ldots, n-1\}$ and for all $i \in \{1, \ldots, k\}$, $C_i(\varphi, y_j) \cap \mathbf{B}(\varphi, x) = \emptyset$ then, y is clearly still in the backward set of x since $h(\varphi \cup \{x_1, \ldots, x_k\}, y_i) = y_{i+1}$ for any $i \in \{1, \ldots, n-1\}$.
- Otherwise, we consider the first index $j_0 \in \{0, \ldots, n-1\}$ such that there exists $i \in \{1, \ldots, k\}$ satisfying $x_1, \ldots, x_k \in C_i(\varphi, y)$. It implies that $v_k(\varphi \cup \{x_1, \ldots, x_k\}, y_{j_0}) \in \{x_1, \ldots, x_k\}$, then, $h(\varphi \cup \{x_1, \ldots, x_k\}, y_{j_0}) \in \{x_1, \ldots, x_k\}$ and y is still in the backward set of x since $h(\varphi \cup \{x_1, \ldots, x_k\}, x_i) = x$ for any $i \in \{1, \ldots, k\}$.

The Loop assumption is proved for this model. □

2.2. Shield assumption

Let us split the hypercube $[-m; m]^d$ into $\kappa = (d\lfloor m^{1/d} \rfloor)^d$ congruent subcubes $Q_1^m, \ldots, Q_\kappa^m$ ($\lfloor \cdot \rfloor$ denotes the integer part). Each of these subcubes has an area equal to

$$\left(\frac{dm}{d\lfloor m^{1/d} \rfloor} \right)^d ,$$

i.e. of order m^{d-1}. Thus, we define the event $\mathscr{E}_m$ as follows:

$$\mathscr{E}_m = \bigcap_{1 \le i \le \kappa} \left\{ \#\mathbf{X}_{Q_i^m} \ge k \right\} .$$

Proposition 2.2. *For $\alpha = 1$, the k-**nearest neighbour walk** satisfies the Shield assumption w.r.t. the family of events $(\mathscr{E}_m)_{m \ge 1}$.*

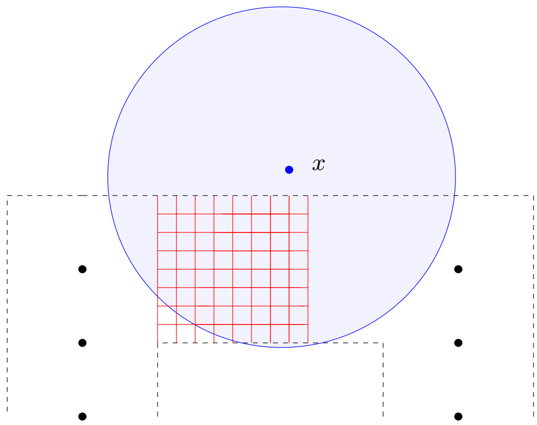

FIGURE 5. Black points are vertices mz for $z \in V$. The event $\mathscr{E}_m$ realized on each $z \oplus [-m;m]^d$ provides a shield between $m\mathcal{A}_1$ and $m\mathcal{A}_2$. Indeed, a given ball centred on x cannot overlap $m\mathcal{A}_2$ without containing a subsquare $z + Q_i^m$.

Proof: Let us first remark that the event $\mathscr{E}_m$ is $\mathscr{S}_{[-m,m]^d}$-measurable and its probability tends to 1. So the first two items of the Shield assumption are satisfied with $\alpha = 1$.

Let us focus on Item (iii). Hence, let us consider $V, A_1, A_2 \subset \mathbf{Z}^d$ such that the topological conditions of the shield assumption occur. Let us consirer $\mathcal{A}_i$ as defined in (1) for $i \in \{1, 2\}$. Let $\varphi \in \mathscr{C}'$ satisfying $\varphi - mz \in \mathscr{E}_m$, for any vertex $z \in V$. Let $x \in \varphi$ be a point which belongs to $m\mathcal{A}_i$. It is sufficient to remark that for any m, any open ball centred on x which overlaps $m\mathcal{A}_i$ contains at least one subsquare $z + Q_i^m$. Hence, the outgoing vertex $h(\varphi, x)$ remains unchanged. □

3. Sketch of the proof of Theorem 1.1

Theorem 1.1 is rigorously proved in [2]. Let us talk about the main steps and arguments of the proof.

We have to prove that any random outdegree-one model which satisfies the Loop and Shield assumptions, does not contain any infinite cluster with probability 1:

$$\mathbf{P}\big(\forall x \in \mathbf{X},\ \#\mathrm{For}(x, \mathbf{X}) < \infty \ \text{ and } \ \#\mathrm{Back}(x, \mathbf{X}) < \infty\big) = 1\ . \tag{4}$$

First, thanks to a standard mass transport argument, we can reduce the proof of the absence of percolation to the absence of forward percolation. This argument is only implied by the stationarity of $\mathbf{X}$ and is proved in [4] and [2] :

$$\mathbf{P}\big(\forall x \in \mathbf{X},\ \#\mathrm{For}(x, \mathbf{X}) < \infty\big) = 1 \implies \mathbf{P}\big(\forall x \in \mathbf{X},\ \#\mathrm{Back}(x, \mathbf{X}) < \infty\big) = 1\ . \tag{5}$$

Reasoning by contradiction, we assume that, with positive probability, an infinite forward branch starts at a typical point 0:

$$\mathbf{P}\big(\#\mathrm{For}(0, \mathbf{X}_0) = \infty\big) > 0\ , \tag{6}$$

In all the rest of this paper, $\mathbf{X}_0$ denotes the configuration $\mathbf{X} \cup \{0\}$. The main part of the proof consists in showing that any infinite forward branch contains an infinite number of particular vertices (called Almost looping points in [2]). These particular vertices have important characteristic: the region where we add the l points creating a loop in their forward contains a ball of radius sufficiently large close to the vertex. To prove this, we use a classical stochastic domination result of Liggett, Schommann and Stacey [5].

$$\mathbf{P}\big(\#\{y \in \mathrm{For}(0, \mathbf{X}_0);\ y \text{ is an almost looping point of } \mathbf{X}_0\} = \infty\big) > 0\ . \tag{7}$$

Heuristically, such event should not occur since it produces an infinite number of opportunities to break the branch by adding points. Then, a result (Proposition 4.5 in [2]) allows to convert the forward result to a backward one. Precisely, the equation (7) implies,

$$\mathbf{E}\big[\#\mathrm{Back}(0,\mathbf{X}_0)\mathbf{1}_{\{0 \text{ is an almost looping point of } \mathbf{x}_0\}}\big] = \infty \ . \tag{8}$$

By adding some suitable marked points, it follows that the mean size of the Backward set of a typical point which has a **finite forward** is infinite (it is the only place where we need the condition (*ii*) of the Loop assumption). But this last result is impossible by another use of the mass transport principle. This contradiction finishes the proof.

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