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SOME MATHEMATICAL PROPERTIES OF A BAROTROPIC MULTIPHASE FLOW MODEL

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Abstract. We study a model for compressible multiphase flows involving N non miscible barotopic phases where N is arbitrary. This model boils down to the barotropic Baer-Nunziato model when $N = 2$. We prove the weak hyperbolicity property, the non-strict convexity of the natural mathematical entropy, and the existence of a symmetric form.

Keywords: Multiphase flows, Compressible flows, Hyperbolic PDEs, Entropy, Symmetrizable systems.

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Introduction

The modeling and numerical simulation of multiphase flows is a relevant approach for a detailed investigation of some patterns occurring in many industrial sectors.

In [4,8,9,14], some modeling efforts have been provided for the design of compressible multiphase flow models allowing unique jump conditions and for which the initial-value problem is well posed. The N-phase flow models developed therein consist in an extension to $N \geq 3$ phases of the well-known Baer-Nunziato two phase flow model [1]. As in the Baer-Nunziato model, the PDEs are composed of a hyperbolic first order convective part consisting in N Euler-like systems coupled through non-conservative terms and zero-th source terms accounting for pressure, velocity and temperature relaxation phenomena between the phases. It is worth noting that the latter models are quite similar to the classical two phase flow models in [2,3,6].

In [5,11], two crucial properties have been proven for a class of two phase flow models containing the Baer-Nunziato model, namely, the convexity of the natural entropy associated with the system, and the existence of a symmetric form. As recalled in that paper, such properties are well understood for systems of conservation laws since Godunov [7] and Mock [13], but remain an open question for non conservative and non strictly hyperbolic models such as those considered here.

In the present paper, we prove the convexity of the entropy and the existence of a symmetric form for a barotropic multiphase flow model with N - where N is arbitrarily large - phases. We restrict the study to the case where the interfacial velocity coincides with one of the phasic material velocities.

1. The barotropic multiphase flow model

We consider the following system of partial differential equations (PDEs) which is the convective part of the system introduced in [9] for the modeling of the evolution of N distinct non-miscible compressible phases in a

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one dimensional space: for $k = 1, .., N$, $x \in \mathbb{R}$ and $t > 0$:

$$\partial_t \alpha_k + u_1 \partial_x \alpha_k = 0, \tag{1a}$$

$$\partial_t (\alpha_k \rho_k) + \partial_x (\alpha_k \rho_k u_k) = 0, \tag{1b}$$

$$\partial_t (\alpha_k \rho_k u_k) + \partial_x \left(\alpha_k \rho_k u_k^2 + \alpha_k p_k\right) + \textstyle\sum_{\substack{l=1\\ l\neq k}}^N \mathscr{P}_{kl}(U) \partial_x \alpha_l = 0. \tag{1c}$$

The model consists in N coupled Euler-type systems. The quantities α_k, ρ_k and u_k represent the mean statistical fraction, the mean density and the mean velocity in phase k (for $k = 1, .., N$). The quantity p_k is the pressure in phase k. We assume barotropic pressure laws for each phase so that the pressure p_k is a given function of the density $p_k : \rho_k \mapsto p_k(\rho_k)$ with the classical assumption that $p_k'(\rho_k) > 0$. The mean statistical fractions and the mean densities are positive and the following saturation constraint, which expresses the non-miscibility of the phases, holds everywhere at every time:

$$\sum_{k=1}^N \alpha_k = 1. \tag{2}$$

Thus, among the N equations (1a), $N-1$ are independent and the main unknown U is expected to belong to the physical space:

$$\Omega_U = \Big\{ U = (\alpha_2, .., \alpha_N, \alpha_1\rho_1, .., \alpha_N\rho_N, \alpha_1\rho_1 u_1, .., \alpha_N\rho_N u_N)^T \in \mathbb{R}^{3N-1},$$
$$\text{such that } 0 < \alpha_2, .., \alpha_N, \textstyle\sum_{k=2}^N \alpha_k < 1 \text{ and } \alpha_k\rho_k > 0 \text{ for all } k = 1, .., N \Big\}.$$

Following [9], we make the following choice for the closure laws of the so-called interface pressures $\mathscr{P}_{kl}(U)$:

$$\begin{array}{lll} \text{for } k = 1, & \mathscr{P}_{1l}(U) = p_l(\rho_l), & \text{for } l = 2, .., N \\ \text{for } k \neq 1, & \mathscr{P}_{kl}(U) = p_k(\rho_k), & \text{for } l = 1, .., N,\ l \neq k. \end{array} \tag{3}$$

Observing that the saturation constraint gives $\sum_{l=1, l\neq k}^N \partial_x \alpha_l = -\partial_x \alpha_k$ for all $k = 1, .., N$ the momentum equations (1c) can be simplified as follows:

$$\partial_t (\alpha_1 \rho_1 u_1) + \partial_x \left(\alpha_1 \rho_1 u_1^2 + \alpha_1 p_1(\rho_1)\right) + \textstyle\sum_{l=2}^N p_l(\rho_l) \partial_x \alpha_l = 0, \tag{4}$$

$$\partial_t (\alpha_k \rho_k u_k) + \partial_x \left(\alpha_k \rho_k u_k^2 + \alpha_k p_k(\rho_k)\right) - p_k(\rho_k) \partial_x \alpha_k = 0, \quad k = 2, .., N. \tag{5}$$

2. Eigenstructure of the system

The following result characterizes the wave structure of system (1):

Theorem 2.1. *System (1) is weakly hyperbolic on Ω_U : it admits the following $3N-1$ real eigenvalues: $\sigma_1(U) = .. = \sigma_{N-1}(U) = u_1$, $\sigma_{N-1+k}(U) = u_k - c_k(\rho_k)$ for $k = 1, .., N$ and $\sigma_{2N-1+k}(U) = u_k + c_k(\rho_k)$ for $k = 1, .., N$, where $c_k(\rho_k) = \sqrt{p_k'(\rho_k)}$. The corresponding right eigenvectors are linearly independent if, and only if,*

$$|u_1 - u_k| \neq c_k(\rho_k), \quad \forall k = 2, .., N. \tag{6}$$

The characteristic field associated with $\sigma_1(U), .., \sigma_{N-1}(U)$ is linearly degenerate while the characteristic fields associated with $\sigma_{N-1+k}(U)$ and $\sigma_{2N-1+k}(U)$ for $k = 1, .., N$ are genuinely non-linear. When (6) is violated, the system is said to be resonant.

Proof. In the following, we denote p_k and c_k instead of $p_k(\rho_k)$ and $c_k(\rho_k)$ for $k=1,..N$ in order to ease the notations. Choosing the variable $\mathcal{U}=(\alpha_2,..,\alpha_N,u_1,p_1,..,u_N,p_N)^T$, the smooth solutions of system (1) satisfy the following equivalent system:

$$\partial_t\mathcal{U}+\mathscr{A}(\mathcal{U})\partial_x\mathcal{U}=0,$$

where $\mathscr{A}(\mathcal{U})$ is the block matrix:

$$\mathscr{A}(\mathcal{U})=\left(\begin{array}{c|ccc} A & & \mathbf{0} & \\ \hline B_1 & C_1 & & \\ \vdots & & \ddots & \\ B_N & & & C_N \end{array}\right). \tag{7}$$

Defining $M_k=(u_k-u_1)/c_k$ the Mach number of phase k relatively to phase 1 for $k=2,..,N$, the matrices A, $B_1,..,B_N$ and $C_1,..,C_N$ are given as follows.

$$\begin{aligned}
&A=\mathrm{diag}(u_1,..,u_1)\in\mathbb{R}^{(N-1)\times(N-1)},\\
&B_1=\left(\frac{1}{\alpha_1\rho_1}\sum_{k=2}^{N}(p_k-p_1)\delta_{i,1}\,\delta_{j+1,k}\right)_{\substack{1\le i\le 2\\1\le j\le N-1}}\in\mathbb{R}^{2\times(N-1)},\\
&B_k=\left(\frac{\rho_k\,M_k\,c_k^3}{\alpha_k}\delta_{i,2}\,\delta_{j+1,k}\right)_{\substack{1\le i\le 2\\1\le j\le N-1}}\in\mathbb{R}^{2\times(N-1)},\quad\text{for } k=2,..,N,\\
&C_k=\begin{pmatrix}u_k & 1/\rho_k\\ \rho_k c_k^2 & u_k\end{pmatrix},\quad\text{for } k=1,..,N,
\end{aligned}$$

where $\delta_{p,q}$ is the Kronecker symbol: for $p,q\in\mathbb{N}$, $\delta_{p,q}=1$ if $p=q$ and $\delta_{p,q}=0$ otherwise. Since A is diagonal and C_k is $\mathbb{R}$-diagonalizable with eigenvalues u_k-c_k and u_k+c_k, the matrix $\mathscr{A}(\mathcal{U})$ admits the eigenvalues u_1 (with multiplicity $N-1$), u_k-c_k and u_k+c_k for $k=1,..,N$. In addition, $\mathscr{A}(\mathcal{U})$ is $\mathbb{R}$-diagonalizable provided that the corresponding right eigenvectors span $\mathbb{R}^{3N-1}$. The right eigenvectors are the columns of the following block matrix:

$$\mathcal{R}(\mathcal{U})=\left(\begin{array}{c|ccc} A' & & \mathbf{0} & \\ \hline B'_1 & C'_1 & & \\ \vdots & & \ddots & \\ B'_N & & & C'_N \end{array}\right),$$

where A', $B'_1,..,B'_N$ and $C'_1,..,C'_N$ are matrices defined by:

$$\begin{aligned}
&A=\mathrm{diag}(1-M_2^2,..,1-M_N^2)\in\mathbb{R}^{(N-1)\times(N-1)}\\
&B'_1=\left(-\frac{1}{\alpha_1}\sum_{k=2}^{N}(p_k-p_1)(1-M_k^2)\delta_{i,2}\,\delta_{j+1,k}\right)_{\substack{1\le i\le 2\\1\le j\le N-1}}\in\mathbb{R}^{2\times(N-1)},\\
&B'_k=\left(\Big(-\frac{M_k c_k}{\alpha_k}\delta_{i,1}+\frac{\rho_k(c_k M_k)^2}{\alpha_k}\delta_{i,2}\Big)\,\delta_{j+1,k}\right)_{\substack{1\le i\le 2\\1\le j\le N-1}}\in\mathbb{R}^{2\times(N-1)},\\
&\text{for } k=2,..,N,\\
&C'_k=\begin{pmatrix}-1 & 1\\ \rho_k c_k & \rho_k c_k\end{pmatrix},\quad\text{for } k=1,..,N.
\end{aligned}$$

The first $N-1$ columns are the eigenvectors associated with the eigenvalue u_1. For $k=1,..,N$, the $\big(N+2(k-1)\big)$-th and $\big(N+(2k-1)\big)$-th columns are the eigenvectors associated with u_k-c_k and u_k+c_k respectively. We can see that $\mathcal{R}(\mathcal{U})$ is invertible if and only if $M_k \neq 1$ for all $k = 2, .., N$ *i.e.* if and only if inequations (6) hold. Denote $(\mathcal{R}_j(\mathcal{U}))_{1\leq j\leq 3N-1}$ the columns of $\mathcal{R}(\mathcal{U})$. If $1 \leq j \leq N-1$, we can see that the N-th component of $\mathcal{R}_j(\mathcal{U})$ is zero. This implies that for all $1 \leq j \leq N-1$, $\mathcal{R}_j(\mathcal{U}) \cdot \nabla_{\mathcal{U}}(u_1) = 0$. Hence, the field associated with the eigenvalue u_1 is linearly degenerated. Now we observe that all the acoustic fields are genuinely non linear since for all $k = 1, .., N$:

$$\mathcal{R}_{N+2(k-1)}(\mathcal{U}) \cdot \nabla_{\mathcal{U}}(u_k - c_k) = -1 - \rho_k c_k \frac{\partial c_k}{\partial p_k} \neq 0,$$
$$\mathcal{R}_{N+(2k-1)}(\mathcal{U}) \cdot \nabla_{\mathcal{U}}(u_k + c_k) = 1 + \rho_k c_k \frac{\partial c_k}{\partial p_k} \neq 0.$$

□

Proposition 2.2. *The linearly degenerated field* $\sigma_1(U) = .. = \sigma_{N-1}(U) = u_1$ *admits the following* $2N$ *independent Riemann invariants:*

$$\psi_1(U) = u_1,$$
$$\psi_2(U) = \sum_{l=1}^{N} \left(\alpha_l p_l(\rho_l) + \alpha_l \rho_l (u_l - u_1)^2\right),$$
$$\psi_{1+k}(U) = \alpha_k \rho_k (u_k - u_1), \quad \textit{for } k = 2, .., N,$$
$$\psi_{N+k}(U) = e_k(\rho_k) + \frac{p_k(\rho_k)}{\rho_k} + \frac{1}{2}(u_k - u_1)^2, \quad \textit{for } k = 2, .., N.$$

Proof. Denoting $\mathcal{U} = (\alpha_2, .., \alpha_N, u_1, p_1, .., u_N, p_N)^T$, one must check that for $p = 1, .., 2N$, $\nabla_{\mathcal{U}}\psi_p(\mathcal{U}) \cdot \mathcal{R}_j(\mathcal{U}) = 0$ for all $j = 1, .., N-1$ where $(\mathcal{R}_j(\mathcal{U}))_{1\leq j\leq N-1}$ are the eigenvectors associated with the eigenvalue $\sigma_1(\mathcal{U}) = .. = \sigma_{N-1}(\mathcal{U}) = u_1$. The computation is tedious but straightforward. □

3. Mathematical Entropy

An important consequence of the closure law (3) for the interface pressures $\mathscr{P}_{kl}(U)$ is the existence of an additional conservation law for the smooth solutions of (1). Defining the specific internal energy of phase k, e_k by $e_k'(\rho_k) = p_k(\rho_k)/\rho_k^2$ and the specific total energy of phase k by $E_k = u_k^2/2 + e_k(\rho_k)$, the smooth solutions of (1) satisfy the following identities:

$$\partial_t\left(\alpha_1\rho_1 E_1\right) + \partial_x\left(\alpha_1\rho_1 E_1 u_1 + \alpha_1 p_1(\rho_1) u_1\right) + u_1 \textstyle\sum_{l=2}^{N} p_l(\rho_l)\partial_x\alpha_l = 0, \quad (8)$$
$$\partial_t\left(\alpha_k\rho_k E_k\right) + \partial_x\left(\alpha_k\rho_k E_k u_k + \alpha_k p_k(\rho_k) u_k\right) - u_1 p_k(\rho_k)\partial_x\alpha_k = 0, \quad k = 2, .., N. \quad (9)$$

Summing for $k = 1, .., N$,the smooth solutions of (1) are seen to satisfy the following additional conversation equation which expresses the conservation of the total mixture energy :

$$\partial_t\left(\textstyle\sum_{k=1}^{N} \alpha_k\rho_k E_k\right) + \partial_x\left(\textstyle\sum_{k=1}^{N}\left(\alpha_k\rho_k E_k u_k + \alpha_k p_k(\rho_k) u_k\right)\right) = 0. \quad (10)$$

As regards the non-smooth weak solutions of (1), one has to add a so-called *entropy criterion* in order to select the relevant physical solutions. For this purpose, we prove the following result.

Theorem 3.1. *For all* $k = 1, .., N$*, the fractional specific energy of phase* k *defined by*

$$(\alpha_k\rho_k E_k) : U \mapsto (\alpha_k\rho_k E_k)(U),$$

is a non strictly convex function of U*. Consequently, the total mixture energy, defined by* $\left(\sum_{k=1}^{N}\alpha_k\rho_k E_k\right)(U)$ *is also a non strictly convex function of* U*. In the light of* (10)*, the total mixture energy is a mathematical entropy of system* (1)*.*

Proof. For all $k = 1,..,N$, define $V_k = (\rho_k, \rho_k u_k)^T$ the monophasic state vector of phase k and define $U_k = (\alpha_k, \alpha_k\rho_k, \alpha_k\rho_k u_k)^T = (\alpha_k, \alpha_k V_k^T)^T$. The monophasic mathematical entropy of phase k is given by:

$$\mathcal{S}_k(\rho_k, \rho_k u_k) = \mathcal{S}_k(V_k) = \rho_k\Big(\frac{(\rho_k u_k)^2}{2\rho_k^2} + e_k(\rho_k)\Big).$$

Defining $\mathscr{S}_k(U_k) = \alpha_k \mathcal{S}_k\left(\frac{\alpha_k V_k}{\alpha_k}\right)$, we have $(\alpha_k\rho_k E_k)(U) = \mathscr{S}_k(U_k)$ for $k = 1,..,N$. Without loss of generality, we can rearrange the components of U and assume that: $U = \left(\alpha_1\rho_1, \alpha_1\rho_1 u_1, U_2^T, U_3^T, .., U_N^T\right)^T$. Thus, for $k = 2,..,N$, $(\alpha_k\rho_k E_k)(U)$ solely depends on U_k while $(\alpha_1\rho_1 E_1)(U)$ depends on $(\alpha_1\rho_1, \alpha_1\rho_1 u_1)$ and on all U_k for $k = 2,..,N$ through its dependence on $\alpha_1 = 1 - \sum_{k=2}^N \alpha_k$.

Case 1: Convexity of $(\alpha_k\rho_k E_k)(U)$ *for* $k = 2,..,N$: The matrix $(\alpha_k\rho_k E_k)''(U)$ has the following block-diagonal structure for $k = 2,..,N$:

$$(\alpha_k\rho_k E_k)''(U) = \text{block-diag}\left(0_{\mathbb{R}^{2\times 2}}, 0_{\mathbb{R}^{3\times 3}}, .., 0_{\mathbb{R}^{3\times 3}}, \mathscr{S}_k''(U_k), 0_{\mathbb{R}^{3\times 3}}, .., 0_{\mathbb{R}^{3\times 3}}\right).$$

Hence, $(\alpha_k\rho_k E_k)''(U)$ is a positive matrix if and only if $\mathscr{S}_k''(U_k)$ is a positive matrix. Since $\mathscr{S}_k(U_k) = \alpha_k \mathcal{S}_k\left(\frac{\alpha_k V_k}{\alpha_k}\right)$, differentiating twice, we obtain that the matrix $\mathscr{S}_k''(U_k)$ is the 3×3 matrix given by:

$$\mathscr{S}_k''(U_k) = \left(\begin{array}{c|c} A_k & B_k^T \\ \hline B_k & C_k \end{array}\right)$$

with

$$\begin{aligned} A_k &= \frac{1}{\alpha_k} V_k^T \mathcal{S}_k''(V_k) V_k \in \mathbb{R}, \\ B_k &= -\frac{1}{\alpha_k} \mathcal{S}_k''(V_k) V_k \in \mathbb{R}^{2\times 1}, \\ C_k &= \frac{1}{\alpha_k} \mathcal{S}_k''(V_k) \in \mathbb{R}^{2\times 2}. \end{aligned} \tag{11}$$

Let be given $(a, \mathbf{b}^T)^T \in \mathbb{R}^{3\times 1}$ with $a \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^{2\times 1}$. Then, we easily see that:

$$\begin{aligned} &(a, \mathbf{b}^T)\mathscr{S}_k''(U_k)(a, \mathbf{b}^T)^T \\ &\quad = a^2 A_k + 2aB_k^T\mathbf{b} + \mathbf{b}^T C_k \mathbf{b} \\ &\quad = a^2\frac{1}{\alpha_k} V_k^T \mathcal{S}_k''(V_k) V_k - 2a\frac{1}{\alpha_k}\mathbf{b}^T \mathcal{S}_k''(V_k) V_k + \frac{1}{\alpha_k}\mathbf{b}^T \mathcal{S}_k''(V_k)\mathbf{b} \\ &\quad = \frac{1}{\alpha_k}(aV_k - \mathbf{b})^T \mathcal{S}_k''(V_k)(aV_k - \mathbf{b}). \end{aligned}$$

Since $\mathcal{S}_k''(V_k)$ is a positive matrix by the strict convexity of the monophasic mathematical entropy $\mathcal{S}_k$, the right hand side is positive, which yields the positivity of the matrix $\mathscr{S}_k''(U_k)$ and hence the (non-strict) convexity of $(\alpha_k\rho_k E_k)(U)$ for $k = 2,..,N$.

Case 2: Convexity of $(\alpha_1\rho_1 E_1)(U)$: We have

$$(\alpha_1\rho_1 E_1)(U) = \big(1 - \sum_{k=2}^N \alpha_k\big)\mathcal{S}_1\left(\frac{\alpha_1 V_1}{1 - \sum_{k=2}^N \alpha_k}\right).$$

Thus, the Hessian matrix $(\alpha_1\rho_1 E_1)''(U)$ has the following structure:

$$(\alpha_1\rho_1 E_1)''(U) = \left(\begin{array}{c|ccc} C_1 & -\mathbf{B_1} & \cdots & -\mathbf{B_1} \\ \hline -\mathbf{B_1}^{\mathbf{T}} & \mathbf{A_1} & \cdots & \mathbf{A_1} \\ \vdots & \vdots & & \vdots \\ -\mathbf{B_1}^{\mathbf{T}} & \mathbf{A_1} & \cdots & \mathbf{A_1} \end{array}\right).$$

Defining A_1, B_1 and C_1 as in (11), the matrices $\mathbf{A_1}$ and $\mathbf{B_1}$ are given by:

$$\mathbf{A_1} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3}, \qquad -\mathbf{B_1} = \left(\begin{array}{c|cc} -B_1 & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{array}\right) \in \mathbb{R}^{2\times 3}.$$

Let be given $\mathbf{x} = (\mathbf{b}_1^T, a_2, \mathbf{b}_2^T, a_3, \mathbf{b}_3^T, .., a_N, \mathbf{b}_N^T)^T \in \mathbb{R}^{(3N-1)\times 1}$ with $a_k \in \mathbb{R}$ for $k = 2, .., N$ and $\mathbf{b}_k \in \mathbb{R}^{2\times 1}$ for all $k = 1, .., N$. An easy computation gives:

$$\mathbf{x}^T(\alpha_1\rho_1 E_1)''(U)\mathbf{x} = \mathbf{b}_1^T C_1 \mathbf{b}_1 + \sum_{p=2}^{N}\left((a_p, \mathbf{b}_p^T)(-\mathbf{B_1}^T\mathbf{b}_1) + \sum_{k=2}^{N}(a_p, \mathbf{b}_p^T)\mathbf{A_1}(a_k, \mathbf{b}_k^T)^T\right).$$

We easily check that

$$(a_p, \mathbf{b}_p^T)(-\mathbf{B_1}^T\mathbf{b}_1) = (a_p, \mathbf{b}_p^T)(-\mathbf{b}_1^T B_1, 0, 0)^T = a_p\mathbf{b}_1^T(-B_1),$$
$$(a_p, \mathbf{b}_p^T)\mathbf{A_1}(a_k, \mathbf{b}_k^T)^T = (a_p, \mathbf{b}_p^T)(a_k A_1, 0, 0)^T = a_p a_k A_1.$$

Hence,

$$\begin{aligned} \mathbf{x}^T(\alpha_1\rho_1 E_1)''(U)\mathbf{x} &= \mathbf{b}_1^T C_1\mathbf{b}_1 + \sum_{p=2}^{N} a_p\mathbf{b}_1^T(-B_1) + \sum_{p=2}^{N}\sum_{k=2}^{N} a_p a_k A_1 \\ &= \frac{1}{\alpha_1}\mathbf{b}_1^T\mathcal{S}_1''(V_1)\mathbf{b}_1 + \frac{1}{\alpha_1}\Big(\sum_{p=2}^{N} a_p\Big)\mathbf{b}_1^T\mathcal{S}_1''(V_1)V_1 \\ &+ \frac{1}{\alpha_1}\Big(\sum_{p=2}^{N} a_p\Big)V_1^T\mathcal{S}_1''(V_1)\Big(\sum_{k=2}^{N} a_k\Big)V_1 \\ &= \frac{1}{\alpha_1}\left(\Big(\sum_{k=2}^{N} a_k\Big)V_1 + \mathbf{b}_1\right)^T \mathcal{S}_1''(V_1)\left(\Big(\sum_{k=2}^{N} a_k\Big)V_1 + \mathbf{b}_1\right). \end{aligned}$$

Since $\mathcal{S}_1''(V_1)$ is a positive matrix by the strict convexity of the monophasic mathematical entropy $\mathcal{S}_1$, the right hand side is positive, which yields the positivity of the matrix $(\alpha_1\rho_1 E_1)''(U)$. Since $\mathbf{x}^T(\alpha_1\rho_1 E_1)''(U)\mathbf{x}$ does not depend on $\mathbf{b}_k$ for $k = 2, .., N$, $(\alpha_1\rho_1 E_1)''(U)$ is not positive definite and $(\alpha_1\rho_1 E_1)(U)$ is non strictly convex.

The convexity of the total mixture energy is a direct consequence of the convexity of all the fractional specific energies and we have:

$$\mathbf{x}^T(\sum_{k=1}^{N}\alpha_k\rho_k E_k)''(U)\mathbf{x} = 0 \iff \quad \mathbf{x} = \Big(-\big(\sum_{k=2}^{N} a_k\big)V_1^T, a_2, a_2V_2^T, .., a_N, a_NV_N^T\Big)^T \text{ with } (a_2, .., a_N) \in \mathbb{R}^{N-1}.$$

Thus, the total mixture energy in non strictly convex. □

4. Symmetrizability

Definition 4.1. *The system* (1) *is said to be symmetrizable if there exists a* C^1*-diffeomorphism* $\mathbb{R}^{3N-1} \to \mathbb{R}^{3N-1}$, $U \mapsto \mathcal{U}$, *a symmetric positive definite matrix* $\mathcal{P}(\mathcal{U})$, *and a symmetric matrix* $\mathcal{Q}(\mathcal{U})$ *such that the smooth solutions of* (1) *satisfy:*

$$\mathcal{P}(\mathcal{U})\partial_t \mathcal{U} + \mathcal{Q}(\mathcal{U})\partial_x \mathcal{U} = 0.$$

Since system (1) admits no conservative form and since the total mixture energy defined in the previous section is not strictly convex, we cannot use it to prove the symmetrizability of the system by multiplication by its hessian matrix. However we can find a suitable positive definite matrix $\mathcal{P}(\mathcal{U})$ which symmetrizes the system.

Theorem 4.1. *System* (1) *is symmetrizable as long as the non resonance condition* (6) *holds.*

Proof. Let us define $\mathcal{U} = (\alpha_2, .., \alpha_N, u_1, p_1, .., u_N, p_N)^T$. The smooth solutions of system (1) satisfy

$$\partial_t \mathcal{U} + \mathscr{A}(\mathcal{U})\partial_x \mathcal{U} = 0,$$

where the matrix $\mathscr{A}(\mathcal{U})$ is given in (7). Let us seek for a symmetric positive definite matrix $\mathcal{P}(\mathcal{U})$ that symmetrizes the system. We seek for $\mathcal{P}(\mathcal{U})$ in the form:

$$\mathcal{P}(\mathcal{U}) = \left(\begin{array}{c|cccc} \theta \mathbb{I}_{N-1} & D_1^T & \dots & D_N^T \\ \hline D_1 & P_1 & & \\ \vdots & & \ddots & \\ D_N & & & P_N \end{array}\right), \quad \text{with} \quad P_k = \begin{pmatrix} (\rho_k c_k)^2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\theta \in \mathbb{R}^+$, $\mathbb{I}_{N-1}$ is the $(N-1)\times(N-1)$ identity matrix and for $k = 1, .., N$, D_k is a $2 \times (N-1)$ matrix. The associated convection matrix is $\mathcal{Q}(\mathcal{U}) = \mathcal{P}(\mathcal{U})\mathscr{A}(\mathcal{U})$ with:

$$\mathcal{Q}(\mathcal{U}) = \left(\begin{array}{c|cccc} \theta u_1 \mathbb{I}_{N-1} + \sum_{k=1}^N D_k^T B_k & D_1^T C_1 & \dots & D_N^T C_N \\ \hline u_1 D_1 + P_1 B_1 & P_1 C_1 & & \\ \vdots & & \ddots & \\ u_1 D_N + P_N B_N & & & P_N C_N \end{array}\right).$$

We can easily see that the matrix $P_k C_k$ is symmetric for all $k = 1, .., N$. A necessary and sufficient condition for $\mathcal{Q}(\mathcal{U})$ to be symmetric is:

$$(i) \quad (C_k^T - u_1 \mathbb{I}_2) D_k = P_k B_k, \qquad \text{for all } k = 1, .., N,$$

$$(ii) \quad \sum_{k=1}^N D_k^T B_k \text{ is symmetric.}$$

The matrix $C_k^T - u_1 \mathbb{I}_2$ is a 2×2 matrix the determinant of which is $c_k^2(M_k^2 - 1)$ where $M_k = (u_k - u_1)/c_k$ is the relative Mach number of phase k. Hence, the matrices $C_k^T - u_1 \mathbb{I}_2$ are invertible if and only if the non resonance condition (6) holds. Assuming (6), the matrix D_k is therefore given by:

$$D_k = (C_k^T - u_1 \mathbb{I}_2)^{-1} P_k B_k.$$

An easy computation shows that the matrix $(C_k^T - u_1 \mathbb{I}_2)^{-1} P_k$ is symmetric and we get that $D_k^T B_k = B_k^T (C_k^T - u_1 \mathbb{I}_2)^{-1} P_k B_k$ is also symmetric. Thus, condition (6) is a necessary and sufficient condition for matrix $\mathcal{Q}(\mathcal{U})$ to be symmetric. The matrix $\mathcal{P}(\mathcal{U})$ is clearly symmetric. Therefore, it remains to prove that there exists $\theta > 0$

such that $\mathcal{P}(\mathcal{U})$ is positive definite. Let $\mathbf{x} = (\mathbf{a}^T, \mathbf{b}_1^T, .., \mathbf{b}_N^T)^T \in \mathbb{R}^{(3N-1)\times 1}\backslash\{0\}$ with $\mathbf{a} \in \mathbb{R}^{(N-1)\times 1}$ and for $k = 1, .., N$, $\mathbf{b}_k \in \mathbb{R}^{2\times 1}$. We have:

$$\begin{aligned}\mathbf{x}^T\mathcal{P}(\mathcal{U})\mathbf{x} &= \theta\mathbf{a}^T\mathbf{a} + 2\mathbf{a}^T\sum_{k=1}^N D_k^T\mathbf{b}_k + \sum_{k=1}^N \mathbf{b}_k^T P_k\mathbf{b}_k \\ &\geq \theta|\mathbf{a}|^2 - 2|\mathbf{a}|\Big|\sum_{k=1}^N D_k^T\mathbf{b}_k\Big| + \sum_{k=1}^N \mathbf{b}_k^T P_k\mathbf{b}_k.\end{aligned}$$

by the Cauchy-Schwarz inequality. The right hand side of this inequality is a polynomial of degree 2 in $|\mathbf{a}|$ and its second discriminant Δ' is given by:

$$\begin{aligned}\Delta' &= \Big|\sum_{k=1}^N D_k^T\mathbf{b}_k\Big|^2 - \theta\sum_{k=1}^N \mathbf{b}_k^T P_k\mathbf{b}_k \\ &\leq N\sum_{k=1}^N \big|D_k^T\mathbf{b}_k\big|^2 - \theta\sum_{k=1}^N \mathbf{b}_k^T P_k\mathbf{b}_k \\ &= N\sum_{k=1}^N \mathbf{b}_k^T D_k D_k^T\mathbf{b}_k - \theta\sum_{k=1}^N \mathbf{b}_k^T P_k\mathbf{b}_k,\end{aligned}$$

again by the Cauchy-Schwarz inequality. Since $D_k D_k^T$ is symmetric and P_k is symmetric positive definite, there exists an invertible 2×2 matrix Q_k which simultaneously diagonalizes the two associated quadratic forms. More precisely, we have $Q_k^T P_k Q_k = \mathbb{I}_2$ and $Q_k^T D_k D_k^T Q_k = \delta_k$ where δ_k is a diagonal matrix. Defining $\bar{\mathbf{b}}_k = Q_k^{-1}\mathbf{b}_k$ we obtain:

$$\Delta' \leq \sum_{k=1}^N \bar{\mathbf{b}}_k^T(N\delta_k - \theta\mathbb{I}_2)\bar{\mathbf{b}}_k.$$

Hence, choosing θ larger than the two eigenvalues of $N\delta_k$ for all $k = 1, .., N$ (observe that these eigenvalues only depend on $\mathcal{U}$ and not on the vector $\mathbf{x}$), we get that $\Delta' < 0$ and therefore $\mathbf{x}^T\mathcal{P}(\mathcal{U})\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^{(3N-1)\times 1}\backslash\{0\}$. □

Conclusion

We have proven the weak hyperbolicity, the existence of a convex mathematical entropy as well as the existence of a symmetric form. This last property is valid only far from resonance, *i.e.* as long as the model remain in its domain of hyperbolicity. These properties have been obtained for any admissible phasic equations of state (increasing phasic pressure laws). What is more, the proven properties can be extended to the two and three dimensional version of the model thanks to the frame invariance.

An important consequence of the symmetrisability and Kato's theorem on quasi-linear symmetric systems ([12]) is that, far from resonance, there exists a unique local-in-time smooth solution to the Cauchy problem. The loss of regularity in finite time still holds, but with the additional restriction due to the non resonance condition (6).

It is worth mentioning that the techniques developed herein can be extended to similar systems modelling miscible compressible mixtures. We refer to [10] where symmetrisability results have been obtained for a three-phase flow model where two of the present phases are miscible.

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