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REMARK ON THE WELL-POSEDNESS OF WEAKLY DISPERSIVE EQUATIONS

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Abstract. We improve the results about the well-posedness of the regularized fractional dispersive equation $(1 + D_x^\alpha)u_t + u_x + uu_x = 0$ when $0 < \alpha \leq 1$. When $\alpha < 1$, the existence and uniqueness of vanishing viscosity solution is proved.

Introduction

Fractional Benjamin-Bona-Mahony (fBBM) equation

$$u_t + u_x + uu_x + D_x^\alpha u_t = 0$$

was introduced by Linares, Pilod and Saut [16] to investigate the role of weak dispersion ($0 < \alpha < 1$) on the solution of the Burgers equation

$$u_t + u_x + uu_x = 0.$$

They showed the local in time well-posedness in Sobolev spaces using energy estimates. Nevertheless, this method does not provide the uniqueness. Indeed, the difference $w = u - v$ between two solutions satisfies (see inequality (4.31) of [16])

$$\begin{aligned} \frac{d}{dt}||w||^2_{H^{s+\frac{\alpha}{2}}} \quad \leq \quad & \int_{\mathbb{R}} |\left((-\Delta)^{s/2}(u+v)\right) w_x (-\Delta)^{s/2} w| dx + \frac{1}{2}\int_{\mathbb{R}} |(u+v)_x (-\Delta)^{s/2} w|^2 dx \\ & + \int_{\mathbb{R}} |R(-\Delta)^{s/2} w| dx + \int_{\mathbb{R}} |\left((-\Delta)^{s/2}(u+v)_x\right) w (-\Delta)^{s/2} w| dx, \end{aligned}$$

where $\int_{\mathbb{R}} |R(-\Delta)^{s/2} w| dx \leq ||u+v||_{H^{s+\frac{\alpha}{2}}} ||w||^2_{H^{s+\frac{\alpha}{2}}}$. But the last term can not be uniformly controlled by the $H^{s+\frac{\alpha}{2}}-$norm if $0 < \alpha < 1$.

When $0 < \alpha < 1$, it seems difficult to obtain the global well-posedness. It has been shown by Bona and Saut [3] that the linearization around 0 has blow-up solution and Klein, Saut numerically observe blow-up when $0 < \alpha < \frac{1}{3}$ [15]. While $\alpha = 1$, global well-posedness can be obtained thanks to Brezis-Gallouët estimates [5, 18].

The paper is organized as follows. In Section 2, we improve the regularity of the global well-posedness when $\alpha = 1$. In Section 3, we deal with the uniqueness of vanishing viscosity solution for $0 < \alpha < 1$.

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1. The regularized Benjamin-Ono equation

When $\alpha = 1$, the equation can be rewritten as the Benjamin-Ono equation under the form

$$u_t + H(u_x)_t + u_x + uu_x = 0,$$

where H is the Hilbert defined by its Fourier symbol $\widehat{H(u)}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{u}(\xi)$.

Theorem 1.1. *Let $\alpha = 1$ and $u_0 \in H^{\frac{1}{2}}(\mathbb{R})$. There exists a unique solution $u \in \mathcal{C}(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{R}))$ of the initial value problem*

$$\begin{cases} u_t + u_x + uu_x + D_x u_t = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1}$$

Moreover, for all $t \in \mathbb{R}$

$$\|u(t)\|_{H^{\frac{1}{2}}} = \|u_0\|_{H^{\frac{1}{2}}},$$

and the map $u_0 \in H^{\frac{1}{2}}(\mathbb{R}) \to u \in \mathcal{C}(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{R}))$ is continuous.

The proof is done in two steps: first, a compactness argument is used to obtain a weak solution, then the uniqueness of the weak solution provides the strong continuity of the weak solution.

Let $(u_{n,0})_{n\in\mathbb{N}}$ be a sequence of $H^1(\mathbb{R})$ such that $u_{n,0} \to u_0$ in $H^{\frac{1}{2}}(\mathbb{R})$. We denote by $u_n(t) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}))$ the solution of the initial value problem associated with the initial datum $u_{n,0}$. Then it is proved that the energy [18]

$$E(t) = \int_{\mathbb{R}} u_n^2 + ||D_x|^{\frac{1}{2}} u_n|^2 dx$$

is preserved with respect to time, *i.e.*

$$\|u_n\|^2_{H^{\frac{1}{2}}} = \|u_{n,0}\|^2_{H^{\frac{1}{2}}} \leq C,$$

and the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{R}))$.
On the other hand, multiplying (1) by $\partial_t u_n$ and integrating over space gives

$$\int_{\mathbb{R}} |\partial_t u_n|^2 dx = -\int_{\mathbb{R}} \partial_t u_n \partial_x (1 + D_x)^{-1}(u_n + \frac{u_n^2}{2}) dx,$$

and Young's inequality provides, for $\varepsilon > 0$.

$$\begin{aligned}
\|\partial_t u_n\|^2_{L^2} = \int_{\mathbb{R}} |\partial_t u_n|^2 dx &\leq \Big| \int_{\mathbb{R}} \partial_t u_n \partial_x (1 + D_x)^{-1} u_n dx + \int_{\mathbb{R}} \partial_t u_n \partial_x (1 + D_x)^{-1} \frac{u_n^2}{2} dx \Big| \\
&\leq \varepsilon \int_{\mathbb{R}} |\partial_t u_n|^2 dx + \frac{1}{2\varepsilon} \int_{\mathbb{R}} |\partial_x (1 + D_x)^{-1} u_n|^2 dx + \frac{1}{2\varepsilon} \int_{\mathbb{R}} |\partial_x (1 + D_x)^{-1} \frac{u_n^2}{2}|^2 dx \\
&\leq \varepsilon ||\partial_t u_n||^2_{L^2} + \frac{1}{2\varepsilon} ||\partial_x (1 + D_x)^{-1} u_n||^2_{L^2} + \frac{1}{2\varepsilon} \left\| \partial_x (1 + D_x)^{-1} \frac{u_n^2}{2} \right\|^2_{L^2} \\
&\leq \varepsilon \|\partial_t u_n\|^2_{L^2} + \frac{1}{4\varepsilon} \big(\|u_n\|^2_{L^2} + \|u_n^2\|^2_{L^2} \big) \\
&\leq \varepsilon \|\partial_t u_n\|^2_{L^2} + \frac{1}{4\varepsilon} \big(\|u_n\|^2_{L^2} + \|u_n\|^4_{L^4} \big) \\
&\leq \varepsilon \|\partial_t u_n\|^2_{L^2} + \frac{1}{4\varepsilon} \big(\|u_n\|^2_{L^2} + \|u_n\|^4_{H^{\frac{1}{2}}} \big).
\end{aligned}$$

Taking $\varepsilon = \frac{1}{2}$, it gets

$$\|\partial_t u_n\|_{L^2} \leq \|u_n\|^2_{H^{\frac{1}{2}}} \leq \|u_{n,0}\|^2_{H^{\frac{1}{2}}} \leq C.$$

Thus,

$$||u_n(t) - u_n(s)||_{L^2} = \left\| \int_s^t \partial_t u_n(r) dr \right\|_{L^2} \leq \int_s^t \|\partial_t u_n(r)\|_{L^2} \, dr \leq C|t-s|,$$

and the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded and equicontinuous.
We deduce according to the Rellich theorem for all $T > 0$, there exists $u \in \mathcal{C}_w([0,T]; H^{\frac{1}{2}}(\mathbb{R})) \bigcap \mathcal{C}([0,T]; L^2(\mathbb{R}))$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $\forall t \in [0,T]$

$$u_{n_k}(t) \rightharpoonup u(t) \text{ in } H^{\frac{1}{2}}(\mathbb{R}) \text{ and } u_{n_k}(t) \to u(t) \text{ in } L^2(\mathbb{R}).$$

Since the subsequence $(u_{n_k})_{k\in\mathbb{N}}$ satisfies for all $v \in \mathcal{C}_c^\infty(\mathbb{R})$

$$\int_0^T \int_{\mathbb{R}} v\partial_t u_{n_k} + v\partial_t D_x u_{n_k} + v\partial_x \left(u_{n_k} + \frac{u_{n_k}^2}{2} \right) dx\, dt = 0,$$

the limit verifies

$$\int_0^T \int_{\mathbb{R}} u\partial_t v + u\partial_t D_x v + \left(u + \frac{u^2}{2} \right) \partial_x v \, dx\, dt = 0.$$

The function $u \in \mathcal{C}_w([0,T]; H^{\frac{1}{2}}(\mathbb{R})) \bigcap \mathcal{C}([0,T]; L^2(\mathbb{R}))$ is a weak solution of the equation (1) and

$$\|u(t)\|_{H^{\frac{1}{2}}} \leq \liminf_{k\to\infty} \|u_{n_k}(t)\|_{H^{\frac{1}{2}}} \leq \|u_0\|_{H^{\frac{1}{2}}}. \tag{2}$$

Suppose now that the weak solution of the Cauchy problem $u \in \mathcal{C}_w([0,T]; H^{\frac{1}{2}}(\mathbb{R})) \bigcap \mathcal{C}([0,T]; L^2(\mathbb{R}))$ is unique. By reversing time in the regularized Benjamin-Ono equation (1), we have

$$\|u(-t)\|_{H^{\frac{1}{2}}} \leq \|u(0)\|_{H^{\frac{1}{2}}} \text{ or } \|u(0)\|_{H^{\frac{1}{2}}} \leq \|u(t)\|_{H^{\frac{1}{2}}}.$$

From inequality (2) and from the uniqueness of the weak solution, we obtain

$$\|u(t)\|_{H^{\frac{1}{2}}} = \|u_0\|_{H^{\frac{1}{2}}}$$

and the solution u belongs to $\mathcal{C}([0,T]; H^{\frac{1}{2}}(\mathbb{R}))$. Note that we also obtain the continuity with respect to the initial data since we proved that for $u_{n,0} \to u_0$ in $H^{\frac{1}{2}}(\mathbb{R})$, then the respective solution (u_n) verifies $u_n \to u$ in $\mathcal{C}([0,T]; H^{\frac{1}{2}}(\mathbb{R}))$.

It remains to prove the uniqueness of the weak solution. We are inspired by the method introduced by Yudovich [21]. We need a Trudinger-type estimates proved by Gérard and Grellier [9] in the torus $\mathbb{T}$.

Lemma 1.2. *There exists a constant $C > 0$ such that for all $2 < p < \infty$, we have*

$$\|u\|_{L^p(\mathbb{R})} \leq C\sqrt{p}\|u\|_{H^{\frac{1}{2}}(\mathbb{R})}.$$

Proof. The proof is similar to [9] except that u is split as

$$u = u_{>\lambda} + u_{<\lambda} = \frac{1}{2\pi} \int_{|\xi|\leq\lambda} e^{i\xi x} \widehat{u}(\xi) d\xi + \frac{1}{2\pi} \int_{|\xi|\geq\lambda} e^{i\xi x} \widehat{u}(\xi) d\xi.$$

□

Let u and $v \in \mathcal{C}_w([0,T];H^{\frac{1}{2}}(\mathbb{R})) \cap \mathcal{C}([0,T];L^2(\mathbb{R}))$ be two weak solutions of (1) starting from the same initial datum. Consider the function g defined as

$$g(t) = \|u(t) - v(t)\|_{L^2}^2.$$

Then, for $w := u - v$ and $P(D_x) := \partial_x(1 + D_x)^{-1}$, we have

$$\begin{aligned} g'(t) &= 2\int_{\mathbb{R}} (u-v)(u_t - v_t)dx = -2\int_{\mathbb{R}} (u-v)P(D_x)\left(u + \frac{u^2}{2} - v - \frac{v^2}{2}\right) \\ &= -2\int_{\mathbb{R}} wP(D_x)(w) + wP(D_x)\left(w(u+v)\right)dx = -2\int_{\mathbb{R}} wP(D_x)\left(w(u+v)\right)dx \\ &= -\int_{\mathbb{R}} w^2 P(D_x)(u+v)dx - \int_{\mathbb{R}} w(P(D_x)w)(u+v)dx - \int_{\mathbb{R}} wRdx \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III} \end{aligned}$$

where

$$R = P(D_x)(w(u+v)) - (u+v)P(D_x)w - wP(D_x)(u+v).$$

Since $w^2 = w^{2\left(1-\frac{1}{p}\right)} w^{\frac{2}{p}}$, the Hölder inequality provides

$$\begin{aligned} |\mathrm{I}| = \int_{\mathbb{R}} w^2 P(D_x)(u+v)dx &\leq \|w^{2\left(1-\frac{1}{p}\right)}\|_{L^{\frac{p}{p-1}}} \|w^{\frac{2}{p}} P(D_x)(u+v)\|_{L^p} \\ &\lesssim \|w\|_{L^2}^{2\left(1-\frac{1}{p}\right)} \left(\|u\|_{L^4}^{\frac{2}{p}} + \|v\|_{L^4}^{\frac{2}{p}}\right) \|P(D_x)(u+v)\|_{L^{2p}}. \end{aligned}$$

We note that the operator $P(D_x)$ is bounded in L^p, for $1 < p < +\infty$ according to the Mikhlin-Hörmander theorem [17, 19]. The Sobolev and the Trudinger inequalities imply

$$|\mathrm{I}| \lesssim \|w\|_{L^2}^{2\left(1-\frac{1}{p}\right)} \left(\|u\|_{L^4}^{\frac{2}{p}} + \|v\|_{L^4}^{\frac{2}{p}}\right) (\|u\|_{L^{2p}} + \|v\|_{L^{2p}}) \lesssim \sqrt{2p}\, g(t)^{1-\frac{1}{p}} \left(\|u\|_{L^4}^{\frac{2}{p}} + \|v\|_{L^4}^{\frac{2}{p}}\right) (\|u\|_{H^{\frac{1}{2}}} + \|v\|_{H^{\frac{1}{2}}})$$

and from inequality (2)

$$|\mathrm{I}| \lesssim \sqrt{p} g(t)^{1-\frac{1}{p}}.$$

Similarly, we have

$$\begin{aligned} |\mathrm{II}| = \int_{\mathbb{R}} w(P(D_x)w)(u+v)dx &\leq \|w^{1-\frac{2}{p}} P(D_x)w\|_{L^{\frac{p}{p-1}}} \|w^{\frac{2}{p}}(u+v)\|_{L^p} \\ &\lesssim \|w^{1-\frac{2}{p}}\|_{L^{\frac{2p}{p-2}}} \|P(D_x)w\|_{L^2} \left(\|u\|_{L^4}^{\frac{2}{p}} + \|v\|_{L^4}^{\frac{2}{p}}\right) \|u+v\|_{L^{2p}} \\ &\lesssim \|w\|_{L^2}^{1-\frac{2}{p}} \|w\|_{L^2} (\|u\|_{L^{2p}} + \|v\|_{L^{2p}}) \\ &\lesssim g(t)^{1-\frac{1}{p}} (\|u\|_{L^{2p}} + \|v\|_{L^{2p}}) \lesssim \sqrt{p} g(t)^{1-\frac{1}{p}} \|u\|_{H^{\frac{1}{2}}} \lesssim \sqrt{p} g(t)^{1-\frac{1}{p}}. \end{aligned}$$

We can write thanks to the fractional Leibniz rule [14].

Lemma 1.3. *We have for* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$

$$\|P(D)(uv) - vP(D)(u) - uP(D)(v)\|_{L^p} \lesssim \|u\|_{L^q} \|v\|_{L^r}.$$

We deduce

$$|\mathrm{III}| = \int_{\mathbb{R}} wRdx \lesssim \|w\|_{L^2}\|u+v\|_{L^q}\|w\|_{L^r} \lesssim \|w\|_{L^2}(\|u\|_{L^q}+\|v\|_{L^q})\|w\|_{L^2}^{1-\frac{2}{p}}\|w\|_{L^p}^{\frac{2}{p}}$$

where

$$\frac{1}{r}=\frac{1}{2}-\frac{1}{q} \text{ or } \frac{1}{r}=\frac{1}{2}-\frac{1}{p}\left(1-\frac{2}{p}\right).$$

Thus

$$|\mathrm{III}| \lesssim \sqrt{q}\|w\|_{L^2}^{2-\frac{2}{p}}\|w\|_{L^p}^{\frac{2}{p}}\|u\|_{H^{\frac{1}{2}}} \lesssim \sqrt{q}g(t)^{1-\frac{1}{p}}(\|u\|_{L^p}^{\frac{2}{p}}+\|v\|_{L^p}^{\frac{2}{p}}) \lesssim p^{\frac{1}{p}}\sqrt{q}g(t)^{1-\frac{1}{p}}\|u\|_{H^{\frac{1}{2}}}^{\frac{2}{p}} \lesssim p^{\frac{1}{p}}\sqrt{q}g(t)^{1-\frac{1}{p}}.$$

Taking $p>2$ large enough so that

$$|\mathrm{III}| \lesssim p^{\frac{1}{p}}\sqrt{\frac{p}{1-\frac{2}{p}}}g(t)^{1-\frac{1}{p}} \lesssim \sqrt{p}g(t)^{1-\frac{1}{p}}$$

implies

$$|g'(t)| \le C\sqrt{p}g(t)^{1-\frac{1}{p}}, \quad \forall 2<p<\infty,$$

with $C=C(\|u_0\|_{H^{\frac{1}{2}}},T)$ independent of p. Then

$$g(t)^{\frac{1}{p}} \le C\frac{t}{\sqrt{p}},$$

or

$$g(t) \le C\frac{t^p}{p^{\frac{p}{2}}} \to 0, \quad \text{as } p\to\infty.$$

Finally, $g(t)\equiv 0$, and $u=v$.

2. The weak dispersive equation

Let us come back to the initial value problem, for $0<\alpha<1$,

$$\begin{cases} u_t+u_x+uu_x+D_x^{\alpha}\partial_t u=0, \\ u(x,0)=u_0(x). \end{cases} \tag{3}$$

The existence is proved in [16] using energy estimates.

Theorem 2.1. *Let $0<\alpha<1, r>\max\left(1,\frac{3}{2}-\alpha\right)$ and $u_0\in H^r(\mathbb{R})$. Then the Cauchy problem has at least one solution in $H^r(\mathbb{R})$.*

We briefly remind the proof in order to highlight the loss of uniqueness (see [16] for details).

Proof. For $r=s+\frac{\alpha}{2}$, we define $J^s=(I-\Delta)^{\frac{s}{2}}$ by its Fourier transform

$$\widehat{J^s u}(\xi) := \widehat{(I-\Delta)^{\frac{s}{2}}u}(\xi)=(1+\xi^2)^{\frac{s}{2}}\widehat{u}(\xi).$$

Applying the operator J^s to (3), multiplying by $J^s u$ and integrating by part over space, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}|J^s u|^2+|J^s D_x^{\frac{\alpha}{2}}u|^2dx &= \int_{\mathbb{R}}-J^s uJ^s(uu_x)dx \\ &= \int_{\mathbb{R}}|J^s u|^2u_x+uJ^s uJ^s\partial_x u+RJ^s udx = \frac{1}{2}\int_{\mathbb{R}}|J^s u|^2u_x+\int_{\mathbb{R}}RJ^s udx, \end{aligned}$$

where, thanks to Leibniz's rule,

$$J^s(uu_x) = uJ^su_x + u_xJ^su + R,$$

with, for all $0 < \epsilon < s$,

$$\|R\|_{L^{\frac{2}{1+\alpha}}} \lesssim \|J^{s-\epsilon}u\|_{L^{\frac{4}{1+\alpha}}} \|J^\epsilon u_x\|_{L^{\frac{4}{1+\alpha}}}.$$

The Hölder inequality provides

$$\int_{\mathbb{R}} |(J^su)^2u_x|dx \le \|u_x\|_{L^{\frac{2}{1+\alpha}}} \|J^su\|^2_{L^{\frac{2}{1-\alpha}}}$$

and

$$\int_{\mathbb{R}} |RJ^su|dx \le \|R\|_{L^{\frac{2}{1+\alpha}}} \|J^su\|_{L^{\frac{2}{1-\alpha}}}.$$

Using Sobolev's embedding

$$H^{\frac{\alpha}{2}}(\mathbb{R}) \hookrightarrow L^{\frac{2}{1-\alpha}},\ H^{\frac{\alpha}{2}+\epsilon} \hookrightarrow L^{\frac{4}{1+\alpha}},\ H^{s+\frac{\alpha}{2}-\epsilon-1} \hookrightarrow L^{\frac{4}{1+\alpha}},$$

we finally find

$$\frac{d}{dt}\|u\|^2_{H^{s+\frac{\alpha}{2}}} \lesssim ||u||^2_{H^{s+\frac{\alpha}{2}}}$$

and we conclude with the Gronwall lemma. □

Let us try to prove the uniqueness. Let u and v be two solutions of (3) and denote $w = u - v$. We have

$$w_t + \partial_x(1 + D^\alpha_x)^{-1}\left(w + \frac{1}{2}w(u + v)\right) = 0.$$

Using the fractional Leibniz rule and integrating by parts, it gets

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}} |J^sw|^2 + |J^{s+\frac{\alpha}{2}}w|^2dx \quad &\le \quad \int_{\mathbb{R}} |J^s(u+v)w_xJ^sw|dx + \frac{1}{2}\int_{\mathbb{R}} |(u+v)_x||J^sw|^2dx \\ &\quad + \int_{\mathbb{R}} |RJ^sw|dx + \int_{\mathbb{R}} |J^s(u+v)_xwJ^sw|dx.\end{aligned}$$

Here, for all $0 < \epsilon < s$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$,

$$\|R\|_{L^2} \le \|J^{s-\epsilon}(u + v)\|_{L^p}\|J^\epsilon w\|_{L^q}.$$

Even if the three first terms are bounded by $||u+v||_{H^{s+\frac{\alpha}{2}}}||w||^2_{H^{s+\frac{\alpha}{2}}}$, the last one can not be uniformly controlled by the $H^{s+\frac{\alpha}{2}}-$norm when $0 < \alpha < 1$.

To avoid this difficulty, we propose the following regularization [11]

$$\begin{cases} u^\varepsilon_t + u^\varepsilon_x + u^\varepsilon u^\varepsilon_x + D^\alpha_x u^\varepsilon_t - \varepsilon u^\varepsilon_{xx} = 0, \\ u^\varepsilon(0, x) = u^\varepsilon_0(x). \end{cases} \tag{4}$$

Lemma 2.2. *Let $0 < \alpha < 1$ and $r \ge 0$. We have*

$$\left\|\partial_x(1 + D^\alpha_x)^{-1}S_t(uv)\right\|_{H^r} \le C(\varepsilon, t)\,\|u\|_{H^r}\,\|v\|_{H^r},$$

where

$$S_tu = \mathcal{F}^{-1}\left(\mathrm{e}^{-\frac{i\xi+\varepsilon\xi^2}{1+|\xi|^\alpha}t}\right) * u(x), \quad \textit{and} \quad C(\varepsilon, t) = C(\varepsilon t)^{-\frac{\frac{7}{4}-\alpha}{2-\alpha}}.$$

Proof. The idea of the proof is introduced in [4, 13]. By duality, it is enough to show that for all function $w \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} \partial_x (1+D_x^\alpha)^{-1} S_t\big(u(x)v(x)\big)\overline{w}(x)dx \le C(t)\,\|u\|_{H^r}\|v\|_{H^r}\|w\|_{H^{-r}}.$$

The Plancherel identity provides

$$\begin{aligned}\int_{\mathbb{R}} \partial_x (1+D_x^\alpha)^{-1} S_t\big(u(x)v(x)\big)\overline{w}(x)dx &= \int_{\mathbb{R}} \frac{i\xi}{1+|\xi|^\alpha} \mathrm{e}^{-\frac{i\xi+\varepsilon\xi^2}{1+|\xi|^\alpha}t}\widehat{u(\xi)v(\xi)}\overline{\widehat{w}}(\xi)d\xi \\ &= \int_{\mathbb{R}} \frac{i\xi}{1+|\xi|^\alpha} \mathrm{e}^{-\frac{i\xi+\varepsilon\xi^2}{1+|\xi|^\alpha}t}\widehat{u}*\widehat{v}(\xi)\overline{\widehat{w}}(\xi)d\xi \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{i\xi}{1+|\xi|^\alpha} \mathrm{e}^{-\frac{i\xi+\varepsilon\xi^2}{1+|\xi|^\alpha}t}\widehat{u}(\xi-\eta)\widehat{v}(\eta)d\eta\,\overline{\widehat{w}}(\xi)d\xi.\end{aligned}$$

Let us denote $\langle\xi\rangle = (1+\xi^2)^{\frac12}$, $\widehat{U} = \langle\xi\rangle^r\widehat{u}$, $\widehat{V} = \langle\xi\rangle^r\widehat{v}$, $\widehat{W} = \langle\xi\rangle^{-r}\widehat{w}$, it is necessary to prove

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\frac{i\xi+\varepsilon\xi^2}{1+|\xi|^\alpha}t}\frac{|\xi|\langle\xi\rangle^r}{\langle\xi\rangle^\alpha\langle\xi-\eta\rangle^r\langle\eta\rangle^r}\widehat{U}(\xi-\eta)\widehat{V}(\eta)\overline{\widehat{W}}(\xi)d\xi d\eta\right| \le C(\varepsilon,t)\,\|u\|_{H^r}\|v\|_{H^r}\|w\|_{H^{-r}}.$$

The triangle inequality $\langle\xi\rangle^r \lesssim \langle\xi-\eta\rangle^r\langle\eta\rangle^r$ implies

$$\begin{aligned}&\left|\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\frac{i\xi+\varepsilon\xi^2}{1+|\xi|^\alpha}t}\frac{|\xi|\langle\xi\rangle^r}{\langle\xi\rangle^\alpha\langle\xi-\eta\rangle^r\langle\eta\rangle^r}\widehat{U}(\xi-\eta)\widehat{V}(\eta)\overline{\widehat{W}}(\xi)d\xi d\eta\right| \\ &\qquad\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}}\langle\xi\rangle^{1-\alpha}e^{-\frac{\varepsilon\xi^2}{1+|\xi|^\alpha}t}|\widehat{U}(\xi-\eta)\widehat{V}(\eta)\overline{\widehat{W}}(\xi)|d\xi d\eta \\ &\qquad\lesssim \|W\|_{L^2_\xi}\left\|\langle\xi\rangle^{1-\alpha}e^{-\frac{\varepsilon\xi^2}{1+|\xi|^\alpha}t}\int_{\mathbb{R}}\widehat{U}(\xi-\eta)\widehat{V}(\eta)d\eta\right\|_{L^2_\xi} \\ &\qquad\lesssim \|W\|_{L^2_\xi}\left\|\langle\xi\rangle^{1-\alpha}e^{-\frac{\varepsilon\xi^2}{1+|\xi|^\alpha}t}\right\|_{L^2_\xi}\left\|\int_{\mathbb{R}}\widehat{U}(\xi-\eta)\widehat{V}(\eta)d\eta\right\|_{L^\infty_\xi} \\ &\qquad\lesssim \|W\|_{L^2_\xi}\|U\|_{L^2_\xi}\|V\|_{L^2_\xi}\left\|\langle\xi\rangle^{1-\alpha}e^{-\frac{\varepsilon\xi^2}{1+|\xi|^\alpha}t}\right\|_{L^2_\xi} \\ &\qquad\lesssim \|w\|_{H^{-r}}\|u\|_{H^r}\|v\|_{H^r}\left\|\langle\xi\rangle^{1-\alpha}e^{-\frac{\varepsilon\xi^2}{1+|\xi|^\alpha}t}\right\|_{L^2_\xi}.\end{aligned}$$

Since for $x\in\mathbb{R},\theta\in\mathbb{R}$, we have $e^{-x}\le\frac{1}{x^\theta}$, we deduce

$$\left\|\langle\xi\rangle^{1-\alpha}e^{-\frac{\varepsilon\xi^2}{1+|\xi|^\alpha}t}\right\|_{L^2_\xi} \le \left\|\langle\xi\rangle^{1-\alpha}\frac{1}{(\frac{\varepsilon\xi^2 t}{1+|\xi|^\alpha})^\theta}\right\|_{L^2_\xi} \le \frac{1}{(\varepsilon t)^\theta}\left\|\frac{1}{\langle\xi\rangle^{(2-\alpha)\theta-(1-\alpha)}}\right\|_{L^2_\xi} \lesssim \frac{1}{(\varepsilon t)^\theta},$$

if $(2-\alpha)\theta-(1-\alpha) > 1/2$, *i.e.* $\theta > \frac{\frac32-\alpha}{2-\alpha}$. □

Lemma 2.3. *Let $0<\alpha<1$, $r\ge 0$ and $u_0^\varepsilon\in H^r(\mathbb{R})$, then there exist a time $T_\varepsilon = T(\varepsilon) > 0$ and a unique solution $u^\varepsilon\in\mathcal{C}([0,T_\varepsilon];H^r(\mathbb{R}))$.*

Proof. Thanks to the Duhamel formula, u^ε is solution of (4) if and only if u^ε is the fixed point of Φ^ε defined as

$$\Phi^\varepsilon u^\varepsilon(t) := S_t u_0^\varepsilon - \frac12\int_0^t S_{t-\tau}\left((1+D_x^\alpha)^{-1}\partial_x u^\varepsilon(\tau)^2\right)d\tau$$

where $S_t u = \mathcal{F}^{-1}\Big(e^{-\frac{i\xi+\varepsilon\xi^2}{1+|\xi|^\alpha}t}\Big) * u(x)$. Let $\overline{B}_T$ be the closed ball

$$\overline{B}_T := \left\{u \in \mathcal{C}([0,T];H^r(\mathbb{R})), \|u\|_{L^\infty([0,T],H^r(\mathbb{R}))} \leq 2\|u_0\|_{H^r}\right\}.$$

We prove that $\Phi^\varepsilon(\overline{B}_T) \subseteq \overline{B}_T$ and Φ^ε is a contraction mapping on $\overline{B}_T$. Let $u^\varepsilon \in \overline{B}_T$, we have

$$\begin{aligned}\|\Phi^\varepsilon u^\varepsilon(t)\|_{H^r(\mathbb{R})} &= \left\|S_t u_0 - \frac{1}{2}\int_0^t S_{t-\tau}(1+D_x^\alpha)^{-1}\partial_x u^\varepsilon(\tau)^2 d\tau\right\|_{H^r} \\ &\leq \|u_0^\varepsilon\|_{H^r} + \int_0^t \left\|S_{t-\tau}(1+D_x^\alpha)^{-1}\partial_x u^\varepsilon(\tau)^2\right\|_{H^r} d\tau.\end{aligned}$$

Lemma 2.2 gives

$$\left\|S_{t-\tau}(1+D_x^\alpha)^{-1}\partial_x u^\varepsilon(\tau)^2\right\|_{H^r} \leq C\big(\varepsilon(t-\tau)\big)^{-\frac{\frac{7}{4}-\alpha}{2-\alpha}} \|u^\varepsilon(\tau)\|_{H^r}^2.$$

Note that $0 < \frac{\frac{7}{4}-\alpha}{2-\alpha} < 1$, and

$$\|\Phi^\varepsilon u^\varepsilon(t)\|_{H^r} \leq \|u_0^\varepsilon\|_{H^r} + C_{\varepsilon,r}T\|u^\varepsilon\|_{L^\infty([0,T],H^r)}^2.$$

Thus, choosing $T < \frac{1}{C_{\varepsilon,r}\|u_0^\varepsilon\|_{H^r}}$ implies

$$\|\Phi^\varepsilon u^\varepsilon(t)\|_{L^\infty([0,T],H^r)} \leq 2\|u_0^\varepsilon\|_{H^r}.$$

Let $u_1^\varepsilon, u_2^\varepsilon \in \overline{B}_T$, one gets from Lemma 2.2

$$\begin{aligned}\|\Phi^\varepsilon u_1^\varepsilon(t) - \Phi^\varepsilon u_2^\varepsilon(t)\|_{H^r} &\leq \int_0^t \left\|S_{t-\tau}(1+D_x^\alpha)^{-1}\partial_x\big((u_1^\varepsilon)^2-(u_2^\varepsilon)^2\big)\right\|_{H^r} d\tau \\ &\leq \int_0^t C\big(\varepsilon(t-\tau)\big)^{-\frac{\frac{7}{4}-\alpha}{2-\alpha}} \|u_1^\varepsilon+u_2^\varepsilon\|_{H^r}\|u_1^\varepsilon-u_2^\varepsilon\|_{H^r} d\tau \\ &\leq C_{\varepsilon,r}T\|u_0^\varepsilon\|_{H^r}\|u_1^\varepsilon-u_2^\varepsilon\|_{L^\infty([0,T];H^r)},\end{aligned}$$

and Φ^ε is a contraction mapping on $\overline{B}_T$ as soon as $T < \frac{1}{C_{\varepsilon,r}\|u_0^\varepsilon\|_{H^r}}$. Finally, there exists a unique fixed point $u^\varepsilon := \Phi^\varepsilon(u^\varepsilon)$ in $\overline{B}_T$ for $T < \frac{1}{C_{\varepsilon,s}\|u_0^\varepsilon\|_{H^r}}$.

To obtain the continuity with respect to the initial data, for u_0^ε and v_0^ε in $H^r(\mathbb{R})$ with $\|u_0^\varepsilon\|_{H^r} \leq M$, $\|v_0^\varepsilon\|_{H^r} \leq M$, we consider $u^\varepsilon, v^\varepsilon$ the respective solution. Let $0 \leq t \leq T = \frac{1}{C_{\varepsilon,s}M}$, we find similarly

$$\begin{aligned}\|u^\varepsilon(t)-v^\varepsilon(t)\|_{H^r} &\leq \|S_t(u_0^\varepsilon-v_0^\varepsilon)\|_{H^r} + \int_0^t \left\|S_{t-\tau}(1+D_x^\alpha)^{-1}\partial_x((u^\varepsilon)^2-(v^\varepsilon)^2)\right\|_{H^r} d\tau \\ &\leq \|u_0^\varepsilon-v_0^\varepsilon\|_{H^r} + C_{\varepsilon,r}TM\|u^\varepsilon-v^\varepsilon\|_{L^\infty([0,T];H^r)},\end{aligned}$$

or in other words, since $1 - C_{\varepsilon,r}TM > 0$,

$$\|u^\varepsilon(t)-v^\varepsilon(t)\|_{H^r} \leq \frac{1}{1-C_{\varepsilon,r}TM}\|u_0^\varepsilon-v_0^\varepsilon\|_{H^r}.$$

□

Lemma 2.4. *Let $0 < \alpha < 1$, $r \geq 2 - \frac{\alpha}{2}$ and $u_0^\varepsilon \in H^r(\mathbb{R})$, then T_ε can be chosen independent of ε.*

Proof. Let $r = s + \frac{\alpha}{2}$. Applying J^s to (4), multiplying by $J^s u^\varepsilon$ and integrating by parts over space, it comes

$$\frac{d}{dt}\left(\int_{\mathbb{R}} (J^s u^\varepsilon)^2 + (J^s D_x^{\frac{\alpha}{2}} u^\varepsilon)^2 dx\right) + 2\varepsilon \int_{\mathbb{R}} (J^s u_x^\varepsilon)^2 dx = \int_{\mathbb{R}} -J^s u^\varepsilon J^s (u^\varepsilon u_x^\varepsilon) dx.$$

We obtain from Kato-Ponce commutator estimates [12]

$$\int_{\mathbb{R}} J^s u^\varepsilon J^s (u^\varepsilon u_x^\varepsilon) dx = \frac{1}{2}\int_{\mathbb{R}} u_x^\varepsilon (J^s u^\varepsilon)^2 dx + \int_{\mathbb{R}} R J^s u^\varepsilon dx \leq C \left\| u^\varepsilon \right\|^3_{H^{s+\frac{\alpha}{2}}} .$$

It follows that

$$\frac{d}{dt} \left\| u^\varepsilon \right\|^2_{H^{s+\frac{\alpha}{2}}} \leq C \left\| u^\varepsilon \right\|^3_{H^{s+\frac{\alpha}{2}}}$$

or in other words $\left\| u^\varepsilon \right\|^2_{H^{s+\frac{\alpha}{2}}} \leq y(t)$ where $y(t)$ satisfies the ordinary differential equation

$$\begin{cases} y'(t) = C y(t)^{\frac{3}{2}} \\ y(0) = \left\| u_0^\varepsilon \right\|^2_{H^{s+\frac{\alpha}{2}}(\mathbb{R})}, \end{cases}$$

which solution is given by

$$y(t) = \frac{y(0)}{(1 - C y(0)^{1/2} t)^2} = \frac{\left\| u_0^\varepsilon \right\|^2_{H^{s+\frac{\alpha}{2}}}}{(1 - C \left\| u_0^\varepsilon \right\|_{H^{s+\frac{\alpha}{2}}} t)^2},$$

and u^ε can be extended until $T = \frac{1}{2C\left\| u_0^\varepsilon \right\|_{H^{s+\frac{\alpha}{2}}}}$. □

Theorem 2.5. *Let $0 < \alpha < 1$, $r \geq 2 - \frac{\alpha}{2}$ and $u_0 \in H^r(\mathbb{R})$. Then there exist a time $T > 0$ and a unique vanishing viscosity solution $u \in \mathcal{C}([0,T]; H^r(\mathbb{R}))$ of the initial value problem (3). In other words, the solution u^ε of the initial value problem* (4) *converges, when $\varepsilon \to 0$, uniformly to u solution of* (3) *in $\mathcal{C}([0,T]; H^r(\mathbb{R}))$.*

Proof. We prove that $(u^\varepsilon)_{\varepsilon \geq 0}$ is a Cauchy sequence in the Sobolev space $H^r(\mathbb{R})$. Let u^ε and v^δ be two solutions of (4). The difference $w = u^\varepsilon - v^\delta$ satisfies

$$w_t + w_x + D_x^\alpha \partial_t w - \delta w_{xx} + \big(u^\varepsilon w - \frac{w^2}{2}\big)_x = (\varepsilon - \delta) u_{xx}^\varepsilon. \tag{5}$$

One multiplies (5) by w and integrates over space to obtain

$$\frac{d}{dt}\int_{-\infty}^{+\infty} w^2 + (D_x^{\frac{\alpha}{2}} w)^2 dx \quad = \quad \int_{-\infty}^{+\infty} -2\delta w_x^2 - u_x^\varepsilon w^2 + 2(\varepsilon - \delta) u_{xx}^\varepsilon w dx \leq \int_{-\infty}^{+\infty} -u_x^\varepsilon w^2 + 2(\varepsilon - \delta) u_{xx}^\varepsilon w dx$$

and Sobolev's inequality provide

$$\frac{d}{dt}\int_{-\infty}^{+\infty} w^2 + (D_x^{\frac{\alpha}{2}} w)^2 dx \quad \leq \quad C||u||_{H^{2-\frac{\alpha}{2}}} \left\| w \right\|^2_{H^{\frac{\alpha}{2}}} + C(\varepsilon - \delta)||u||_{H^{2-\frac{\alpha}{2}}} \left\| w \right\|_{H^{\frac{\alpha}{2}}} .$$

Finally

$$\frac{d}{dt} \left\| w \right\|^2_{H^{\frac{\alpha}{2}}} \leq C \left\| w \right\|^2_{H^{\frac{\alpha}{2}}} + C(\varepsilon - \delta) \left\| w \right\|_{H^{\frac{\alpha}{2}}}$$

and the Gronwall Lemma offers a limit of $(u^\varepsilon)_{\varepsilon \geq 0}$ in $H^{\frac{\alpha}{2}}(\mathbb{R})$.

From the preceding lemma, the map $t \in [0,T] \to u^\varepsilon(t)$ is continuous and uniformly bounded. In particular, the sequence $(u^\varepsilon(t))_{\varepsilon \geq 0}$ is weakly convergent in $H^s(\mathbb{R}), s \geq \frac{\alpha}{2}$ to $u(t)$ a weakly continuous and uniformly bounded function. We deduce

$$t \in [0,T] \to (1 + D_x^\alpha)^{-1} (\partial_x u + u \partial_x u) \in H^{s+1-\alpha}(\mathbb{R})$$

is weakly continuous and

$$u(t) = u_0 - \int_0^t (1 + D_x^\alpha)^{-1} (\partial_x u + u \partial_x u) \text{ in } H^{s+1-\alpha}(\mathbb{R})$$

is unique. Indeed, let $v \in H^{s+1-\alpha}(\mathbb{R})$ defined by

$$v(t) = v_0 - \int_0^t (1 + D_x^\alpha)^{-1} (\partial_x v + v \partial_x v),$$

we obtain with similar computations

$$\frac{d}{dt} \|u - v\|^2_{H^{\frac{\alpha}{2}}} \leq C \|u - v\|^2_{H^{\frac{\alpha}{2}}}$$

and the Gronwall lemma allows to conclude. □

References

[1] J.L. Bona, and R. Smith, *The initial-value problem for the Korteweg-de Vries equation*, Philos. Trans. Roy. Soc. London Ser. A 278, 1287 (1975) 555–601.
[2] T.B. Benjamin, J.L. Bona, and J.J Mahony, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Roy. Soc. London Ser. A 272, 1220 (1972) 47-78.
[3] J.L. Bona, and J. C. Saut, *Dispersive blow-up II. Schrödinger-type equations, optical and oceanic rogue waves*, Chinese Ann. Math. Ser. B 31, 6 (2010) 793-818.
[4] J.L. Bona, and N. Tzvetkov, *Sharp well-posedness results for the BBM equation*, Discrete Contin. Dyn. Syst 23, 7 (2009) 1241-1252.
[5] H. Brézis, and T. Gallouët, *Nonlinear Schrödinger evolution equations*, Nonlinear Anal. 4 (1980) 677-681.
[6] N. Burq, and F. Planchon, *On well-posedness for the Benjamin-Ono equation*, Mathematische Annalen 340, 3 (2008) 497-542.
[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$*, Journal American Math. Soc. 16, 3 (2003) 705-749.
[8] A. Friedman, *Partial differential equations of parabolic type*, Courier Corporation, 2013.
[9] P. Gérard, and S. Grellier, *The cubic Szegö equation*, Annales École Normale Sup. 43, 5 (2010) 761-810.
[10] A. Ionescu, and C. Kenig, *Global well-posedness of the Benjamin-Ono equation in low-regularity spaces*, Journal American Math. Soc. 20,3 (2007) 753-798.
[11] R.J. Iório Jr., *KdV, BO and friends in weighted Sobolev spaces*, Functional-Analytic Methods for Partial Differential equations, Lecture Notes in Math. 1450, Springer-Verlag, Berlin, (1990) 104-121.
[12] T. Kato, and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Commun. Pure Appl. Math. 41,7 (1988) 891-907.
[13] C.E. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. American Math. Soc. 9,2 (1996) 573-603.
[14] C.E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Commun. Pure Appl. Math. 46, 4 (1993) 527-620.
[15] C. Klein, and J.C. Saut, *A numerical approach to blow-up issues for dispersive perturbations of Burgers' equation*, Physica D 295 (2015) 46-65.
[16] F. Linares, D. Pilod, and J.C. Saut, *Dispersive perturbations of Burgers and hyperbolic equations I: local theory*, SIAM Journal Math. Ana. 46, 2 (2014) 1505-1537.
[17] Y. Mammeri, *On the decay in time of solutions of some generalized regularized long waves equations*, CPAA Comm. Pure Appl. Ana. 7, 3, (2008), 513-532.
[18] Y. Mammeri, *Long time bounds for the periodic Benjamin-Ono-BBM equation*, Nonlinear Analysis TMA 71, 10 (2009) 5010-5021.
[19] E. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, (1993).
[20] T. Tao, *Global well-posedness of the Benjamin-Ono equation in $H^1(R)$*, Journal Hyperbolic Diff. Eq. 1 (2004) 27-49.
[21] V.I. Yudovich, *Non-stationary flows of an ideal incompressible fluid*, Zhurnal Vychislitel'noi Matematikii Matematicheskoi Fiziki, 3, 6 (1963) 1032-1066.