FREE BOUNDARY VALUE PROBLEMS AND HJB EQUATIONS FOR THE STOCHASTIC OPTIMAL CONTROL OF ELASTO-PLASTIC OSCILLATORS

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Abstract. We consider the optimal stopping and optimal control problems related to stochastic variational inequalities modeling elasto-plastic oscillators subject to random forcing. We formally derive the corresponding free boundary value problems and Hamilton-Jacobi-Bellman equations which belong to a class of nonlinear partial differential equations with nonlocal Dirichlet boundary conditions. Then, we focus on solving numerically these equations by employing a combination of Howard's algorithm and the numerical approach [A backward Kolmogorov equation approach to compute means, moments and correlations of non-smooth stochastic dynamical systems; Mertz, Stadler, Wylie; 2017] for this type of boundary conditions. Numerical experiments are given.

Résumé. Nous considérons les problèmes de l'arrêt optimal et du contrôle optimal associés à des inéquations variationnelles stochastiques modélisant des oscillateurs élasto-plastiques sous forçage aléatoire. Nous obtenons formellement les problèmes à frontière libre et les équations de Hamilton-Jacobi-Bellman correspondantes qui appartiennent à une classe d'équations aux dérivées partielles nonlinéaires avec des conditions aux bord de Dirichlet nonlocales. Ensuite, nous nous concentrons sur la résolution numérique de ces équations en employant une combinaison de l'algorithme d'Howard et de l'approche numérique [A backward Kolmogorov equation approach to compute means, moments and correlations of non-smooth stochastic dynamical systems; Mertz, Stadler, Wylie; 2017] pour ce type de conditions aux bord. Des expériences numériques sont présentées.

1. MOTIVATIONS AND BACKGROUND

In an extremely wide range of applications, mechanical systems are fundamentally affected by vibrations. For instance, components (such as pipes) in a power plant are designed to be structurally robust under seismic vibrations. Vibrations also cause mechanical systems to accumulate plastic deformations and eventually fail. The importance of these issues has motivated a lot of effort in modeling. From the modeling point of view, keeping track of the impact of past vibrations requires a specific description of the mechanical systems under consideration. The state of the system must be described by a randomly forced dynamical system with memory. Random forcing here expresses the stochastic nature of the vibrations that apply to the mechanical structures. In this framework, the difficulty is to handle dynamical systems with memory. A huge engineering literature has been devoted to this topic [14, 17, 18, 25, 29, 31, 34–36, 40, 41]. This paper mainly focuses on an important model, referred to as the elastic-perfectly-plastic oscillator (EPPO), which was introduced in [28] and appears

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in various applications [16, 19–21, 23, 24]. In terms of rheology, the response of such model is a combinations of elementary rheological models : springs, Saint-Venant's elements, dashpots and material points as shown in Figure 1.

FIGURE 1. Rheological model of an elasto-perfectly-plastic oscillator. A mass (black box) is associated in series with elements which are themselves an association in parallel or in series of elementary rheological models. The whole system is excited by a time-dependent force η .

In [12, 13], the stochastic variational inequalities (SVIs) have been identified as the right mathematical tool to deal with this model. Here, our objective is to derive and solve numerically a new class of free boundary problems and Hamilton-Jacobi-Bellman equations related to the control of these SVIs. Below, we provide background on and identify research issues on a stochastically driven EPPO.

1.1. An elasto-perfectly-plastic problem with noise

The dynamics focuses on two quantities: the deformation supported by the structure, namely x_t , when subjected to the vibrations, and its velocity, namely $y_t = \dot{x}_t$. In the simplest case, the dynamics of the velocity is given by $\dot{y}_t + c_0 y_t + f_t = \eta_t$, where $c_0 > 0$ is a damping coefficient, η_t describes the random vibrations and f_t describes the restoring force arising from the structure. The exact nature of f_t depends on the particular structure under consideration. The so-called perfect model corresponds to the case in which the force f_t is a linear function of x_t in the elastic phase and a constant in the plastic phase. In this case, the permanent (plastic) deformation at time t can be written as $\Delta_t \triangleq \int_0^t \mathbf{1}_{\{|f_s|=kY\}} y_s ds$, where kY describes the plastic phase through a stiffness coefficient k and an elasto-plastic bound, that is normalized to $Y = 1$. The variable $z_t = x_t - \Delta_t$ describes the deformation without plastic deformation. When $|z_t| < 1$, the structure is in the elastic regime and when $|z_t| = 1$, it is in the plastic regime. In both cases, the restoring force f_t is given by kz_t . The dynamics is then described by the pair (y_t, z_t) that satisfies

$$\dot{y}_t + c_0 y_t + kz_t = \eta_t, (\dot{z}_t - y_t)(\phi - z_t) \geq 0, \forall |\phi| \leq 1, |z_t| \leq 1, \forall t. \quad (1)$$

1.2. Research Issues

From a mathematical perspective, the point is to understand the behavior of the model in order to predict the behavior of the mechanical structure of interest. For engineering applications, this is fundamental in assessing the risk of failure of the structure. In [5, 6, 12], the ergodic behavior of the couple velocity-restoring force (y_t, z_t) of an elasto-perfectly-plastic oscillator (EPPO) excited by white noise and by a filtered noise has been established. The invariant probability can be described, through its density, as the solution of a partial differential equation (PDE), with specific boundary conditions arising from the plastic properties of the model. It is worth mentioning that ergodicity does not hold for the plastic deformation Δ_t . Thus, the behavior of Δ_t can be investigated through different quantities such as

$$(\text{growth rate}) \text{ variance of the plastic deformation } \frac{\mathbb{E}[\Delta_T^2]}{T} \quad \text{and the failure risk } \mathbb{P}\left(\max_{0 \leq t \leq T} |\Delta_t| \geq b\right) \quad (2)$$

where b is a critical threshold of deformation considered as dangerous for the system and $(0, T)$ is a given interval of time. For large time, analysis of the variance for the plastic deformation (2) has been carried out in [8] for an EPPO excited by white noise and a probabilistic formula has been derived. Also, in [7] an analytical formula has been proven. Concerning the failure risk, the mathematical techniques that have been developed so far provide an asymptotic formula [30]. Analysis of the risk raises other important problems for which no answer has been given so far.

- The risk probability (2) can be reformulated in terms of an optimal stopping problem as follows:

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{P}(|\Delta_\tau| \geq b), \quad \text{where } \mathcal{T}_{0,T} \text{ is the set of stopping time } \tau \text{ in the interval } [0, T].$$

This reformulation relies on a general result on reachability for Markov processes [10]. The problem is technically delicate¹. Here, we rather focus on a similar problem for the plastic state, that is,

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{P}(|z_\tau| = 1) = \mathbb{P}(\exists t \in [0, T], |z_t| = 1). \quad (3)$$

In subsection 2.1, we describe a general framework based on a new class of free boundary value problems characterizing optimal stopping problems of the form

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E} \left(\int_0^\tau e^{-\lambda s} g(z_s, y_s) ds + e^{-\lambda \tau} f(z_\tau, y_\tau) \right).$$

- Another important concept related to the notion of critical excitation [38] must be considered. The major point is to answer the following question: can we find the most likely random force, of the form $\eta_t = \alpha_t + \dot{w}_t$ where w_t is a Wiener process, that would maximize the accumulation of the plastic deformation or the time spent in plastic regime or in the sense

$$\sup_{\alpha(\cdot)} \mathbb{E} \left(\int_0^T \mathbf{1}_{\{|z_s|=1\}} |y_s| ds \right) \quad \text{or} \quad \sup_{\alpha(\cdot)} \mathbb{E} \left(\int_0^T \mathbf{1}_{\{|z_s|=1\}} ds \right). \quad (4)$$

We aim at translating the problem within the framework of stochastic control. In subsection 2.2, we describe a general framework based on a new class of Hamilton-Jacobi-Bellman Equations aimed at characterizing the control α in the form

$$\sup_{\alpha(\cdot)} \mathbb{E} \left(\int_0^T e^{-\lambda s} g(z_s, y_s) ds + f(z_T, y_T) \right).$$

1.3. Notations

Given $T > 0$, set $D_T \triangleq (-1, 1) \times \mathbb{R} \times [0, T]$, $D_T^+ \triangleq \{1\} \times \mathbb{R} \times [0, T]$ and $D_T^- \triangleq \{-1\} \times \mathbb{R} \times [0, T]$. Then, define three differential operators A , B_+ and B_- as follows

$$A \triangleq \frac{1}{2} \frac{\partial^2}{\partial y^2} - (c_0 y + k z) \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \quad \text{and} \quad B_\pm \triangleq \frac{1}{2} \frac{\partial^2}{\partial y^2} - (c_0 y \pm k) \frac{\partial}{\partial y} \pm \min(0, \pm y) \frac{\partial}{\partial z}.$$

We use the notation C_T^* for the set of continuous functions on $[-1, 1] \times \mathbb{R} \times [0, T]$ that are C^1 -regular with respect to z , C^2 -regular with respect to y and C^1 -regular with respect to t .

¹In order to treat the optimal stopping problem related to (2), we would need to extend the state variable from (z_t, y_t) to (z_t, y_t, x_t) (or (z_t, y_t, Δ_t)) and thus leading to a three dimensional free boundary value problem. From a computational viewpoint, that would become expensive. Therefore, in order to verify our approach which remains the same anyway, we rather prefer working with the random time that maximizes the probability of the plastic state over the interval $[0, T]$ as it leads to a two dimensional problem.

1.4. Overview of our numerical study

The PDEs related to the optimal stopping and optimal stochastic control of the elasto-plastic oscillator are non-standard free boundary value and HJB problems. To the best of our knowledge, it seems they have not been treated in the literature. In the linear case that is without control (i.e backward Kolmogorov equations), only partial existence and uniqueness results are available [9, 13]. This is because standard PDE theory techniques do not apply due to the non-standard boundary conditions and the degeneracy of these problems. Therefore, in the note, we mostly study the behavior of these PDEs numerically in order to gather insight in the solution behavior.

2. DERIVATION OF THE FREE BOUNDARY VALUE AND HJB PROBLEMS

In this section, we derive formally a new class of free boundary value problems for the optimal stopping (see subsection 2.1) and a new class of Hamilton-Jacobi-Bellman Equations for the optimal control (see subsection 2.2) of the elasto-plastic oscillator (1). These two problems are time dependent on a finite time interval with a terminal condition. Infinite horizon and stationary versions of these two problems are also given (see subsection 2.3).

2.1. Free boundary value problem for the optimal stopping of (1)

We are not aware of mathematical studies regarding optimal stopping problems for SVIs modeling elasto-plastic oscillators and their applications to the risk analysis of failure. Thus, to address this issue, the strategy here is to rely on the connection between optimal stopping and variational inequalities as established by Bensoussan-Lions [3]. For an EPPO excited by white noise, this approach leads to a new type of free boundary value problem governed by a variational inequality with a nonlocal Dirichlet boundary condition.

Theorem 1 (a free boundary value problem with nonlocal Dirichlet boundary condition for an EPPO excited by a white noise). *Let $\lambda \in \mathbb{R}$ and $u \in C_T^*$, assume that u satisfies*

$$\max \left(\frac{\partial u}{\partial t} + Au - \lambda u + g, f - u \right) = 0 \text{ in } D_T \quad \text{and} \quad \max \left(\frac{\partial u}{\partial t} + B_{\pm} u - \lambda u + g, f - u \right) = 0 \text{ in } D_T^{\pm} \quad (5)$$

with the terminal condition $u(z, y; T) = f(z, y)$. Then $u(z, y, t)$ has a probabilistic interpretation in terms of an optimal stopping problem, that is

$$u(z, y, t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E} \left(\int_t^{\tau} e^{-\lambda(s-t)} g(z_s, y_s) ds + e^{-\lambda(\tau-t)} f(z_{\tau}, y_{\tau}) | (z_t, y_t) = (z, y) \right)$$

where (z_s, y_s) satisfies (1) with η is a white noise and $\mathcal{T}_{t,T}$ is the set of stopping times between t and T .

For the optimal stopping rule, we use the notation

$$\hat{a}(z, y, t) \triangleq \begin{cases} 0, & \text{if } \left(\frac{\partial u}{\partial t} + Au - \lambda u + g \right)(z, y, t) \geq f(z, y) - u(z, y, t), \\ 1, & \text{if } \left(\frac{\partial u}{\partial t} + Au - \lambda u + g \right)(z, y, t) < f(z, y) - u(z, y, t) \end{cases} \quad (6)$$

for $|z| < 1$ and

$$\hat{a}(\pm 1, y, t) \triangleq \begin{cases} 0, & \text{if } \left(\frac{\partial u}{\partial t} + B_{\pm} u - \lambda u + g \right)(\pm 1, y, t) \geq f(\pm 1, y) - u(\pm 1, y, t), \\ 1, & \text{if } \left(\frac{\partial u}{\partial t} + B_{\pm} u - \lambda u + g \right)(\pm 1, y, t) < f(\pm 1, y) - u(\pm 1, y, t) \end{cases} \quad (7)$$

for $z = \pm 1$.

Proof. Without loss of generality, we can assume $t = 0$ and prove

$$u(z, y, 0) = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E} \left[\int_0^\tau e^{-\lambda s} g(z_s, y_s) ds + e^{-\lambda \tau} f(z_\tau, y_\tau) \right]$$

where (z_s, y_s) satisfies (1). Define

$$\begin{aligned} M_t &\triangleq e^{-\lambda t} u(z_t, y_t, t) - \int_0^t e^{-\lambda s} \left(\frac{\partial u}{\partial t} + Au - \lambda u \right) (z_s, y_s, s) 1_{\{|z_s| < 1\}} ds \\ &\quad - \int_0^t e^{-\lambda s} \left(\frac{\partial u}{\partial t} + B_\pm u - \lambda u \right) (\pm 1, y_s, s) 1_{\{z_s = \pm 1\}} ds. \end{aligned}$$

We claim that M_t is a martingale. Since $\tau \in \mathcal{T}_{0,T}$ is bounded, by optional sampling theorem, $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$. Also using the inequalities in the hypothesis on u , we have for any $t \in [0, T]$, $M_t \geq e^{-\lambda t} u(z_t, y_t, t) + \int_0^t e^{-\lambda s} g(z_s, y_s) ds$. So on one hand, $u(z, y, 0) = M_0 = \mathbb{E}[M_0]$, on the other hand, $\mathbb{E}[M_\tau] \geq \mathbb{E}[e^{-\lambda \tau} u(z_\tau, y_\tau, \tau) + \int_0^\tau e^{-\lambda s} g(z_s, y_s) ds]$. Therefore, since τ is arbitrary, we have: $u(z, y, 0) \geq \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[e^{-\lambda \tau} u(z_\tau, y_\tau, \tau) + \int_0^\tau e^{-\lambda s} g(z_s, y_s) ds]$. Define $\tau_{opt} \triangleq \inf \{t \geq 0 \mid u(z_t, y_t, t) = f(z_t, y_t)\}$. From the terminal condition of u , we know $\tau_{opt} \in \mathcal{T}_{0,T}$. Meanwhile for any $s \in [0, \tau_{opt})$, $u(z_s, y_s, s) > f(z_s, y_s)$, so from the hypothesis we must have $\frac{\partial u}{\partial t} + Au - \lambda u + g = 0$ on D_T and $\frac{\partial u}{\partial t} + B_\pm u - \lambda u + g = 0$ on $D_T^\pm$. Therefore using optional sampling theorem again and the continuity of u , we have

$$\begin{aligned} u(z, y, 0) &= \mathbb{E}[M_{\tau_{opt}}] \\ &= \mathbb{E} \left[e^{-\lambda \tau_{opt}} u(z_{\tau_{opt}}, y_{\tau_{opt}}, \tau_{opt}) + \int_0^{\tau_{opt}} g(z_s, y_s) ds \right] \\ &= \mathbb{E} \left[e^{-\lambda \tau_{opt}} f(z_{\tau_{opt}}, y_{\tau_{opt}}) + \int_0^{\tau_{opt}} e^{-\lambda s} g(z_s, y_s) ds \right] \\ &\leq \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E} \left[e^{-\lambda \tau} f(z_\tau, y_\tau) + \int_0^\tau e^{-\lambda s} g(z_s, y_s) ds \right]. \end{aligned}$$

Collecting results, the probabilistic interpretation is obtained. $\square$

2.2. HJB equations for the stochastic control problem

A natural approach to address the problem of the critical excitation [38] is the Bellman principle.

2.2.1. Statement of the problem

Let $0 \leq t \leq T$, $\alpha : [t, T] \rightarrow \mathbb{R}$ be a given function and consider the state variable (z^α, y^α) satisfying (1) with $\eta = \alpha + \dot{w}$, that is

$$\dot{y}_t^\alpha + c_0 y_t^\alpha + k z_t^\alpha = \alpha_t + \dot{w}_t, \quad (\dot{z}_t^\alpha - y_t^\alpha)(\phi - z_t^\alpha) \geq 0, \quad \forall |\phi| \leq 1, |z_t^\alpha| \leq 1. \quad (8)$$

Remark 1. By changing the probability measure, which is indeed possible by means of Girsanov transform, we can easily get rid of the control α (say when it is bounded). This permits to address the solvability of the SVI in the weak sense, provided that the underlying filtration is chosen accordingly. This technique is standard for SDEs and for the so-called “weak formulation” of related optimal stochastic control problems.

In this context, we consider α as a control variable for the state variable (z^α, y^α) . Next, we define the functional

$$J_{z,y,t}(\alpha(\cdot)) \triangleq \mathbb{E} \left[\int_t^T e^{-\lambda(s-t)} g(z_s^\alpha, y_s^\alpha) ds + e^{-\lambda(T-t)} f(z_T^\alpha, y_T^\alpha) \mid (z_t, y_t) = (z, y) \right], \quad (9)$$

where $(z_s^\alpha, y_s^\alpha)_{s \geq t}$ satisfies (8) with $(z_s^\alpha, y_s^\alpha) = (z, y)$. For instance, if $\lambda = 0, f = 0$ and $g(z, y) = |y| \mathbf{1}_{|z|=1}$ then the integrand in J represents the accumulated plastic deformation. We say that the noise $\hat{\alpha}_{z,y,t} + \dot{w}$ is a critical excitation when $\hat{\alpha}_{z,y,t}$ is an optimal control for Problem (8)–(9) maximizing the functional $J_{z,y,t}$, that is, $\sup_{\alpha(\cdot) \in \mathcal{A}_{t,T}} J_{z,y,t}(\alpha(\cdot)) = J_{z,y,t}(\hat{\alpha}(\cdot))$. Here $\mathcal{A}_{t,T}$ is a set of functions taking values in $\mathbf{A} = [-1, 1]$. The problem is to find the value function $v(z, y, t) = \sup_{\alpha(\cdot) \in \mathcal{A}_{t,T}} J_{z,y,t}(\alpha(\cdot))$ and an optimal control $\hat{\alpha}_{z,y,t}$.

2.2.2. Bellman principle and HJB equation

Bellman principle can be formulated in our context as follows: if we consider an optimal control $\hat{\alpha}_{|[t,T]}$ for Problem (8)–(9) starting at time t with initial condition (z, y) , then for any intermediate time $t + h$ between the times t and T , the portion of control enclosed by the times $t + h$ and T , $\hat{\alpha}_{|[t+h,T]}$ remains optimal for the problem starting at time $t + h$ with initial condition $(z_{t+h}^\alpha, y_{t+h}^\alpha)$ resulting from $\hat{\alpha}_{|[t,t+h]}$. Mathematically, it reads

$$\begin{cases} v(z, y, t) = \sup_{\alpha(\cdot) \in \mathcal{A}_{t,t+h}} \mathbb{E} \left[\int_t^{t+h} e^{-\lambda(s-t)} g(z_s^\alpha, y_s^\alpha) ds + v(z_{t+h}^\alpha, y_{t+h}^\alpha, t+h) \mid (z_t^\alpha, y_t^\alpha) = (z, y) \right] \\ v(z, y, T) = f(z, y). \end{cases}$$

The infinitesimal version of the Bellman principle leads to a new type of HJB equation with a nonlocal Dirichlet boundary condition.

Theorem 2 (A nonlocal HJB problem related to an EPPO). *Let $v \in C_T^*$ and assume that it satisfies*

$$\max_{\alpha \in \mathbf{A}} \left\{ \frac{\partial v}{\partial t} + A^\alpha v - \lambda v + g \right\} = 0 \text{ in } D_T \quad \text{and} \quad \max_{\alpha \in \mathbf{A}} \left\{ \frac{\partial v}{\partial t} + B_\pm^\alpha v - \lambda v + g \right\} = 0 \text{ in } D_T^\pm \quad (10)$$

with the terminal condition $v(z, y, T) = f(z, y)$ and where, for any $\alpha \in \mathbf{A}$, $A^\alpha \triangleq \alpha \frac{\partial}{\partial y} + A$, $B_\pm^\alpha \triangleq \alpha \frac{\partial}{\partial y} + B_\pm$. Moreover, we assume that (a) there exists $\hat{a}(z, y, t)$ maximizing $a \mapsto \left(\frac{\partial v}{\partial t} + A^\alpha v - \lambda v + g \right)(z, y, t)$ and $\hat{a}(\pm 1, y, t)$ maximizing $a \mapsto \left(\frac{\partial v}{\partial t} + B_\pm^\alpha v - \lambda v + g \right)(\pm 1, y, t)$ such that

$$\left(\frac{\partial v}{\partial t} + A^{\hat{a}(z,y,t)} v - \lambda v + g \right)(z, y, t) = 0 \text{ in } D_T, \quad \text{and} \quad \left(\frac{\partial v}{\partial t} + B_\pm^{\hat{a}(\pm 1,y,t)} v - \lambda v + g \right)(\pm 1, y, t) = 0 \text{ in } D_T^\pm,$$

(b) the process $\hat{\alpha}_t \triangleq \hat{a}(z_t, y_t, t)$ is a well-defined control process in $\mathcal{A}$ and (c) the SVI in (8) with $\alpha_t = \hat{\alpha}_t$ defines a unique solution $(\hat{z}_s, \hat{y}_s)_{s \geq t}$ for each given initial data $(\hat{z}_t, \hat{y}_t) = (z, y)$. Then $v(z, y, t)$ has the following probabilistic interpretation as the value function of an optimal stochastic control problem, $v(z, y, t) = \sup_{\alpha \in \mathcal{A}} J_{z,y,t}(\alpha(\cdot))$.

Going back to the case where $\lambda = 0, f = 0$ and $g(z, y) = |y| \mathbf{1}_{|z|=1}$, the objective functional

$$J_{z,y,t}(\alpha(\cdot)) \triangleq \mathbb{E} \left[\int_t^T |y_s^\alpha| \mathbf{1}_{|z_s^\alpha|=1} ds \mid (z_t, y_t) = (z, y) \right] \quad (11)$$

so finding $\hat{\alpha}$ would allow us to know a certain type of random forcing that would maximize the accumulated plastic deformation over the interval of time $(0, T)$.

Proof. In our context, the proof follows the line of [39]. Let $\alpha(\cdot) \in \mathcal{A}_{t,T}$ be an arbitrary control, $(z_s^\alpha, y_s^\alpha)_{s \geq t}$ the associated state process with initial condition $(z_t, y_t) = (z, y)$. By Itô's lemma applied to v between t and T ,

$$\begin{aligned} v(z, y, t) = & e^{-\lambda(T-t)} f(z_T^\alpha, y_T^\alpha) + \int_t^T e^{-\lambda(s-t)} g(z_s^\alpha, y_s^\alpha) ds - \int_t^T e^{-\lambda(s-t)} \frac{\partial v}{\partial y}(s, z_s^\alpha, y_s^\alpha) dW_s \\ & - \int_t^T e^{-\lambda(s-t)} \left(\frac{\partial v}{\partial t} + A^{\alpha_s} v - \lambda v + g \right) (s, z_s^\alpha, y_s^\alpha) \mathbf{1}_{\{|z_s^\alpha| < 1\}} ds \\ & - \int_t^T e^{-\lambda(s-t)} \left(\frac{\partial v}{\partial t} + B_{\pm}^{\alpha_s} v - \lambda v + g \right) (s, z_s^\alpha, y_s^\alpha) \mathbf{1}_{\{z_s^\alpha = \pm 1\}} ds \end{aligned}$$

But since for every $s \in [t, T]$, $\frac{\partial v}{\partial t} + A^{\alpha_s} v - \lambda v + g \leq \max_{a \in A} \left\{ \frac{\partial v}{\partial t} + A^a v - \lambda v + g \right\} = 0$ in D_T and $\frac{\partial v}{\partial t} + B_{\pm}^{\alpha_s} v - \lambda v + g \leq \max_{a \in A} \left\{ \frac{\partial v}{\partial t} + B_{\pm}^a v - \lambda v + g \right\} = 0$ in $D_T^\pm$. We obtain $v(z, y, t) \geq J(\alpha(\cdot))$ which implies

$$v(z, y, t) \geq \sup_{\alpha} J(\alpha(\cdot)). \quad (12)$$

Finally, from the (a)-(b)-(c) hypotheses, the control achieves the equality in (12). $\square$

2.3. Infinite horizon problems

A similar set of problems as those presented in equations (5) and (10) can be obtained in the infinite time horizon and stationary case. We give the two statements without proof since they are similar to what was done for the time dependent case. As the time horizon is infinite, there are a few additional assumptions.

Theorem 3 (an infinite horizon free boundary value problem with nonlocal Dirichlet boundary condition for an EPPO excited by a white noise). *Let $\lambda \geq 0$ and $u \in C_{T=\infty}^*$, assume that u satisfies*

$$\max (Au - \lambda u + g, f - u) = 0 \text{ in } D \quad \text{and} \quad \max (B_{\pm} u - \lambda u + g, f - u) = 0 \text{ in } D^{\pm}. \quad (13)$$

Then $u(z, y)$ has a probabilistic interpretation in terms of an optimal stopping problem, that is

$$u(z, y) = \sup_{\tau \in \mathcal{T}} \mathbb{E} \left(\int_0^\tau e^{-\lambda s} g(z_s, y_s) ds + e^{-\lambda \tau} f(z_\tau, y_\tau) \mid (z_t, y_t) = (z, y) \right)$$

where (z_s, y_s) satisfies (1) with η is a white noise and $\mathcal{T}$ is the set of stopping times almost surely finite.

Theorem 4 (an infinite horizon nonlocal HJB problem related to an EPPO). *Let $\lambda \geq 0$ and $v \in C_{T=\infty}^*$, assume that it satisfies*

$$\max_{\alpha \in \mathbf{A}} \{A^\alpha v - \lambda v + g\} = 0 \text{ in } D \quad \text{and} \quad \max_{\alpha \in \mathbf{A}} \{B_{\pm}^\alpha v - \lambda v + g\} = 0 \text{ in } D^{\pm} \quad (14)$$

where, for any $\alpha \in \mathbf{A}$, $A^\alpha \triangleq \alpha \frac{\partial}{\partial y} + A$, $B_{\pm}^\alpha \triangleq \alpha \frac{\partial}{\partial y} + B_{\pm}$. Moreover, we assume that $(a)_\infty$ there exists $\hat{a}(z, y)$ maximizing $a \mapsto (A^a v - \lambda v + g)(z, y)$ and $\hat{a}(\pm 1, y)$ maximizing $a \mapsto (B_{\pm}^a v - \lambda v + g)(\pm 1, y)$ such that

$$\left(A^{\hat{a}(z, y)} v - \lambda v + g \right) (z, y) = 0 \text{ in } D, \quad \text{and} \quad \left(B_{\pm}^{\hat{a}(\pm 1, y)} v - \lambda v + g \right) (\pm 1, y) = 0 \text{ in } D^{\pm},$$

(b) $_\infty$ the process $\hat{\alpha}_t \triangleq \hat{a}(z_t, y_t)$ is a well-defined control process in $\mathcal{A}$ and (c) $_\infty$ the SVI in (8) with $\alpha_t = \hat{\alpha}_t$ defines a unique solution $(\hat{z}_s, \hat{y}_s)_{s \geq t}$ for each given initial data $(\hat{z}_t, \hat{y}_t) = (z, y)$. Then $v(z, y)$ has the following probabilistic interpretation as the value function of an infinite horizon optimal stochastic control problem, $v(z, y) = \sup_{\alpha \in \mathcal{A}} J_{z, y, t}(\alpha(\cdot))$ where

$$J_{z, y}(\alpha(\cdot)) \triangleq \mathbb{E} \left[\int_0^\infty e^{-\lambda s} g(z_s^\alpha, y_s^\alpha) ds \mid (z_0, y_0) = (z, y) \right]. \quad (15)$$

Let us comment on the shape of the optimizer $\hat{a}$. Since g does not depend on α , Equation (14) can be simply written as

$$\max_{\alpha \in \mathbf{A}} \left\{ \alpha \frac{\partial v}{\partial y} \right\} + Av - \lambda v + g = 0 \quad \text{in } D \quad \text{and} \quad \max_{\alpha \in \mathbf{A}} \left\{ \alpha \frac{\partial v}{\partial y} \right\} + B_{\pm} v - \lambda v + g = 0 \quad \text{in } D^{\pm}.$$

Since $\mathbf{A} = [-1, 1]$, we must have

$$\max_{\alpha \in \mathbf{A}} \left\{ \alpha \frac{\partial v}{\partial y} \right\} = \left| \frac{\partial v}{\partial y} \right|$$

and the maximizer is given by $\hat{a}(z, y) = \text{sign} \left(\frac{\partial v}{\partial y}(z, y) \right)$, where $\text{sign}(x) \triangleq \frac{x}{|x|}$, if $x \neq 0$ and 0 if $x = 0$. It turns out that the (feedback) control $\hat{a}_t = \hat{a}(z_t, y_t)$ takes values in $\{-1, 1\}$ and thus belongs to the class of bang-bang controls.

Furthermore, when f, g are symmetric functions i.e $f(z, y) = f(-z, -y)$ and $g(z, y) = g(-z, -y)$, it can be seen that $\hat{a}$ must be an antisymmetric function with respect to (y, z) . Indeed, denoting $\tilde{v} : (z, y) \mapsto v(-z, -y)$, it can be readily checked that $\tilde{v}$ solves the same PDE as v (using the fact that $\mathbf{A}$ is a symmetric interval around 0). Assuming uniqueness of the solution to this PDE, we have $\tilde{v} = v$. Hence v is symmetric and thus $\frac{\partial v}{\partial y}$ and $\hat{a}$ are antisymmetric. A similar discussion holds for the time dependent problem in Theorem 2.

Numerically, in the case of accumulated plastic deformation (see Figures 8-9 below), we have indeed found an antisymmetric function that seems to take the value 1 (resp. -1) on a set of the form $\{(x, y), \gamma(x) > y\}$ (resp. $\{(x, y), \gamma(x) < y\}$). The interface γ seems to be of the form $\gamma(x) = cx$ where c is a certain constant. The study of this interface is an interesting question that we leave for future work.

3. NUMERICAL SCHEME

3.1. Howard's algorithm

In this section, we recall Howard's algorithm [1, 2, 26] which treats the following typical problem: find $v \in \mathbb{R}^N$, satisfying the componentwise minimization

$$\min_{\alpha \in \mathbf{A}^N} (B(\alpha)v - c(\alpha)) = 0_{\mathbb{R}^N} \quad (16)$$

where $\mathbf{A}$ is a nonempty compact set and for all $\alpha \in \mathbf{A}^N$, $B(\alpha)$ is a $N \times N$ matrix and $c(\alpha)$ is a vector of size N . As a particular case, it also includes the following problem: find $v \in \mathbb{R}^N$, satisfying the componentwise minimization

$$\min \left(\tilde{B}v - g, v - f \right) = 0 \quad (17)$$

where $\tilde{B}$ is a $N \times N$ matrix and h, g are vectors of size N . Indeed, it can be written under the form (16) with, for all $1 \leq i \leq N$,

$$B_{i,\bullet}(\alpha) = \begin{cases} \tilde{B}_{i,\bullet} & \text{if } \alpha_i = 0 \\ I_{i,\bullet} & \text{if } \alpha_i = 1 \end{cases} \quad \text{and} \quad c_i(\alpha) = \begin{cases} g_i & \text{if } \alpha_i = 0 \\ f_i & \text{if } \alpha_i = 1. \end{cases}$$

Here $\tilde{B}_{i,\bullet}$ is the line i of the matrix $\tilde{B}$ and I is the $N \times N$ identity matrix. As expressed for example in [11], Howard's algorithm for (16) is as follows:

- (1) Initialize $\alpha^{(0)} \in \mathbf{A}^N$
- (2) Iterate for $k \geq 0$:
 - (i) Find $v^{(k)} \in \mathbb{R}^N$ solution of $B(\alpha^{(k)})v^{(k)} - c(\alpha^{(k)}) = 0$.
If $k \geq 1$ and $v^{(k)} = v^{(k-1)}$ then stop. Otherwise go to (ii).

- (ii) $\alpha^{(k+1)} \triangleq \arg \min_{a \in \mathbf{A}^N} (B(a)v^{(k)} - c(a))$. Set $k \triangleq k + 1$ and go to (i).

For the specific case (17), the algorithm writes:

- (1) Initialize $\alpha^{(0)} \in \mathbf{A}^N \triangleq \{0, 1\}^N$.
- (2) Iterate for $k \geq 0$:
 - (i) Find $v^{(k)} \in \mathbb{R}^N$ solution of $B(\alpha^{(k)})v^{(k)} - c(\alpha^{(k)}) = 0$.
If $k \geq 1$ and $v^{(k)} = v^{(k-1)}$ then stop. Otherwise go to (ii).
 - (ii) For every $1 \leq i \leq N$, $\alpha_i^{(k+1)} \triangleq \begin{cases} 0 & \text{if } (\tilde{B}v^{(k)} - g)_i \leq (v^{(k)} - f)_i \\ 1 & \text{otherwise.} \end{cases}$
Set $k \triangleq k + 1$ and go to (i).

In the numerical implementation, we replace the stopping condition $v^{(k)} = v^{(k-1)}$ by a criterion of the form

$$\sqrt{\sum_{i=1}^N |v_i^{(k)} - v_i^{(k-1)}|^2} < \epsilon_H, \quad \text{for a fixed tolerance } \epsilon_H > 0.$$

For convergence results related to this algorithm, we refer the reader to [11]. Now we explain how to solve numerically equations (5), (10), (13) and (14) using this algorithm. To this end, we need to discretize these equations and rewrite them under the form (16) and (17).

3.2. Discretization of PDEs problems following [32]

To numerically approximate the solutions of (5), (10), (13) and (14), we use a finite difference scheme. We truncate the unbounded domains D and $D^\pm$ to obtain $D_Y \triangleq (-1, 1) \times (-Y, Y)$ and $D_Y^\pm \triangleq \{\pm 1\} \times (-Y, Y)$, where Y is chosen sufficiently large that the probability of finding the underlying process outside D_Y and $D_Y^\pm$ is negligible. We apply a homogeneous Neumann boundary condition at $y = \pm Y$. We consider a two-dimensional rectangular finite difference grid,

$$\mathcal{G} \triangleq \{(z_i, y_j) \triangleq (-1 + (i-1)\delta z, -Y + (j-1)\delta y)\}_{1 \leq i \leq I, 1 \leq j \leq J},$$

where $\delta z \triangleq \frac{2}{I-1}$, $\delta y \triangleq \frac{2Y}{J-1}$. Here, I, J are odd integers of the form $2\tilde{I} + 1, 2\tilde{J} + 1$. The total number of nodes in $\mathcal{G}$ is $N = IJ$. The numerical approximations of $u(z_i, y_j, t_n)$, $v(z_i, y_j, t_n)$, $u_\lambda(z_i, y_j)$ and $v_\lambda(z_i, y_j)$ are denoted by $u_{i,j}^n$, $v_{i,j}^n$, $u_{i,j}^\lambda$ and $v_{i,j}^\lambda$ and the corresponding vectors collecting all the unknowns are $\mathbf{u}^n$, $\mathbf{v}^n$, $\mathbf{u}^\lambda$ and $\mathbf{v}^\lambda$. We use the notation $f_{i,j}, g_{i,j}$ for $f(x_i, y_j), g(x_i, y_j)$ and the corresponding vectors are $\mathbf{f}, \mathbf{g}$. Here, $t_n \triangleq n\delta t$ discretizes the time and $N_T\delta t = T$.

3.2.1. Case of the free boundary problems (5) and (13)

For the time dependent problem (5), we use an implicit Euler method to discretize in time together with finite differences in space and, for the stationary problem (13), we only use finite differences in space. For both problems, the condition on D in (5) and in (13) at the black points in Figure 2 result in

$$u_{i,j}^0 = f_{i,j} \quad \text{and} \quad \min \left(\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\delta t} - (L\mathbf{u}^{n+1})_{i,j} - g_{i,j}, u_{i,j}^{n+1} - f_{i,j} \right) = 0, \quad 0 \leq n \leq N_T - 1, \quad (18)$$

and

$$\min \left(\lambda u_{i,j}^\lambda - (L\mathbf{u}^\lambda)_{i,j} - g_{i,j}, u_{i,j}^\lambda - f_{i,j} \right) = 0, \quad (19)$$

where $(L\mathbf{u})_{i,j}$ is of the form

$$-(L\mathbf{u})_{i,j} \triangleq -(L_y\mathbf{u})_{i,j} - \max(0, y_j) \left(\frac{u_{i+1,j} - u_{i,j}}{\delta z} \right) - \min(0, y_j) \left(\frac{u_{i,j} - u_{i-1,j}}{\delta z} \right) \quad (20)$$

FIGURE 2. Discretization of $D_Y = (-1, 1) \times (-Y, Y)$ and $D_Y^\pm = \{\pm 1\} \times (-Y, Y)$. At black points, the discretized equation is satisfied. At grey points, homogeneous Neumann boundary conditions are used, and at red points non-standard boundary conditions are employed.

with

$$-(L_y \mathbf{u})_{i,j} \triangleq -\frac{1}{2} \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\delta y^2} \right) - \max(0, b_{i,j}) \left(\frac{u_{i,j+1} - u_{i,j}}{\delta y} \right) - \min(0, b_{i,j}) \left(\frac{u_{i,j} - u_{i,j-1}}{\delta y} \right)$$

where $b_{i,j} = -(c_0 y_j + k z_i)$. The nonlocal Dirichlet boundary conditions (second condition) in (5) and in (13) at the red points in Figure 2 are discretized by the same formulae with L replaced by $L_\pm$, where $(L_\pm \mathbf{u})_{i,j}$ are defined by

$$-(L_+ \mathbf{u})_{i,j} \triangleq -(L_y \mathbf{u})_{i,j} - \min(0, y_j) \left(\frac{u_{i,j} - u_{i-1,j}}{\delta z} \right) \quad (21)$$

and

$$-(L_- \mathbf{u})_{i,j} \triangleq -(L_y \mathbf{u})_{i,j} - \max(0, y_j) \left(\frac{u_{i+1,j} - u_{i,j}}{\delta z} \right). \quad (22)$$

Note that, unlike [11, 27], we are not able to guess the value of u and u_λ or its derivative on the boundaries $y = \pm Y$. Instead, we proceed as in [32] in imposing an homogeneous Neumann condition. This is based on the intuition that, when Y is chosen sufficiently large, the probability of finding the underlying process outside the truncated domain is negligible. This idea is supported by Monte Carlo simulations as shown below.

The Neumann boundary conditions at the points shown in Figure 2 results in

$$(N_2 \mathbf{u})_{i,j} = 0, \quad \text{where} \quad (N_2 \mathbf{u})_{i,j} \triangleq \begin{cases} \frac{u_{i,j+1} - u_{i,j}}{\delta y} & \text{if } j = 1, \\ \frac{u_{i,j} - u_{i,j-1}}{\delta y} & \text{if } j = J. \end{cases} \quad (23)$$

We also do the following modification : $g_{i,j} = 0$ at the grey points. For the time dependent problem, this results in the following nonlinear system to be solved in each time step:

$$\min(M_{\delta t} \mathbf{u}^{n+1} - \mathbf{g}_{\delta t}^n, \mathbf{u}^{n+1} - \mathbf{f}) = 0, \quad \mathbf{u}^0 = \mathbf{f}, \quad (24)$$

and for the stationary problem

$$\min(M_\lambda \mathbf{u}^\lambda - \mathbf{g}, \mathbf{u}^\lambda - \mathbf{f}) = 0. \quad (25)$$

Here $M_{\delta t}$ is a sparse $N \times N$ -matrix that depends on δt but not on n . Precisely, using the notation

$$\begin{aligned} l(i, j) &\triangleq j + (i - 1)J, \quad \mathbf{G} \triangleq \{(i, j), i \in \{1, \dots, I\}, j \in \{1, J\}\}, \quad \mathbf{R}_- \triangleq \{(1, j), j \in \{2 \dots \tilde{J}\}\}, \\ \mathbf{R}_+ &\triangleq \{(I, j), j \in \{\tilde{J} + 2 \dots J - 1\}\}, \quad \mathbf{B} \triangleq \{1, \dots, I\} \times \{1, \dots, J\} - (\mathbf{G} \cup \mathbf{R}_- \cup \mathbf{R}_+). \end{aligned}$$

- for every $n \in \{1, \dots, N_T\}$ and for every point $(i, j) \in \mathbf{G}$, $(g_{\delta t}^n)_{i,j} = 0$. Otherwise, $(g_{\delta t}^n)_{i,j} = (\mathbf{u}^n + \delta t \mathbf{g})_{i,j}$.
- for every grey point labeled by $(i, j) \in \mathbf{G}$,
 - $M_{\delta t}(l(i, 1), l(i, 1)) = \frac{1}{\Delta y}$, $M_{\delta t}(l(i, 1), l(i, 2)) = -\frac{1}{\Delta y}$
 - $M_{\delta t}(l(i, J), l(i, J)) = \frac{1}{\Delta y}$, $M_{\delta t}(l(i, J), l(i, J - 1)) = -\frac{1}{\Delta y}$
- for every black point labeled by $(i, j) \in \mathbf{B}$,
 - $M_{\delta t}(l(i, j), l(i - 1, j)) = \frac{\Delta t}{\Delta z} \min(0, y_j)$
 - $M_{\delta t}(l(i, j), l(i, j - 1)) = -\Delta t \left(\frac{1}{2\Delta y^2} - \frac{\min(0, b_{i,j})}{\Delta y} \right)$
 - $M_{\delta t}(l(i, j), l(i, j)) = 1 + \Delta t \left(\frac{1}{\Delta y^2} + \frac{|b_{i,j}|}{\Delta y} + \frac{|y_j|}{\Delta z} \right)$
 - $M_{\delta t}(l(i, j), l(i, j + 1)) = -\Delta t \left(\frac{1}{2\Delta y^2} + \frac{\max(0, b_{i,j})}{\Delta y} \right)$
 - $M_{\delta t}(l(i, j), l(i + 1, j)) = -\frac{\Delta t}{\Delta z} \max(0, y_j)$
- for every red point labeled by $(i, j) \in \mathbf{R}_-$,
 - $M_{\delta t}(l(1, j), l(1, j - 1)), M_{\delta t}(l(1, j), l(1, j + 1)), M_{\delta t}(l(1, j), l(2, j))$ have the same expression than in the black points case but $M_{\delta t}(l(1, j), l(1, j)) = 1 + \Delta t \left(\frac{1}{\Delta y^2} + \frac{|b_{1,j}|}{\Delta y} + \frac{\min(0, y_j)}{\Delta z} \right)$.
- for every red point labeled by $(i, j) \in \mathbf{R}_+$,
 - $M_{\delta t}(l(I, j), l(I - 1, j)), M_{\delta t}(l(I, j), l(I, j - 1)), M_{\delta t}(l(I, j), l(I, j)), M_{\delta t}(l(I, j), l(I, j + 1))$ have the same expression than in the black points case but $M_{\delta t}(l(I, j), l(I, j)) = 1 + \Delta t \left(\frac{1}{\Delta y^2} + \frac{|b_{I,j}|}{\Delta y} + \frac{\max(0, y_j)}{\Delta z} \right)$.

The construction of M_λ is very similar. For the computational results presented in the remainder of this paper, we use a C code that implements a Monte Carlo (MC) probabilistic simulation to approximate the solution of (1). We use a MATLAB implementation for the PDE approach (24). Implementations are available upon request.

3.2.2. Case of the HJB problems (10) and (14)

We proceed as what was done above, except that we need to incorporate the control in the upwinding differences. The first conditions in (10) and (14) at the black points in Figure 2 result in

$$v_{i,j}^0 = f_{i,j} \quad \text{and} \quad \min_{\alpha \in \mathbf{A}} \left(\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\delta t} - (L^\alpha \mathbf{v}^{n+1})_{i,j} - g_{i,j} \right) = 0, \quad 0 \leq n \leq N_T - 1, \quad (26)$$

and

$$\min_{\alpha \in \mathbf{A}} \left(\lambda v_{i,j}^\lambda - (L^\alpha \mathbf{v}^\lambda)_{i,j} - g_{i,j} \right) = 0, \quad (27)$$

where

$$-(L^\alpha \mathbf{u})_{i,j} \triangleq -(L_y^\alpha \mathbf{u})_{i,j} - \max(0, y_j) \left(\frac{u_{i+1,j} - u_{i,j}}{\delta z} \right) - \min(0, y_j) \left(\frac{u_{i,j} - u_{i-1,j}}{\delta z} \right)$$

with

$$-(L_y^\alpha \mathbf{u})_{i,j} \triangleq -\frac{1}{2} \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\delta y^2} \right) - \max(0, \alpha_{i,j} + b_{i,j}) \left(\frac{u_{i,j+1} - u_{i,j}}{\delta y} \right) - \min(0, \alpha_{i,j} + b_{i,j}) \left(\frac{u_{i,j} - u_{i,j-1}}{\delta y} \right).$$

Again, the nonlocal Dirichlet boundary conditions (second condition) in (10) and (14) at the red points in Figure 2 are discretized by the same formulae with L^α replaced by $L_\pm^\alpha$ defined by

$$-(L_+^\alpha \mathbf{u})_{i,j} \triangleq -(L_y^\alpha \mathbf{u})_{i,j} - \min(0, y_j) \left(\frac{u_{i,j} - u_{i-1,j}}{\delta z} \right)$$

and

$$-(L_-^\alpha \mathbf{u})_{i,j} \triangleq -(L_y^\alpha \mathbf{u})_{i,j} - \max(0, y_j) \left(\frac{u_{i+1,j} - u_{i,j}}{\delta z} \right).$$

Also, we impose a Neumann boundary condition at the points shown in grey as in (23). Here, for the time dependent problem, this results in the following nonlinear system to be solved in each time step:

$$\min_{\alpha} (M_{\delta t}^\alpha \mathbf{u}^{n+1} - \mathbf{g}_{\delta t}^n) = 0, \quad \mathbf{u}^0 = \mathbf{f}, \quad (28)$$

and for the stationary problem

$$\min_{\alpha} (M_{\lambda}^\alpha \mathbf{u}^\lambda - \mathbf{g}) = 0, \quad (29)$$

where the matrices $M_{\delta t}^\alpha$ and M_{λ}^α are almost the same matrices as $M_{\delta t}$ and M_{λ} , except that b is replaced by $b + \alpha$.

4. NUMERICAL RESULTS

4.1. Free boundary value problems

We consider the case shown in (3) which is related to the problem (5) with $\lambda = 0.1$, $g = 0$ and $f = \mathbf{1}_{\{|z|=1\}}$. This quantity is the probability that the elasto-plastic system has entered, at least once, in the plastic phase before the instant T . Roughly speaking, as long as it remains small, it is somehow reasonable to approximate statistics of the elasto-plastic system by using those of a linear system. By solving (24), a numerical approximation for $u(0, 0, t_n)$ of (5) is obtained. For comparison, a probabilistic numerical approximation of the left hand side in (3) is provided as an alternative method to the deterministic approach. The numerical results are shown in Figure 3. Note that in contrast with the Monte Carlo approach, the PDE approach also computes $u(x_i, y_j, t)$, $\forall i, j$, as presented in Figure 4 (with $\lambda = 0.1$). Figure 5 displays the shape of the optimal stopping decision. We see that, except for the initial time, the optimal stopping decision is constant in time. Recall that, following the notations introduced in Section 3, 1 corresponds to stopping and 0 to continuing. We see that the optimal decision is to stop when the process reaches the plastic boundary and to continue otherwise. Note that the process never reaches the regions where $\text{sign}(y)z = -1$ hence the value of the control in these regions does not matter.

4.2. HJB problems

We consider both cases shown in (4) which are related to the problem (10) with $\lambda = 0.1$, $f(z, y) = 0$ and

- either $g(z, y) = |y| \mathbf{1}_{\{|z|=1\}}$ (*accumulated plastic deformation*),
- or $g(z, y) = \mathbf{1}_{\{|z|=1\}}$ (*time spent on the plastic boundary*).

By solving (28), a numerical approximation for $v(0, 0, t_n)$ of (10) is obtained. We do not have explicit solutions for the problems we are studying here and we cannot compute them using a straightforward probabilistic approach. However, for checking, we consider a numerical approximation of an optimal control $\alpha_{opt}(\cdot)$ satisfying

$$J(\alpha_{opt}(\cdot)) = \sup_{\alpha} \mathbb{E} \left(\int_0^T g(z_s^\alpha, y_s^\alpha) ds \right).$$

The approximation of α_{opt} is obtained from the PDE solution. Then we have used it as an input in the probabilistic numerical scheme to approximate $\mathbb{E} \left(\int_0^T g(z_s^{\alpha_{opt}}, y_s^{\alpha_{opt}}) ds \right)$. This expectation is computed by means

FIGURE 3. Comparison of the PDE (solid) approach solving the free boundary value problem (5) and MC (dashed) approach based on the right hand side of (3). Approximation of the values $u(0, 0, T)$ and $\mathbb{P}_{0,0}(\exists t \in [0, T], |z_t| = 1)$ are shown for $T \in [0, 7]$.

of Monte Carlo simulations based on an Euler discretization of the dynamics (8) controlled by α_{opt} , see e.g. Bernardin [15] for more details. This method has given satisfactory results in the sense that the empirical expectation computed by the probabilistic method matches the corresponding value of $\mathbf{u}$ solving (28) computed by the deterministic method. In order to check the convergence of the method, we compute the residual in the sup-norm for (28) with (u_{opt}, α_{opt}) obtained numerically with the above PDE method, that is

$$\max_{i,j,n} ((I + \delta t(M + M^{\alpha_{opt}}))\mathbf{u}_{opt}^{n+1} - \mathbf{u}_{opt}^n - \delta t \mathbf{g})_{i,j}.$$

For a grid of 400 time steps and $I = J = 150$, we obtain a residual of the order of 10^{-12} .

Figure 6, shows the graph of $v(0, 0, T)/T$ for $T \in [0, 100]$, computed by the PDE method described in Section 3, for the two test cases *accumulated plastic deformation* (i.e. $g(z, y) = |y|\mathbf{1}_{\{|z|=1\}}$) and *time spent on the plastic boundary* (i.e. $g(z, y) = \mathbf{1}_{\{|z|=1\}}$). One can see that for large T , the time spent in plastic deformation is smaller than the accumulated plastic deformation. This is because for large time, even when it starts from $(0, 0)$, the process has a high probability to reach large values of $|y|$ on the boundary $|z| = 1$ and when $|y| > 1$, the running cost of the test case *time spent on the plastic boundary* is smaller than the running cost of the other test case. Figure 7 shows the time evolution of the value function for the *accumulated plastic deformation* test case (with $\lambda = 0.1$). The range of the value function keeps increasing with time because with longer time horizon, the process will accumulate more plastic deformation. However, for a given z , the value function is increasing with respect to $|y|$. This can be explained as follows: when the process starts closer to the plastic boundary, it is more likely to accumulate a lot of plastic deformation, which is reflected by the value function of this problem. The shape optimal control (at the final time $t = 100$) is displayed in Figure 8 (3d plot) and Figure 9 (2d plot). One can see that, roughly speaking, the optimal decision is to push the process towards the closest portion of the plastic boundary. One can see that the control is -1 in a region and $+1$ in another region. The line between them crosses the quadrant $\{(y, z) : y > 0, z < 0\}$ and $\{(y, z) : y < 0, z > 0\}$ and, as t increases, it becomes closer to the line $y = 0$. The exact evolution of this line is an interesting question that we leave for future work. Using the optimal control computed from this PDE method, we can then simulate trajectories of the optimally controlled process. Figure 10 shows, for one realization of the noise, the trajectory with the optimal control (for the sake of completeness, we also show the non-controlled trajectory), and the

FIGURE 4. FB problem as shown in (5) : evolution of the value function in time $u(y, z, t)$. The shots are taken at different times $t = 0, 1, 5, 10$. The terminal condition is $f = \mathbf{1}_{\{|z|=1\}}$, the right hand side $g = 0$ and $\lambda = 0.1$.

corresponding evolution of the optimal control. One can see that the value of the optimal control switches between -1 and $+1$. Moreover, the value $+1$ is applied, roughly speaking, in the upper half of the phase space, while the value -1 is applied in the lower part. This is consistent with Figures 8–9.

The value function and the optimal control for the problem of the time spent in the plastic deformation are very similar to those obtained for the case of accumulated plastic deformation so we omit them. Furthermore, we have also solved the ergodic problems (free boundary value problem and Hamilton-Jacobi-Bellman problem) and checked that the value functions of dynamic problems converge to the one of the corresponding ergodic problems.

5. CONCLUSION

In this work we have investigated control problems for elasto-plastic oscillators. We have derived the PDEs for free boundary value problems and Hamilton-Jacobi-Bellman problems. We then implemented numerical methods based on Howard algorithm to compute approximate solutions. From here, several directions of research are still open. First, it would be interesting to clarify the structure of the optimal control for the problem of

FIGURE 5. FB problem as shown in (5): evolution of the optimal stopping decision $\hat{a}(z, y, t)$ as shown in (6) and (7). The shots are taken at different times $t = 0, 1, 5, 10$. It consists in stopping (value 1) when the process reaches the plastic boundary and continuing otherwise (value 0). The terminal condition is $f = \mathbf{1}_{\{|z|=1\}}$, the right hand side is $g = 0$ and $\lambda = 0.1$.

accumulated plastic deformation (see Figure 8). Second, we could consider more general control problems (in this work, as a first step we have considered only a control acting linearly on the drift, see (8)). In particular, computing solutions for the problems (2) remains a challenging task.

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FIGURE 6. Approximation of the values of $v(0, 0, T)/T$ for $T \in [0, 100]$ obtained by the PDE approach solving the HJB problem (10) for both problems *accumulated plastic deformation* (solid line) and *time spent on the plastic boundary* (dashed line).

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FIGURE 7. HJB problem as shown in (10): evolution of the value function in time $v(z, y, t)$ for the accumulated plastic deformation. The shots are taken at different times $t = 0, 1, 5, 10$. The terminal condition is $f = 0$, the right hand side is $g = |y|1_{\{|z|=1\}}$ and $\lambda = 0.1$.

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FIGURE 8. HJB problem as shown in (10): evolution of the optimal control in time $\hat{a}(z, y, t)$ for the accumulated plastic deformation. The shots are taken at different times $t = 0, 1, 5, 10$. The terminal condition is $f = 0$, the right hand side is $g = |y|\mathbf{1}_{\{|z|=1\}}$ and $\lambda = 0.1$.

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FIGURE 9. HJB problem as shown in (10): evolution of the optimal control in time $\hat{a}(z, y, t)$ for the accumulated plastic deformation (view from above). The shots are taken at different times $t = 0, 1, 5, 10$. The terminal condition is $f = 0$, the right hand side is $g = |y|1_{\{|z|=1\}}$ and $\lambda = 0.1$.

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(A) Trajectories

(B) Optimal control

FIGURE 10. HJB problem as shown in (10): Left: one realization of a trajectory with the optimal control (full line) and without control (dashed line) for the same realization of the noise; right: corresponding evolution of the optimal control in time $\hat{\alpha}_t = \hat{a}(z_t, y_t, t)$ for the accumulated plastic deformation. The terminal condition is $f = 0$, the right hand side is $g = |y| \mathbf{1}_{\{|z|=1\}}$ and $\lambda = 0.1$.