

‘UNCERTAINTY’ PRINCIPLE IN TWO FLUID–MECHANICS

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Abstract. Hamilton’s principle (or principle of stationary action) is one of the basic modelling tools in finite-degree-of-freedom mechanics. It states that the reversible motion of mechanical systems is completely determined by the corresponding Lagrangian which is the difference between kinetic and potential energy of our system. The governing equations are the Euler-Lagrange equations for Hamilton’s action.

Hamilton’s principle can be naturally extended to both one-velocity and multi-velocity continuum mechanics (infinite-degree-of-freedom systems). In particular, the motion of multi-velocity continuum is described by a coupled system of ‘Newton’s laws’ (Euler-Lagrange equations) for each component. The introduction of dissipative terms compatible with the second law of thermodynamics and a natural restriction on the behaviour of potential energy (convexity) allows us to derive physically reasonable and mathematically well posed governing equations.

I will consider a simplest example of two-velocity fluids where one of the phases is incompressible (for example, flow of dusty air, or flow of compressible bubbles in an incompressible fluid). A very surprising fact is that one can obtain different governing equations from the same Lagrangian. Different types of the governing equations are due to the choice of independent variables and the corresponding virtual motions. Even if the total momentum and total energy equations are the same, the equations for individual components differ from each other by the presence or absence of gyroscopic forces (also called ‘lift’ forces). These forces have no influence on the hyperbolicity of the governing equations, but can drastically change the distribution of density and velocity of components.

To the best of my knowledge, such an uncertainty in obtaining the governing equations of multi-phase flows has never been the subject of discussion in a ‘multi-fluid’ community.

INTRODUCTION

There are at least three different approaches for modeling heterogeneous mixtures of fluids. The most common one for studying the heterogeneous two-phase mixtures is the *averaging method* [12], [15], [5]. The averaged equations of motion are obtained by applying an appropriate averaging operator to the balance laws of mass, momentum, angular momentum and energy which are valid inside each phase. The main problem with this approach is the closure of the system obtained: the system contains more unknowns than equations. Different experimental and theoretical hypothesis are necessary for the closure.

The approach known as *Landau method of conservation laws* was initially used for constructing the models of quantum liquids such as superfluid helium [11], [13], [16]. The method consists in the following: the requirement of fulfillment of balance laws complemented by the Galilean relativity principle and the Gibbs thermodynamic identity fully determine the governing equations of motion. Recently this approach was applied to classical mixtures of fluids and suspensions (two-velocity hydrodynamics) in [4], [17], [20], [19]. The method does not

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take into account the geometrical characteristics of the mixture components: the volume concentrations, sizes of particles etc.

Finally, *Hamilton's principle* can be used. In this case kinetic and internal energies should be *a priori* expressed in terms of averaged quantities. Such an approach was successfully applied for the study of wave propagation in fluids containing gas bubbles [2], [3], [8], [9], [7], solid-fluid interaction [14] and shock interaction with droplets [18].

We use here Hamilton's principle to describe two-velocity flows of heterogeneous fluids where one of the phases is incompressible as, for example, wave propagation in dusty clouds or a fluid containing compressible gas bubbles.

In the following vectors are written by bold fonts. The bold letter \mathbf{I} means the identity tensor. We shall use upper index T for the corresponding covectors (line vectors). For any vector \mathbf{a} , \mathbf{b} we shall use the notation $\mathbf{a}^T \mathbf{b}$ for their *scalar product* (the line vector is multiplied by the column vector) and $\mathbf{a} \mathbf{b}^T$ for their *tensor product* (column vector is multiplied by line vector). For vector field $\mathbf{a} = (a^1, a^2, a^3)^T$ depending on $\mathbf{x} = (x^1, x^2, x^3)^T$ the notation $\frac{\partial \mathbf{a}}{\partial \mathbf{x}}$ means the matrix with components $\left(\frac{\partial \mathbf{a}}{\partial \mathbf{x}}\right)_j^i = \frac{\partial a^i}{\partial x^j}$, i means line and j means column. The divergence of a second order tensor is a line vector whose components are the divergence of each column ('french' definition, in the corresponding 'american' definition one takes the divergence of each line). In particular,

$$\operatorname{div}(\mathbf{a} \mathbf{b}^T) = \operatorname{div}(\mathbf{a}) \mathbf{b}^T + \mathbf{a}^T \left(\frac{\partial \mathbf{b}}{\partial \mathbf{x}}\right)^T.$$

The operators ∇ (column operator) and div are always used in the Eulerian coordinates.

1. HAMILTON'S PRINCIPLE

Let us suppose that a mixture of two fluids is well described by the average velocities \mathbf{v}_1 , \mathbf{v}_2 of each component and apparent densities $\rho_1 = \alpha_1 \rho_{10}$, $\rho_2 = \alpha_2 \rho_{20}$. Here α_i , $i = 1, 2$ are the volume fractions of components, $\alpha_1 + \alpha_2 = 1$, $\alpha_i \geq 0$ and ρ_{i0} are the true (physical) densities of each phase. To simplify, we will suppose that one of the densities is constant (for example, ρ_{20}). Let T be kinetic energy, and U be internal energy. In the following, we will consider only mechanical processes by removing thermal evolution. Hence, U is a purely mechanical part of total internal energy. The kinetic energy is represented by a standard formula:

$$T = \rho_1 \frac{|\mathbf{v}_1|^2}{2} + \rho_2 \frac{|\mathbf{v}_2|^2}{2}. \quad (1)$$

The internal energy U is taken in the form :

$$U(\rho_1, \rho_2) = \alpha_1 \rho_{10} \varepsilon_1(\rho_{10}) + \varepsilon_{int}(\alpha_2) = \rho_1 \varepsilon_1 \left(\frac{\rho_1}{1 - \frac{\rho_2}{\rho_{20}}} \right) + \rho_2 \varepsilon_{int} \left(\frac{\rho_2}{\rho_{20}} \right). \quad (2)$$

Here $\varepsilon_1(\rho_{10})$ is the specific energy corresponding to the compressibility of the first phase, while ε_{int} is the interaction energy depending only on the volume fraction α_2 . We formulate Hamilton's action in the form :

$$a = \int_{t_1}^{t_2} \int_{\mathcal{D}} L \, d\mathbf{x} dt, \quad L(\rho_1, \rho_2, \mathbf{v}_1, \mathbf{v}_2) = \rho_1 \frac{|\mathbf{v}_1|^2}{2} + \rho_2 \frac{|\mathbf{v}_2|^2}{2} - U(\rho_1, \rho_2). \quad (3)$$

The mass conservation equations are considered as kinematic constraints :

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 \mathbf{v}_1) = 0, \quad \frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2 \mathbf{v}_2) = 0. \quad (4)$$

Here $[t_1, t_2]$ is a time interval, \mathcal{D} is a fixed bounded domain of the three-dimensional space with the boundary $\partial\mathcal{D}$. We will suppose that the slipping condition on $\partial\mathcal{D}$ is fulfilled for each component.

The internal energy U is supposed to be a convex function of ρ_1, ρ_2 . We will see the consequences of this assumption later.

2. GOVERNING EQUATIONS : FIRST FORM OF THE LAGRANGIAN

We consider the variation of Hamilton's action (3) under the constrains (4). For this we need to introduce the virtual motions of each component. Let \mathbf{x} be the Eulerian coordinates, and \mathbf{X}_i be the Lagrangian coordinates of i -th component, $i = 1, 2$. The relation between the Lagrangian and Eulerian coordinates is given by the diffeomorphisms of the domain \mathcal{D}_{i0} (reference space of i -th component) into \mathcal{D} :

$$\mathbf{x} = \varphi_i(t, \mathbf{X}_i), \quad i = 1, 2. \quad (5)$$

The corresponding deformation gradients are defined as :

$$\mathbf{F}_i = \frac{\partial \varphi_i}{\partial \mathbf{X}_i}.$$

Let us define the virtual motion of the mixture :

$$\mathbf{x} = \Phi_i(t, \mathbf{X}_i, \varepsilon_i), \quad (6)$$

where ε_i varies in a small neighbourhood of zero. The real motion corresponds to $\varepsilon_i = 0$:

$$\Phi_i(t, \mathbf{X}_i, 0) = \varphi_i(t, \mathbf{X}_i).$$

The associated virtual displacements of particles are defined by :

$$\delta_i \mathbf{x}(t, \mathbf{X}_i) = \frac{\partial \Phi_i}{\partial \varepsilon_i}(t, \mathbf{X}_i, 0). \quad (7)$$

They will be denoted as $\zeta_i(t, \mathbf{x})$ in the Eulerian coordinates :

$$\zeta_i(t, \mathbf{x}) = \delta_i \mathbf{x}(t, \varphi_i^{-1}(t, \mathbf{x})). \quad (8)$$

Let

$$\hat{f}_i(t, \mathbf{x}, \varepsilon_i), \quad \tilde{f}_i(t, \mathbf{X}_i, \varepsilon_i) \equiv \hat{f}_i(t, \Phi_i(t, \mathbf{X}_i, \varepsilon_i), \varepsilon_i) \quad (9)$$

be perturbations of unknowns f_i of i -th component (the density, velocity etc.) in the Eulerian and Lagrangian coordinates, respectively. One defines Eulerian (with 'hat' sign) and Lagrangian (with 'tilde' sign) variations of f_i as :

$$\hat{\delta} f_i = \frac{\partial \hat{f}_i}{\partial \varepsilon_i}(t, \mathbf{x}, 0), \quad \tilde{\delta} f_i = \frac{\partial \tilde{f}_i}{\partial \varepsilon_i}(t, \mathbf{X}_i, 0).$$

Relations (7)–(9) imply [3], [7] :

$$\tilde{\delta} f_i = \hat{\delta} f_i + \frac{\partial f_i}{\partial \mathbf{x}} \delta_i \mathbf{x} = \hat{\delta} f_i + \nabla f_i \cdot \delta_i \mathbf{x}. \quad (10)$$

Then the mass conservation law

$$\rho_i \det \mathbf{F}_i = \rho_{0i}(\mathbf{X}_i) \quad (11)$$

implies :

$$\tilde{\delta} \rho_i = -\rho_i \operatorname{div}(\zeta_i).$$

Also, by definition :

$$\tilde{\delta}\mathbf{v}_i = \frac{\partial\delta_i\mathbf{x}}{\partial t}.$$

Using (10) one gets

$$\hat{\delta}\rho_i = -\operatorname{div}(\rho_i\zeta_i), \quad \hat{\delta}\mathbf{v}_i = \frac{D_i\zeta_i}{Dt} - \frac{\partial\mathbf{v}_i}{\partial\mathbf{x}}\zeta_i, \quad (12)$$

with

$$\frac{D_i}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_i^T\nabla. \quad (13)$$

Varying the unknowns (ρ_1, \mathbf{v}_1) and (ρ_2, \mathbf{v}_2) independently and denoting by $\delta_i a$ the corresponding variation of Hamilton's action, we obtain

$$\delta_i a = \int_{t_1}^{t_2} \int_{\mathcal{D}} \left(\frac{\partial L}{\partial\rho_i} \hat{\delta}\rho_i + \frac{\partial L}{\partial\mathbf{v}_i} \hat{\delta}\mathbf{v}_i \right) d\mathbf{x} dt.$$

Taking now into account the formulae (12), and the fact that the variations ζ_i vanish at the boundary of the four-dimensional domain $[t_1, t_2] \times \mathcal{D}$, one obtains integrating by parts :

$$\delta_i a = \int_{t_1}^{t_2} \int_{\mathcal{D}} -\rho_i \zeta_i^T \left(\frac{D_i \mathbf{K}_i}{Dt} + \left(\frac{\partial\mathbf{v}_i}{\partial\mathbf{x}} \right)^T \mathbf{K}_i - \nabla R_i \right) d\mathbf{x} dt, \quad i = 1, 2,$$

where

$$\rho_i \mathbf{K}_i^T = \frac{\partial L}{\partial\mathbf{v}_i}, \quad R_i = \frac{\partial L}{\partial\rho_i}. \quad (14)$$

It implies the equations of motion in the form :

$$\frac{D_i \mathbf{K}_i}{Dt} + \left(\frac{\partial\mathbf{v}_i}{\partial\mathbf{x}} \right)^T \mathbf{K}_i = \nabla R_i. \quad (15)$$

In particular, for the Lagrangian given by (3), one has

$$\mathbf{K}_i = \mathbf{v}_i, \quad R_i = \frac{|\mathbf{v}_i|^2}{2} - \frac{\partial U}{\partial\rho_i},$$

and (15)–(14) are reduced to the system

$$\frac{D_i \mathbf{v}_i}{Dt} + \nabla \left(\frac{\partial U}{\partial\rho_i} \right) = 0, \quad i = 1, 2. \quad (16)$$

The governing equations (15)–(14) have been already obtained in [6], [10] where the case of homogeneous gas mixtures was considered. In the case of heterogeneous mixtures where one of the phases is incompressible, the mathematical models coincide.

3. GOVERNING EQUATIONS : SECOND FORM OF THE LAGRANGIAN

Introducing the mass average variables ρ, \mathbf{v} by

$$\rho = \rho_1 + \rho_2, \quad \rho\mathbf{v} = \rho_1\mathbf{v}_1 + \rho_2\mathbf{v}_2, \quad (17)$$

one can rewrite the Lagrangian (3) in the form :

$$L = \rho_1 \frac{|\mathbf{v}_1|^2}{2} + \rho_2 \frac{|\mathbf{v}_2|^2}{2} - U(\rho_1, \rho_2) = \rho \frac{|\mathbf{v}|^2}{2} + \frac{\rho_1 \rho_2}{2\rho} |\mathbf{v}_2 - \mathbf{v}_1|^2 - U(\rho_1, \rho_2).$$

One can choose (ρ, \mathbf{v}) and, for example, (ρ_2, \mathbf{v}_2) as unknowns. Hence, the mass conservation laws are :

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \quad \frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2 \mathbf{v}_2) = 0. \quad (18)$$

Since

$$\rho_1 = \rho - \rho_2, \quad \mathbf{v}_2 - \mathbf{v}_1 = -\frac{\rho}{\rho_1} (\mathbf{v} - \mathbf{v}_2), \quad (19)$$

one gets

$$L = \rho \frac{|\mathbf{v}|^2}{2} + \frac{\rho \rho_2}{2(\rho - \rho_2)} |\mathbf{v}_2 - \mathbf{v}|^2 - U(\rho - \rho_2, \rho_2). \quad (20)$$

We may introduce in the same way the independent virtual displacements related now with the mean flow \mathbf{v} and the field \mathbf{v}_2 . Formally, it is sufficient to think about the mean field as ‘a new field’ with omitted subscript ‘1’. The equations of motion for the Lagrangian (20) are coming then from (15)–(14) :

$$\frac{D\mathbf{K}}{Dt} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right)^T \mathbf{K} = \nabla R, \quad (21)$$

$$\frac{D_2 \mathbf{K}_2}{Dt} + \left(\frac{\partial \mathbf{v}_2}{\partial \mathbf{x}} \right)^T \mathbf{K}_2 = \nabla R_2, \quad (22)$$

where

$$\begin{aligned} \rho \mathbf{K}^T &= \frac{\partial L}{\partial \mathbf{v}} = \rho \mathbf{v}_1^T, \quad R = \frac{\partial L}{\partial \rho} = \frac{|\mathbf{v}|^2}{2} - \frac{|\mathbf{v} - \mathbf{v}_1|^2}{2} - \frac{\partial U}{\partial \rho_1}, \\ \rho_2 \mathbf{K}_2^T &= \frac{\partial L}{\partial \mathbf{v}_2} = \rho_2 (\mathbf{v}_2 - \mathbf{v}_1)^T, \quad R_2 = \frac{\partial L}{\partial \rho_2} = \frac{|\mathbf{v}_2 - \mathbf{v}_1|^2}{2} + \frac{\partial U}{\partial \rho_1} - \frac{\partial U}{\partial \rho_2}, \end{aligned}$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.$$

After some manipulations, we can rewrite (21)–(22) in the variables \mathbf{v}_1 and \mathbf{v}_2 :

$$\frac{D_1 \mathbf{v}_1}{Dt} + \operatorname{curl}(\mathbf{v}_1) \wedge (\mathbf{v} - \mathbf{v}_1) + \nabla \left(\frac{\partial U}{\partial \rho_1} \right) = 0, \quad (23)$$

$$\frac{D_2 \mathbf{v}_2}{Dt} + \operatorname{curl}(\mathbf{v}_1) \wedge (\mathbf{v} - \mathbf{v}_2) + \nabla \left(\frac{\partial U}{\partial \rho_2} \right) = 0. \quad (24)$$

Complemented by the mass conservation laws (4) they form a closed system of equations. System (16) and (23)–(24) are different from each other by ‘lift’ forces $\mathbf{F}_i = -\operatorname{curl}(\mathbf{v}_1) \wedge (\mathbf{v} - \mathbf{v}_i)$, $i = 1, 2$. Finally, if we use the third set of variables, $(\rho, \rho_1, \mathbf{v}, \mathbf{v}_1)$ we obtain again a new system of equations with different ‘lift’ forces. The ‘lift’ forces appear because we used different variations : one is related to the average velocity, and the other one to the velocity of one of the components. Of course, all systems obtained admit the same full momentum and energy conservation laws :

$$\begin{aligned} \frac{\partial}{\partial t} (\rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2)^T + \operatorname{div} \left(\rho_1 \mathbf{v}_1 \mathbf{v}_1^T + \rho_2 \mathbf{v}_2 \mathbf{v}_2^T + \left(\rho_1 \frac{\partial U}{\partial \rho_1} + \rho_2 \frac{\partial U}{\partial \rho_2} - U \right) \mathbf{I} \right) &= 0, \\ \frac{\partial}{\partial t} \left(\rho_1 \frac{|\mathbf{v}_1|^2}{2} + \rho_2 \frac{|\mathbf{v}_2|^2}{2} + U \right) + \operatorname{div} \left(\rho_1 \mathbf{v}_1 \left(\frac{|\mathbf{v}_1|^2}{2} + \frac{\partial U}{\partial \rho_1} \right) + \rho_2 \mathbf{v}_2 \left(\frac{|\mathbf{v}_2|^2}{2} + \frac{\partial U}{\partial \rho_2} \right) \right) &= 0. \end{aligned}$$

I will now restrict my attention on common features and differences between the corresponding equations.

4. HYPERBOLICITY STUDY

The equations (23)–(24) contain non-conservative terms related with the ‘lift’ forces. The idea is to consider an augmented system containing the vorticity $\text{curl}(\mathbf{v}_1)$ as a new variable. In this case the ‘lift’ force can be considered as the right-hand side. Indeed, the equation (23) can be written as

$$\frac{\partial \mathbf{v}_1}{\partial t} + \text{curl}(\mathbf{v}_1) \wedge \mathbf{v} + \nabla \left(\frac{|\mathbf{v}_1|^2}{2} + \frac{\partial U}{\partial \rho_1} \right) = 0,$$

Taking curl from this equation, one obtains for $\boldsymbol{\omega}_1 = \text{curl}(\mathbf{v}_1)$:

$$\frac{\partial \boldsymbol{\omega}_1}{\partial t} + \text{curl}(\boldsymbol{\omega}_1 \wedge \mathbf{v}) = 0,$$

Using the identity :

$$(\text{curl}(\mathbf{a} \wedge \mathbf{b}))^T = \text{div}(\mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T),$$

one obtains a conservative equation for $\boldsymbol{\omega}_1$:

$$\frac{\partial \boldsymbol{\omega}_1^T}{\partial t} + \text{div}(\mathbf{v}\boldsymbol{\omega}_1^T - \boldsymbol{\omega}_1\mathbf{v}^T) = 0.$$

Since

$$\mathbf{v} - \mathbf{v}_1 = \frac{\rho_2}{\rho}(\mathbf{v}_2 - \mathbf{v}_1), \quad \mathbf{v} - \mathbf{v}_2 = -\frac{\rho_1}{\rho}(\mathbf{v}_2 - \mathbf{v}_1),$$

one obtains the following system of equations:

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \text{div}(\rho_1 \mathbf{v}_1) &= 0, \\ \frac{\partial \rho_2}{\partial t} + \text{div}(\rho_2 \mathbf{v}_2) &= 0, \\ \frac{\partial(\rho_1 \mathbf{v}_1)}{\partial t} + (\text{div}(\rho_1 \mathbf{v}_1 \mathbf{v}_1^T))^T + \nabla \left(\rho_1 \frac{\partial U}{\partial \rho_1} - U \right) + \frac{\rho_1 \rho_2}{\rho} \boldsymbol{\omega}_1 \wedge (\mathbf{v}_2 - \mathbf{v}_1) &= -\frac{\partial U}{\partial \rho_2} \nabla \rho_2, \\ \frac{\partial(\rho_2 \mathbf{v}_2)}{\partial t} + (\text{div}(\rho_2 \mathbf{v}_2 \mathbf{v}_2^T))^T + \nabla \left(\rho_2 \frac{\partial U}{\partial \rho_2} \right) - \frac{\rho_1 \rho_2}{\rho} \boldsymbol{\omega}_1 \wedge (\mathbf{v}_2 - \mathbf{v}_1) &= \frac{\partial U}{\partial \rho_2} \nabla \rho_2, \\ \frac{\partial \boldsymbol{\omega}_1^T}{\partial t} + \text{div}(\mathbf{v}\boldsymbol{\omega}_1^T - \boldsymbol{\omega}_1\mathbf{v}^T) &= 0. \end{aligned} \tag{25}$$

The last equation can also be rewritten in non-conservative form :

$$\frac{\partial}{\partial t} \left(\frac{\boldsymbol{\omega}_1}{\rho} \right) + \frac{\partial}{\partial \mathbf{x}} \left(\frac{\boldsymbol{\omega}_1}{\rho} \right) \mathbf{v} - \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \left(\frac{\boldsymbol{\omega}_1}{\rho} \right) = 0.$$

The equation for the vorticity $\boldsymbol{\omega}_1$ is exactly the Helmholtz equation, but the main feature of this equation is that the vorticity of the phase ‘1’ is transported by the *average velocity* \mathbf{v} . The extended system of equations thus adds new contact characteristics corresponding to the average velocity.

The equations of momentum for each phase are reminiscent of those of the classical Baer-Nunziato model [1], with the interface pressure p_I given by

$$p_I = \rho_{20} \frac{\partial U}{\partial \rho_2}.$$

Indeed, the phase ‘2’ is supposed to be incompressible ($\rho_{20} = \text{const}$), so the right hand side of the momentum equations for each phase represents the classical ‘nozzling’ term [1] :

$$p_I \nabla \alpha_i, \quad i = 1, 2.$$

The only difference is the presence of ‘lift’ forces.

Considering the extended set of variables $\mathbf{W} = (\rho_1, \rho_2, \mathbf{v}_1^T, \mathbf{v}_2^T, \boldsymbol{\omega}_1^T/\rho)^T$ and taking into account the definition of the average velocity $\mathbf{v} = (\rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2)/\rho$, one can easily prove in 1D case that the equations are hyperbolic, if the following polynomial $P(\lambda)$ has real roots :

$$P(\lambda) = (u_1 - \lambda)^2(u_2 - \lambda)^2 - \rho_2 U_{22}(u_1 - \lambda)^2 - \rho_1 U_{11}(u_2 - \lambda)^2 + \rho_1 \rho_2 (U_{11} U_{22} - U_{12}^2).$$

Here

$$U_{ij} = \frac{\partial^2 U}{\partial \rho_i \partial \rho_j}, \quad i, j = 1, 2.$$

and u_1, u_2 are the x -velocity components of \mathbf{v}_1 and \mathbf{v}_2 . The general 3D case is reduced to the 1D case by using the invariance of the governing equations under rotation. If the relative velocity $u_2 - u_1$ is small and U is convex, all eigenvalues λ are real. Let us note that the system (16) complemented by the mass conservation laws has exactly the same characteristic polynomial. It means that the ‘lift’ forces don’t change the hyperbolicity criterion.

Why then the ‘lift’ forces are important? According to the presence or absence of the lift forces, for the same values of the total momentum (and total energy) one has different distributions of the densities and velocities of components. In the following Section we give an example of exact solutions confirming this statement.

5. EXACT SOLUTIONS

We construct here exact solutions to (25). Let $\mathbf{x} = (x, y, z)^T$. Consider shear flows where the horizontal velocity of each component (their x and y components) as well as their densities depend only on the vertical coordinate z :

$$\mathbf{v}_1 = (u_1(z), v_1(z), 0)^T, \quad \mathbf{v}_2 = (u_2(z), v_2(z), 0)^T, \quad \rho_1 = \rho_1(z), \quad \rho_2 = \rho_2(z).$$

The ‘lift’ force has only one component (in z -direction) corresponding to the unit vector \mathbf{k} . Indeed, since

$$\boldsymbol{\omega}_1 = \left(-\frac{dv_1}{dz}, \frac{du_1}{dz}, 0 \right)^T,$$

one has

$$\boldsymbol{\omega}_1 \wedge (\mathbf{v}_2 - \mathbf{v}_1) = H(z) \mathbf{k},$$

where

$$H(z) = -\frac{dv_1}{dz}(v_2 - v_1) - \frac{du_1}{dz}(u_2 - u_1).$$

The momentum equation implies :

$$\rho_1 \frac{\partial U}{\partial \rho_1} + \rho_2 \frac{\partial U}{\partial \rho_2} - U = \text{const}. \quad (26)$$

It is complemented by the differential equation coming from the momentum equation (23) :

$$\frac{d}{dz} \left(\frac{\partial U}{\partial \rho_1} \right) + \frac{\rho_2}{\rho} H(z) = 0. \quad (27)$$

The last one can also be integrated. Indeed, let us consider the Legendre transform $U^*(\rho_1^*, \rho_2^*)$ of $U(\rho_1, \rho_2)$. Here $\rho_i^* = \frac{\partial U}{\partial \rho_i}$, $i = 1, 2$. In particular, (26) implies

$$U^*(\rho_1^*, \rho_2^*) = \text{const}, \quad (28)$$

Then (27) can be integrated in the form

$$\rho_1^* - \rho_2^* + \int^z H(s) ds = \text{const}, \quad (29)$$

since

$$\frac{\partial U^*}{\partial \rho_1^*} \frac{d\rho_1^*}{dz} + \frac{\partial U^*}{\partial \rho_2^*} \frac{d\rho_2^*}{dz} = 0.$$

The relations (28) and (29) give a very simple algorithm to find the solution $\rho_1^*(z)$ and $\rho_2^*(z)$. For this it is sufficient to find the intersection of the closed convex curve defined by (28) with the straight line (29) at any z . Once ρ_i^* are found as functions of z , the densities ρ_i can be found (see Figure 1). When the ‘lift’ forces are absent, such a solution is constant.

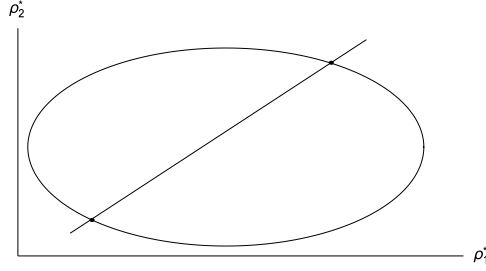


FIGURE 1. The variables ρ_1^* and ρ_2^* are conjugate to ρ_1 and ρ_2 . For a given shear flow, they can be determined at each z as the intersection of the level set of $U^*(\rho_1^*, \rho_2^*)$ and the straight line (29). *A priori*, one has two possible solutions.

6. EXTERNAL AND FRICTION FORCES

One can add external conservative and friction forces into the system with ‘lift’ forces in the following way:

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \text{div}(\rho_1 \mathbf{v}_1) &= 0, \\ \frac{\partial \rho_2}{\partial t} + \text{div}(\rho_2 \mathbf{v}_2) &= 0, \\ \frac{\partial(\rho_1 \mathbf{v}_1)}{\partial t} + (\text{div}(\rho_1 \mathbf{v}_1 \mathbf{v}_1^T))^T + \nabla \left(\rho_1 \frac{\partial U}{\partial \rho_1} - U \right) + \frac{\rho_1 \rho_2}{\rho} \boldsymbol{\omega}_1 \wedge (\mathbf{v}_2 - \mathbf{v}_1) &= -\frac{\partial U}{\partial \rho_2} \nabla \rho_2 + \rho_1 \mathbf{g} + \mu(\mathbf{v}_2 - \mathbf{v}_1), \\ \frac{\partial(\rho_2 \mathbf{v}_2)}{\partial t} + (\text{div}(\rho_2 \mathbf{v}_2 \mathbf{v}_2^T))^T + \nabla \left(\rho_2 \frac{\partial U}{\partial \rho_2} \right) - \frac{\rho_1 \rho_2}{\rho} \boldsymbol{\omega}_1 \wedge (\mathbf{v}_2 - \mathbf{v}_1) &= \frac{\partial U}{\partial \rho_2} \nabla \rho_2 + \rho_2 \mathbf{g} - \mu(\mathbf{v}_2 - \mathbf{v}_1). \\ \frac{\partial \boldsymbol{\omega}_1}{\partial t} + (\text{div}(\mathbf{v} \boldsymbol{\omega}_1^T - \boldsymbol{\omega}_1 \mathbf{v}^T))^T &= \text{curl} \left(\frac{\mu}{\rho_1} (\mathbf{v}_2 - \mathbf{v}_1) \right). \end{aligned} \quad (30)$$

Here μ is a positive parameter, \mathbf{g} is the gravity force, and the friction forces are proportional to the relative velocity. The friction force is responsible for the vorticity creation due to the relative motion of components. The total energy is obviously a Lyapunov function for this system.

CONCLUSION

When two-fluid heterogeneous mixtures are studied, we have a choice in choosing the corresponding reference spaces. One can describe the motion by using the reference spaces related to each component. Another possibility would be to use the reference space for the average motion supplemented by the reference space of one of the components. The corresponding governing equations are different, but they have the same total momentum and total energy equations. The difference is in gyroscopic ('lift') forces, which appear as additional (non-algebraic) internal forces in the governing equations for each component. They don't influence the hyperbolicity condition, but change the distribution of the velocities and densities of components. In particular, one can hypothesize, that under some condition the 'lift' forces can maintain the solution in the hyperbolicity domain if initially it was already inside this domain.

The possibility to obtain three different models from the same Lagrangian is indeed very disturbing. Further study is needed to understand better the nature of gyroscopic forces which are 'hidden' inside the Lagrangian.

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