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RESIDUAL BASED A POSTERIORI ERROR ESTIMATION FOR DIRICHLET BOUNDARY CONTROL PROBLEMS

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Abstract. We study a residual–based a posteriori error estimate for the solution of Dirichlet boundary control problem governed by a convection diffusion equation on a two dimensional convex polygonal domain, using the local discontinuous Galerkin (LDG) method with upwinding for the convection term. With the usage of LDG method, the control variable naturally exists in the variational form due to its mixed finite element structure. We also demonstrate the application of our a posteriori error estimator for the adaptive solution of these optimal control problems.

1. Introduction

In this study, we investigate a numerical approximation of Dirichlet boundary control problems governed by a convection diffusion equation:

$$\text{minimize} \quad J(y,u) = \frac{1}{2}\|y - y^d\|_{0,\Omega}^2 + \frac{\omega}{2}\|u\|_{0,\Gamma}^2 \tag{1}$$

subject to

$$\nabla \cdot (-\epsilon \nabla y + \beta y) + \alpha y = f \quad \text{in } \Omega, \tag{2a}$$
$$y = u \quad \text{on } \Gamma, \tag{2b}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary $\Gamma = \partial\Omega$. We refer to u as the control on the Dirichlet boundary, to y as the state, and to (2) as the state equation. The velocity field is denoted by $\beta \in \left(W^{1,\infty}(\Omega)\right)^2$ and we suppose that it satisfies incompressibility condition, that is, $\nabla \cdot \beta = 0$. The constant coefficients $\epsilon > 0$ and $\alpha > 0$ are corresponding to diffusion and reaction terms, respectively. The regularization parameter ω is a positive constant. For the source function f and the desired state y^d, we assume $f, y^d \in L^2(\Omega)$.

By following the standard arguments in [21, 26], Dirichlet boundary control problem (1)-(2) is equivalent to the following optimality system:

$$\nabla \cdot (-\epsilon \nabla y + \beta y) + \alpha y = f \quad \text{in } \Omega, \qquad y = u \quad \text{on } \Gamma, \tag{3a}$$
$$\nabla \cdot (-\epsilon \nabla z - \beta z) + \alpha z = y - y^d \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \Gamma, \tag{3b}$$
$$\omega u - \epsilon \frac{\partial z}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma, \tag{3c}$$

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where z is the adjoint variable and $\mathbf{n}$ is the unit outer normal to Γ. The adjoint equation (3b) is also a convection diffusion equation, but with convection term $-\beta$ instead of β. The convergence properties of discretization methods applied to the optimal control problems can be substantially different from the convergence properties of discretization methods applied to a single convection dominated PDEs due to the transport of the information in the optimality system with opposite directions.

In such kind of problems (1)-(2), the specific difficulty is that the control variable is not involved in the variational form of the standard finite element setting. To handle this difficulty, various numerical approaches have been proposed including very weak variational setting [7,9,13], modified cost functionals [11,14], approximating the Dirichlet boundary condition with a Robin boundary condition [4], weak boundary penalization (also called as Nitsche's penalty technique) [10], and a mixed formulation [15]. In this study, we employ local discontinuous Galerkin (LDG) method as a discretization technique since discontinuous Galerkin methods exhibit better convergence for the spatial discretization of optimal control problems governed by convection dominated PDEs [20] and the control variable is naturally involved in the variational form thanks to the mixed finite element structure of LDG and the weak enforcement of the boundary conditions. We would like to refer to [1, 24] and references therein for details about local discontinuous Galerkin methods.

To obtain better accuracy with as few of degrees of freedom as possible, one particular way is adaptive finite element method. Although adaptive finite element method, contributed to the pioneer work of Babuška and Rheinboldt [2], has become a popular approach for the efficient solution of boundary and initial value problems for the PDEs, it is quite recent for constrained optimal control problems, initiated by Liu, Yan [23] and Becker, Kapp, Rannacher [3]. Adaptive finite element methods have been applied for various optimal control problems, see, e.g., [16,17,22,27,30,31] for control constrained problems, [5,18,29] for state constrained problems, and [6,19] for Neumann boundary control problems. However, there are a few studies for a posteriori error estimation of Dirichlet boundary control problems. Primal–dual weighted error estimates were derived in [28] for Dirichlet boundary control problem governed by a convection diffusion equation with control constraints. In [11, 14], a residual–type a posteriori error analyses were carried out for Dirichlet boundary control governed by an elliptic equation with the control variable defined on an equivalent form of the norm in $H^{\frac{1}{2}}(\Gamma)$, using continuous finite element discretization. With the present paper, we intent to contribute a residual–based a-posteriori error estimates for the solution of Dirichlet boundary control problem governed by a convection diffusion equation, using the local discontinuous Galerkin method with upwinding for the convection term.

The rest of this paper is as follows: In the next section we present Dirichlet boundary control problem discretized by the local discontinuous Galerkin method. Section 3 is devoted to derivation of a residual–based a posteriori error estimator. Numerical results are given in Section 4 to illustrate the performance of the proposed error estimators.

2. Discretization of Model Problem

Throughout the paper we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,p,\Omega}$ and seminorm $|\cdot|_{m,p,\Omega}$ for $m \geq 0$ and $1 \leq p \leq \infty$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$. The L^2–inner products on $L^2(\Omega)$ and $L^2(\Gamma)$ are defined by

$$(v, w)_{0,\Omega} = \int_\Omega v\, w \; dx \qquad \forall v, w \in L^2(\Omega) \quad \text{and} \quad \langle v, w\rangle_{0,\Gamma} = \int_\Gamma v\, w \; ds \qquad \forall v, w \in L^2(\Gamma),$$

respectively. In addition, C denotes a generic positive constant independent of the mesh size h and differs in various estimates.

The LDG method, one of several discontinuous Galerkin methods, can be considered as a mixed finite element method. As in mixed finite element methods, we rewrite the optimality system as a system of first order equations and discretize it by introducing auxiliary variables. Introducing $\mathbf{q} = \epsilon^{\frac{1}{2}}\nabla y$ and $\mathbf{p} = -\epsilon^{\frac{1}{2}}\nabla z$, the optimality system (3) can be rewritten as a system of the first order equations:

$$\begin{aligned}
\nabla \cdot (\beta y - \epsilon^{\frac{1}{2}} \mathbf{q}) + \alpha y &= f \quad \text{in } \Omega, \qquad y = u \quad \text{on } \Gamma, && \text{(4a)}\\
\mathbf{q} &= \epsilon^{\frac{1}{2}} \nabla y \quad \text{in } \Omega, && \text{(4b)}\\
\nabla \cdot (\epsilon^{\frac{1}{2}} \mathbf{p} - \beta z) + \alpha z &= y - y^d \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \Gamma, && \text{(4c)}\\
\mathbf{p} &= -\epsilon^{\frac{1}{2}} \nabla z \quad \text{in } \Omega, && \text{(4d)}\\
\omega u + \epsilon^{\frac{1}{2}} \mathbf{p} \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma. && \text{(4e)}
\end{aligned}$$

We assume that the domain Ω is polygonal such that the boundary is exactly represented by boundaries of triangles. We denote $\{\mathcal{T}_h\}_h$ as a family of shape-regular simplicial triangulations of Ω. Each mesh $\mathcal{T}_h$ consists of closed triangles such that $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$ holds. The set of all edges $\mathcal{E}_h$ consists of the set $\mathcal{E}_h^0$ of interior edges and the set $\mathcal{E}_h^\partial$ of boundary edges so that $\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^\partial$. The diameter of an element K and the length of an edge E are denoted by h_K and h_E, respectively, and also $h = \max\limits_{K \in \mathcal{T}_h} h_K$. Let the edge E be a common edge for two elements K and K^e. Then, the jump $[\![\cdot]\!]$ and average $\{\!\{\cdot\}\!\}$ of the scalar function y and the vector field $\mathbf{q}$ across the edge E are defined as

$$[\![y]\!] = y|_E \mathbf{n}_K + y^e|_E \mathbf{n}_{K^e}, \quad \{\!\{y\}\!\} = \frac{1}{2}\big(y|_E + y^e|_E\big), \quad [\![\mathbf{q}]\!] = \mathbf{q}|_E \cdot \mathbf{n}_K + \mathbf{q}^e|_E \cdot \mathbf{n}_{K^e}, \quad \{\!\{\mathbf{q}\}\!\} = \frac{1}{2}\big(\mathbf{q}|_E + \mathbf{q}^e|_E\big),$$

where $\mathbf{n}_K$ (resp. $\mathbf{n}_{K^e}$) denotes the unit outward normal to ∂K (resp. ∂K^e). For a boundary edge $E \in K \cap \partial\Omega$, we set $\{\!\{\mathbf{q}\}\!\} = \mathbf{q}$ and $[\![y]\!] = y\mathbf{n}$, where $\mathbf{n}$ is the outward normal unit vector on Γ.

To obtain weak formulation for the state equation in (4a)-(4b), we multiply it by piecewise smooth test functions v and $\mathbf{r}$, respectively, and integrate by parts over the element $K \in \mathcal{T}_h$

$$\begin{aligned}
(\epsilon^{\frac{1}{2}} \mathbf{q} - \beta y, \nabla v)_{0,K} + (\alpha y, v)_{0,K} - \langle (\epsilon^{\frac{1}{2}} \mathbf{q} - \beta y) \cdot \mathbf{n}, v \rangle_{0,\partial K} &= (f, v)_{0,K} \qquad v \in V, && \text{(5a)}\\
(\mathbf{q}, \mathbf{r})_{0,K} + (\epsilon^{\frac{1}{2}} y, \nabla \cdot \mathbf{r})_{0,K} &= \langle \epsilon^{\frac{1}{2}} y, \mathbf{r} \cdot \mathbf{n} \rangle_{0,\partial K} \qquad \mathbf{r} \in \mathbf{W}, && \text{(5b)}
\end{aligned}$$

where $V := \big\{ v \in L^2(\Omega) \, : \, v|_K \in H^1(K), \quad \forall K \in \mathcal{T}_h \big\}$ and $\mathbf{W} := \Big\{ \mathbf{w} \in \big(L^2(\Omega)\big)^2 \, : \, \mathbf{w}|_K \in \big(H^1(K)\big)^2, \quad \forall K \in \mathcal{T}_h \Big\}$. Next, we seek to approximate the state solutions $(y, \mathbf{q})$ with functions $(y_h, \mathbf{q}_h)$ in the following finite element spaces $V_h \times \mathbf{W}_h \subset V \times \mathbf{W}$:

$$\begin{aligned}
V_h &= \big\{ v \in L^2(\Omega) \, : \, v|_K \in \mathbb{S}^1(K), \quad \forall K \in \mathcal{T}_h \big\}, && \text{(6a)}\\
\mathbf{W}_h &= \Big\{ w \in \big(L^2(\Omega)\big)^2 \, : \, w|_K \in \big(\mathbb{S}^1(K)\big)^2, \quad \forall K \in \mathcal{T}_h \Big\}, && \text{(6b)}\\
U_h &= \big\{ u \in U = L^2(\Gamma) \, : \, u|_E \in \mathbb{S}^1(E), \quad \forall E \in \mathcal{E}_h^\partial \big\}, && \text{(6c)}
\end{aligned}$$

where $\mathbb{S}^1(K)$ (resp. $\mathbb{S}^1(E)$) is the local finite element space, which consists of linear polynomials in each element K (resp. on E).

For all $(v, \mathbf{r}) \in V_h \times \mathbf{W}_h$ the approximate solution $(y_h, \mathbf{q}_h)$ of the state solution $(y, \mathbf{q})$ satisfies

$$\begin{aligned}
(\epsilon^{\frac{1}{2}} \mathbf{q}_h - \beta y_h, \nabla v)_{0,K} + (\alpha y_h, v)_{0,K} - \langle (\epsilon^{\frac{1}{2}} \widehat{\mathbf{q}}_h - \beta \widetilde{y}_h) \cdot \mathbf{n}, v \rangle_{0,\partial K} &= (f, v)_{0,K}, && \text{(7a)}\\
(\mathbf{q}_h, \mathbf{r})_{0,K} + (\epsilon^{\frac{1}{2}} y_h, \nabla \cdot \mathbf{r})_{0,K} &= \langle \epsilon^{\frac{1}{2}} \widehat{y}_h, \mathbf{r} \cdot \mathbf{n} \rangle_{0,\partial K}, && \text{(7b)}
\end{aligned}$$

where $\widehat{\mathbf{q}}_h, \widetilde{y}_h, \widehat{y}_h$ denote numerical fluxes. They have to be suitably defined in order to ensure the stability of the method and to enhance its accuracy. The numerical traces of y associated with the diffusion and convection terms are characterized as done in [31]

$$\widehat{y}_h = \begin{cases} \{\!\{y_h\}\!\} + \mathbf{C}_{12} \cdot [\![y_h]\!], & E \in \mathcal{E}_h^0, \\ u_h, & E \in \mathcal{E}_h^\partial, \end{cases} \quad \text{and} \quad \widetilde{y}_h = \begin{cases} u_h, & E \in \Gamma^-, \\ \{\!\{y_h\}\!\} + \mathbf{D}_{11} \cdot [\![y_h]\!], & E \in \mathcal{E}_h^0, \\ y_h, & E \in \Gamma^+, \end{cases} \tag{8}$$

respectively. We note that the numerical trace of y with respect to convection term is the classical upwinding trace. In addition, the numerical flux $\widehat{\mathbf{q}}_h$ is given by

$$\widehat{\mathbf{q}}_h = \begin{cases} \{\!\{\mathbf{q}_h\}\!\} + C_{11}[\![y_h]\!] - \mathbf{C}_{12}[\![\mathbf{q}_h]\!], & E \in \mathcal{E}_h^0, \\ \mathbf{q}_h + C_{11}(y_h - u_h)\cdot\mathbf{n}, & E \in \mathcal{E}_h^\partial. \end{cases} \tag{9}$$

In the numerical implementations, C_{11} is chosen as $C_{11} = \sqrt{\epsilon}/h_E$ for each $E \in \mathcal{E}_h$, whereas we take $\mathbf{C}_{12}$ normal to the edges and modulus $1/2$, i.e., $\mathbf{C}_{12}\cdot\mathbf{n}_E = \frac{1}{2}$, and the vector function $\mathbf{D}_{11}$ is given by $\mathbf{D}_{11}\cdot\mathbf{n} = \frac{1}{2}\operatorname{sign}\big(\mathbf{n}\cdot\beta\big)$.

Employing optimize–discretize approach, see, e.g., [26], to solve the Dirichlet boundary control problem (1)-(2), we obtain discrete optimality system for $\big(y_h, \mathbf{q}_h, z_h, \mathbf{p}_h, u_h\big) \in V_h \times \mathbf{W}_h \times V_h \times \mathbf{W}_h \times U_h$:

$$a(\mathbf{q}_h, \mathbf{r}) + b(y_h, \mathbf{r}) = m_1(u_h, \mathbf{r}) \qquad \forall \mathbf{r} \in \mathbf{W}_h, \tag{10a}$$

$$-b(v, \mathbf{q}_h) + c(y_h, v) = m_2(u_h, v) + (f, v)_{0,\Omega} \qquad \forall v \in V_h, \tag{10b}$$

$$a(\mathbf{p}_h, \psi) - b(z_h, \psi) = 0 \qquad \forall \psi \in \mathbf{W}_h, \tag{10c}$$

$$b(\phi, \mathbf{p}_h) + c(\phi, z_h) = (y_h - y^d, \phi)_{0,\Omega} \qquad \forall \phi \in V_h, \tag{10d}$$

$$\langle \omega u_h + \epsilon^{\frac{1}{2}}\mathbf{p}_h\cdot\mathbf{n}, w\rangle_{0,\Gamma} = 0 \qquad \forall w \in U_h, \tag{10e}$$

where

$$\begin{aligned}
a(\mathbf{q}, \mathbf{r}) &= \int_\Omega \mathbf{q}\cdot\mathbf{r}\, dx, \\
b(y, \mathbf{r}) &= -\sum_{K\in\mathcal{T}_h} \int_K \epsilon^{\frac{1}{2}} \nabla y\cdot\mathbf{r}\, dx + \sum_{E\in\mathcal{E}_h^0} \int_E \epsilon^{\frac{1}{2}} \big(\{\!\{\mathbf{r}\}\!\} - \mathbf{C}_{12}[\![\mathbf{r}]\!]\big)\cdot[\![y]\!]\, ds + \sum_{E\in\mathcal{E}_h^\partial} \int_E \epsilon^{\frac{1}{2}} y\mathbf{r}\cdot\mathbf{n}\, ds, \\
c(y, v) &= \sum_{K\in\mathcal{T}_h} \int_K \big(\alpha y\, v - y\beta\cdot\nabla v\big)\, dx + \sum_{E\in\mathcal{E}_h^0} \int_E \big(\{\!\{y\}\!\} + \mathbf{D}_{11}\cdot[\![y]\!]\big)\beta\cdot[\![v]\!]\, ds \\
&\quad - \sum_{E\in\mathcal{E}_h^0} \int_E \epsilon^{\frac{1}{2}} C_{11}[\![y]\!]\cdot[\![v]\!]\, ds + \sum_{E\in\Gamma^+} \int_E (\mathbf{n}\cdot\beta) y\, v\, ds - \sum_{E\in\mathcal{E}_h^\partial} \int_E \epsilon^{\frac{1}{2}} C_{11} y\, v\, ds, \\
m_1(u, \mathbf{r}) &= \sum_{E\in\mathcal{E}_h^\partial} \int_E \epsilon^{\frac{1}{2}} u\,\mathbf{r}\cdot\mathbf{n}\, ds, \qquad m_2(u, v) = -\sum_{E\in\mathcal{E}_h^\partial} \int_E \epsilon^{\frac{1}{2}} C_{11} u\, v\, ds - \sum_{E\in\Gamma^-} \int_E |\beta\cdot\mathbf{n}| u\, v\, ds.
\end{aligned}$$

We refer to [12,31] for derivation of (bi)–linear forms and references therein. Further, it is easy show that the continuous solution $\big(y, \mathbf{q}, z, \mathbf{p}, u\big)$ satisfies the following optimality system:

$$a(\mathbf{q}, \mathbf{r}) + b(y, \mathbf{r}) = m_1(u, \mathbf{r}) \qquad \forall \mathbf{r} \in \mathbf{W}, \tag{11a}$$

$$-b(v, \mathbf{q}) + c(y, v) = m_2(u, v) + (f, v)_{0,\Omega} \qquad \forall v \in V, \tag{11b}$$

$$a(\mathbf{p}, \psi) - b(z, \psi) = 0 \qquad \forall \psi \in \mathbf{W}, \tag{11c}$$

$$b(\phi, \mathbf{p}) + c(\phi, z) = (y - y^d, \phi)_{0,\Omega} \qquad \forall \phi \in V. \tag{11d}$$

3. A Posteriori Error Estimates

In this section we derive a residual–based a posteriori error estimate for Dirichlet boundary control problem governed by a convection diffusion equation, discretized by the local discontinuous Galerkin (LDG). We first give some known results, which will be needed throughout of the paper.

- Let $i_h : V \to V_h$ and $I_h : \mathbf{W} \to \mathbf{W}_h$ be the L^2 projection operators satisfying (see [8, Chapter III])

$$(y - i_h y, v)_{0,\Omega} = 0, \qquad \big(\nabla\cdot(\mathbf{q} - I_h\mathbf{q}), v)_{0,\Omega} = 0 \qquad \forall v \in V_h. \tag{12a}$$

Then, the following approximation estimates hold

$$\begin{aligned}
\|y - i_h y\|_{-s,2,\Omega} &\leq Ch^{1+s}|y|_{1,2,\Omega}, \quad s = 0,1, \quad y \in W^{s,2}(\Omega), && (13a)\\
\|y - i_h y\|_{-s,2,\Gamma} &\leq Ch^{1/2+s}|y|_{1,2,\Omega}, \quad s = 0,1, \quad y \in W^{s,2}(\Omega), && (13b)\\
\|\mathbf{q} - I_h\mathbf{q}\|_{s,2,\Omega} &\leq Ch^{1-s}|\mathbf{q}|_{1,2,\Omega}, \quad s = 0,1, \quad \mathbf{q} \in \big(W^{s,2}(\Omega)\big)^2, && (13c)\\
\|\mathbf{q} - I_h\mathbf{q}\|_{s,2,\Gamma} &\leq Ch^{1/2-s}|\mathbf{q}|_{1,2,\Omega}, \quad s = 0,1, \quad \mathbf{q} \in \big(W^{s,2}(\Omega)\big)^2. && (13d)
\end{aligned}$$

- Let φ be the solution of a convection diffusion equation in mixed formulation on a convex domain Ω with smooth boundary. Then, for any right–hand side function g the following estimate holds [25, Lemma 1.18]

$$\epsilon^{3/2}\|\varphi\|_{2,\Omega} + \epsilon^{1/2}\|\varphi\|_{1,\Omega} + \|\varphi\|_{0,\Omega} \leq C\|g\|_{0,\Omega}. \tag{14}$$

To prove our reliability result, we need the auxiliary solution $\big(y[u_h], \mathbf{q}[u_h], z[u_h], \mathbf{p}[u_h]\big) \in V \times \mathbf{W} \times V \times \mathbf{W}$, which solves the following system:

$$\begin{aligned}
a(\mathbf{q}[u_h], \mathbf{r}) + b(y[u_h], \mathbf{r}) &= m_1(u_h, \mathbf{r}) && \forall \mathbf{r} \in \mathbf{W}, && (15a)\\
-b(v, \mathbf{q}[u_h]) + c(y[u_h], v) &= m_2(u_h, v) + (f, v)_{0,\Omega} && \forall v \in V, && (15b)\\
a(\mathbf{p}[u_h], \psi) - b(z[u_h], \psi) &= 0 && \forall \psi \in \mathbf{W}, && (15c)\\
b(\phi, \mathbf{p}[u_h]) + c(\phi, z[u_h]) &= (y[u_h] - y^d, \phi)_{0,\Omega} && \forall \phi \in V. && (15d)
\end{aligned}$$

Then, with the help of the bilinear forms (11) and (15) and the corresponding coercivity property of the bilinear forms, we obtain

$$\begin{aligned}
\|\mathbf{q} - \mathbf{q}[u_h]\|_{0,\Omega} + \|y - y[u_h]\|_{0,\Omega} &\leq C\,\|u - u_h\|_{0,\Gamma}, && (16a)\\
\|\mathbf{p} - \mathbf{p}[u_h]\|_{0,\Omega} + \|z - z[u_h]\|_{0,\Omega} &\leq C\,\|y - y[u_h]\|_{0,\Omega}, && (16b)
\end{aligned}$$

where the constant C is dependent on ϵ.

Now, we derive a residual based a posteriori error estimate for the control u, the state y, and the adjoint z.

Theorem 3.1. *Let $\big(y, \mathbf{q}, z, \mathbf{p}, u\big)$ and $\big(y_h, \mathbf{q}_h, z_h, \mathbf{p}_h, u_h\big)$ be the solutions of* (4) *and* (10)*, respectively. Then, it holds that*

$$\|u - u_h\|^2_{0,\Gamma} + \|y - y_h\|^2_{0,\Omega} + \|z - z_h\|^2_{0,\Omega} \leq C\big(\eta_y + \eta_z\big), \tag{17}$$

where

$$\begin{aligned}
\eta_y = \Bigg(&\sum_{K\in\mathcal{T}_h} \frac{h_K^2}{\epsilon}\|f - \nabla\cdot(\beta y_h - \epsilon^{\frac12}\mathbf{q}_h) - \alpha y_h\|^2_{0,K} + \sum_{K\in\mathcal{T}_h}\frac{h_k^2}{\epsilon}\|\epsilon^{\frac12}\nabla y_h - \mathbf{q}_h\|^2_{0,K}\\
&+ \sum_{E\in\mathcal{E}_h^0}\left(\frac{h_E}{2}\|[\![\mathbf{q}_h]\!]\|^2_{0,E} + \left(\frac{h_E}{2\epsilon} + \frac{\epsilon}{h_E}\right)\|[\![y_h]\!]\|^2_{0,E}\right) + \sum_{E\in\mathcal{E}_h^\partial}\left(\frac{h_E}{\epsilon} + \frac{\epsilon}{h_E}\right)\|y_h - u_h\|^2_{0,E}\Bigg), && (18a)\\
\eta_z = \Bigg(&\sum_{K\in\mathcal{T}_h} \frac{h_K^2}{\epsilon}\|y_h - y^d - \nabla\cdot(\epsilon^{\frac12}\mathbf{p}_h - \beta z_h)\|^2_{0,K} + \sum_{K\in\mathcal{T}_h}\frac{h_k^2}{\epsilon}\|\epsilon^{\frac12}\nabla z_h + \mathbf{p}_h\|^2_{0,K}\\
&+ \sum_{E\in\mathcal{E}_h^0}\left(\frac{h_E}{2}\|[\![\mathbf{p}_h]\!]\|^2_{0,E} + \left(\frac{h_E}{2\epsilon} + \frac{\epsilon}{h_E}\right)\|[\![z_h]\!]\|^2_{0,E}\right) \sum_{E\in\mathcal{E}_h^\partial}\left(\frac{h_E}{\epsilon} + \frac{\epsilon}{h_E}\right)\|z_h\|^2_{0,E}\Bigg). && (18b)
\end{aligned}$$

Proof. For $u - u_h \in U$ we have

$$\big(\omega u + \epsilon^{\frac12}\mathbf{p}\cdot\mathbf{n}, u - u_h\big)_{0,\Gamma} - \big(\omega u_h + \epsilon^{\frac12}\mathbf{p}[u_h]\cdot\mathbf{n}, u - u_h\big)_{0,\Gamma} = \omega\|u - u_h\|^2_{0,\Gamma} + \epsilon^{\frac12}\big((\mathbf{p} - \mathbf{p}[u_h])\cdot\mathbf{n}, u - u_h\big)_{0,\Gamma}. \tag{19}$$

The optimality systems (11) and (15) give us

$$(y-y^d, y)_{0,\Omega} = b(y,\mathbf{p}) + c(y,z) + a(\mathbf{p},\mathbf{q}) - b(z,\mathbf{q}) = m_1(u,\mathbf{p}) + m_2(u,z) + (f,z)_{0,\Omega}, \tag{20a}$$

$$(y-y^d, y[u_h])_{0,\Omega} = b(y[u_h],\mathbf{p}) + c(y[u_h],z) + a(\mathbf{p},\mathbf{q}[u_h]) - b(z,\mathbf{q}[u_h]) = m_1(u_h,\mathbf{p}) + m_2(u_h,z) + (f,z)_{0,\Omega} \tag{20b}$$

From (20), we have

$$\big(y-y^d, y-y[u_h]\big)_{0,\Omega} = m_1(u-u_h,\mathbf{p}) + m_2(u-u_h,z). \tag{21}$$

Similarly, we can deduce

$$\big(y[u_h]-y^d, y-y[u_h]\big)_{0,\Omega} = m_1(u-u_h,\mathbf{p}[u_h]) + m_2(u-u_h,z[u_h]). \tag{22}$$

Combining (21) and (22) we obtain

$$\epsilon^{\frac{1}{2}}\big((\mathbf{p}-\mathbf{p}[u_h])\cdot\mathbf{n}, u-u_h\big)_{0,\Gamma} = \|y-y[u_h]\|_{0,\Omega}^2 + m_2(u-u_h, z[u_h]-z). \tag{23}$$

From the optimality conditions (4e), (3c), and definition of $m_2(\cdot,\cdot)$, we have

$$\begin{aligned}\omega\|u-u_h\|_{0,\Gamma}^2 &\le \big(u-u_h, \epsilon^{\frac{1}{2}}(\mathbf{p}_h-\mathbf{p}[u_h])\cdot\mathbf{n}\big)_{0,\Gamma} + \big(u-u_h, \epsilon^{\frac{1}{2}}(\mathbf{p}[u_h]-\mathbf{p})\cdot\mathbf{n}\big)_{0,\Gamma} \\ &\le \underbrace{\big(u-u_h, \epsilon^{\frac{1}{2}}(\mathbf{p}_h-I_h\mathbf{p}[u_h])\cdot\mathbf{n}\big)_{0,\Gamma}}_{M_1} + \underbrace{\big(u-u_h, \epsilon^{\frac{1}{2}}(I_h\mathbf{p}[u_h]-\mathbf{p}[u_h])\cdot\mathbf{n}\big)_{0,\Gamma}}_{M_2} \\ &\quad + \underbrace{\big(u-u_h, (C_{11}\epsilon^{\frac{1}{2}}+|\beta\cdot\mathbf{n}|)(z[u_h]-z)\big)_{0,\Gamma}}_{M_3}.\end{aligned} \tag{24}$$

We first find a bound for the first term in (24) by using trace, inverse, and Young's inequalities as follows

$$M_1 \le \|u-u_h\|_{0,\Gamma}\epsilon^{\frac{1}{2}}\|\mathbf{p}_h - I_h\mathbf{p}[u_h]\|_{0,\Gamma} \le \|u-u_h\|_{0,\Gamma}\epsilon h^{-3/2}\|z[u_h]-z_h\|_{0,\Omega} \le C\delta\|u-u_h\|_{0,\Gamma}^2 + C(\delta)\|z[u_h]-z_h\|_{0,\Omega}^2. \tag{25}$$

Next, the estimate in (13), the bound (14) with $g = y_h - y[u_h]$, and Young's inequality give us

$$M_2 \le \|u-u_h\|_{0,\Gamma}\epsilon^{\frac{1}{2}}\|I_h\mathbf{p}[u_h]-\mathbf{p}[u_h]\|_{0,\Gamma} \le \|u-u_h\|_{0,\Gamma}\epsilon^{\frac{1}{2}}h^{1/2}\|\mathbf{p}[u_h]\|_{0,\Omega} \le C\delta\|u-u_h\|_{0,\Gamma}^2 + C(\delta)\|y_h-y[u_h]\|_{0,\Omega}^2, \tag{26}$$

$$\begin{aligned}M_3 &\le \big(u-u_h, (C_{11}\epsilon^{\frac{1}{2}}+|\beta\cdot\mathbf{n}|)(z[u_h]-i_hz)\big)_{0,\Gamma} + \big(u-u_h, (C_{11}\epsilon^{\frac{1}{2}}+|\beta\cdot\mathbf{n}|)(i_hz-z)\big)_{0,\Gamma} \\ &\le C\delta\|u-u_h\|_{0,\Gamma}^2 + C(\delta)\|z[u_h]-z_h\|_{0,\Omega}^2 + C\delta\|u-u_h\|_{0,\Gamma}^2 + C(\delta)\|y_h-y[u_h]\|_{0,\Omega}^2,\end{aligned} \tag{27}$$

where for any small $\delta > 0$. We note that it is assumed that i_hz approximates z_h.
Putting (25) -(27) into (24), we obtain

$$\|u-u_h\|_{0,\Gamma}^2 \le C\big(\|y_h-y[u_h]\|_{0,\Omega}^2 + \|z[u_h]-z_h\|_{0,\Omega}^2\big). \tag{28}$$

Now, define the following functional

$$\mathcal{A}(y,\mathbf{q};v,\mathbf{r}) = a(\mathbf{q},\mathbf{r}) + b(y,\mathbf{r}) - b(v,\mathbf{q}) + c(y,v). \tag{29}$$

Then, the following Galerkin orthogonality property holds

$$\mathcal{A}(y[u_h]-y_h, \mathbf{q}[u_h]-\mathbf{q}_h; v_h, \mathbf{r}_h) = 0 \qquad \forall v_h \in V_h, \qquad \forall \mathbf{r}_h \in \mathbf{W}_h. \tag{30}$$

By the definitions of (bi)–linear forms in (10) and then applying integration by parts, we obtain

$$
\begin{aligned}
&\mathcal{A}(y[u_h]-y_h,\mathbf{q}[u_h]-\mathbf{q}_h;v,\mathbf{r})\\
&= m_1(u_h,\mathbf{r}) + m_2(u_h,v) + \big(f,v\big)_{0,\Omega} - a(\mathbf{q}_h,\mathbf{r}) - b(y_h,\mathbf{r}) + b(v,\mathbf{q}_h) - c(y_h,v)\\
&= \sum_{K\in\mathcal{T}_h}\int_K \big(f-\nabla\cdot(\beta y_h-\epsilon^{\frac12}\mathbf{q}_h)-\alpha y_h\big)v\,dx - \sum_{E\in\mathcal{E}_h^0}\int_E \Big(\epsilon^{\frac12}\big(\{\!\{v\}\!\}+\mathbf{C}_{12}\cdot[\![v]\!]\big)[\![\mathbf{q}_h]\!] + \big(\mathbf{D}_{11}\cdot[\![v]\!]-\{\!\{v\}\!\}\big)\beta\cdot[\![y_h]\!]\Big)\,ds\\
&\quad + \sum_{E\in\mathcal{E}_h^0}\int_E \epsilon^{\frac12}C_{11}[\![v]\!]\cdot[\![y_h]\!]\,ds + \sum_{E\in\mathcal{E}_h^\partial}\int_E \epsilon^{\frac12}C_{11}(y_h-u_h)v\,ds + \sum_{E\in\Gamma^-}\int_E |\beta\cdot\mathbf{n}|(y_h-u_h)v\,ds\\
&\quad + \sum_{K\in\mathcal{T}_h}\int_K \big(\epsilon^{\frac12}\nabla y_h-\mathbf{q}_h\big)\mathbf{r}\,dx - \sum_{E\in\mathcal{E}_h^0}\int_E \epsilon^{\frac12}\big(\{\!\{\mathbf{r}\}\!\}-\mathbf{C}_{12}[\![\mathbf{r}]\!]\big)[\![y_h]\!]\,ds + \sum_{E\in\mathcal{E}_h^\partial}\int_E \epsilon^{\frac12}(u_h-y_h)\mathbf{r}\,ds. \qquad (31)
\end{aligned}
$$

To obtain an estimate for $\|y[u_h]-y_h\|_{0,\Omega}$, we apply duality argument as follows

$$
\mathcal{A}^*(\phi,\psi;v,\mathbf{r}) = \sum_{K\in\mathcal{T}_h}\int_K (y[u_h]-y_h)v\,dx, \qquad (32)
$$

where $\mathcal{A}^*(z,\mathbf{p};v,\mathbf{r}) = a(\mathbf{p},\mathbf{r}) - b(z,\mathbf{r}) + b(v,\mathbf{p}) + c(v,z),\quad \forall v\in V,\ \forall\mathbf{r}\in\mathbf{W}$. Setting $v = y[u_h]-y_h$ and $\mathbf{r} = \mathbf{q}[u_h]-\mathbf{q}_h$ in (32), and using the operators defined in (12), the definition of $\mathcal{A}(\cdot,\cdot)$, and the Galerkin orthogonality (30), we obtain

$$
\begin{aligned}
\|y[u_h]-y_h\|_{0,\Omega}^2 &= \mathcal{A}^*(\phi,\psi;y[u_h]-y_h,\mathbf{q}[u_h]-\mathbf{q}_h) = \mathcal{A}(y[u_h]-y_h,\mathbf{q}[u_h]-\mathbf{q}_h;\phi,\psi)\\
&= \mathcal{A}(y[u_h]-y_h,\mathbf{q}[u_h]-\mathbf{q}_h;\phi-i_h\phi,\psi-\mathbf{I}_h\psi). \qquad (33)
\end{aligned}
$$

By the residual (31) and Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
\|y[u_h]-y_h\|_{0,\Omega}^2 \le{}& \sum_{K\in\mathcal{T}_h}\|f-\nabla\cdot(\beta y_h-\epsilon^{\frac12}\mathbf{q}_h)-\alpha y_h\|_{0,K}\|\phi-i_h\phi\|_{0,K} + \sum_{K\in\mathcal{T}_h}\|\epsilon^{\frac12}\nabla y_h-\mathbf{q}_h\|_{0,K}\|\psi-\mathbf{I}_h\psi\|_{0,K}\\
&+ \sum_{E\in\mathcal{E}_h^0}\epsilon^{\frac12}\|\{\!\{\phi-i_h\phi\}\!\}+\mathbf{C}_{12}\cdot[\![\phi-i_h\phi]\!]\|_{0,E}\|[\![\mathbf{q}_h]\!]\|_{0,E}\\
&+ \sum_{E\in\mathcal{E}_h^0}\epsilon^{\frac12}\|\{\!\{\psi-\mathbf{I_h}\psi\}\!\}-\mathbf{C}_{12}[\![\psi-\mathbf{I_h}\psi]\!]\|_{0,E}\|[\![y_h]\!]\|_{0,E}\\
&+ \sum_{E\in\mathcal{E}_h^0}\|\{\!\{\phi-i_h\phi\}\!\}-\mathbf{D}_{11}\cdot[\![\phi-i_h\phi]\!]\|_{0,E}\|\beta\cdot[\![y_h]\!]\|_{0,E} + \sum_{E\in\mathcal{E}_h^0}\epsilon^{\frac12}\|C_{11}[\![y_h]\!]\|_{0,E}\|[\![\phi-i_h\phi]\!]\|_{0,E}\\
&+ \sum_{E\in\Gamma^-}|\beta\cdot\mathbf{n}|\|y_h-u_h\|_{0,E}\|\phi-i_h\phi\|_{0,E} + \sum_{E\in\mathcal{E}_h^\partial}\epsilon^{\frac12}C_{11}\|y_h-u_h\|_{0,E}\|\phi-i_h\phi\|_{0,E}\\
&+ \sum_{E\in\mathcal{E}_h^\partial}\epsilon^{\frac12}\|y_h-u_h\|_{0,E}\|\psi-\mathbf{I}_h\psi\|_{0,E}. \qquad (34)
\end{aligned}
$$

The estimates in (13), regularity estimate in (14), Young's inequality, and the assumption $\beta = O(1)$ in (14) yield

$$\begin{aligned}\|y[u_h]-y_h\|_{0,\Omega}^2 \;\leq\; & C\Bigg(\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\epsilon}\|f-\nabla\cdot(\beta y_h-\epsilon^{\frac12}\mathbf{q}_h)-\alpha y_h\|_{0,K}^2+\sum_{K\in\mathcal{T}_h}\frac{h_k^2}{\epsilon}\|\epsilon^{\frac12}\nabla y_h-\mathbf{q}_h\|_{0,K}^2 \\ & +\sum_{E\in\mathcal{E}_h^0}\left(\frac{h_E}{2}\|[\![\mathbf{q}_h]\!]\|_{0,E}^2+\left(\frac{h_E}{2\epsilon}+\frac{\epsilon}{h_E}\right)\|[\![y_h]\!]\|_{0,E}^2\right) \\ & +\sum_{E\in\mathcal{E}_h^\partial}\left(\frac{h_E}{\epsilon^2}+\frac{\epsilon}{h_E}\right)\|y_h-u_h\|_{0,E}^2\Bigg) \;\leq\; C\eta_y^2. \end{aligned} \tag{35}$$

Then, combining (16a) and (35), we obtain

$$\|y-y_h\|_{0,\Omega}^2 \leq C\eta_y^2 + C\|u-u_h\|_{0,\Gamma}^2. \tag{36}$$

Now, we derive an upper bound for $\|z[u_h]-z_h\|_{0,\Omega}$ by following the procedure as in the derivation of (31).

$$\begin{aligned}&\mathcal{A}^*(z[u_h]-z_h,\mathbf{p}[u_h]-\mathbf{p}_h;v,\mathbf{r}) \\ &= (y[u_h]-y^d,v)_{0,\Omega}+\sum_{K\in\mathcal{T}_h}\int_K\big(-\nabla\cdot(\epsilon^{\frac12}\mathbf{p}_h-\beta z_h)-\alpha z_h\big)v\,dx-\sum_{K\in\mathcal{T}_h}\int_K\big(\epsilon^{\frac12}\nabla z_h+\mathbf{p}_h\big)\mathbf{r}\,dx \\ &\quad+\sum_{E\in\mathcal{E}_h^0}\int_E\Big(\epsilon^{\frac12}\big(\{\!\{v\}\!\}+\mathbf{C}_{12}\cdot[\![v]\!]\big)[\![\mathbf{p}_h]\!]-\big(\mathbf{D}_{11}\cdot[\![v]\!]+\{\!\{v\}\!\}\big)\beta\cdot[\![z_h]\!]\Big)\,ds+\sum_{E\in\mathcal{E}_h^0}\int_E\epsilon^{\frac12}C_{11}[\![v]\!]\cdot[\![z_h]\!]\,ds \\ &+\sum_{E\in\mathcal{E}_h^\partial}\int_E\epsilon^{\frac12}C_{11}z_h v\,ds-\sum_{E\in\Gamma^+}\int_E|\beta\cdot\mathbf{n}|z_h v\,ds+\sum_{E\in\mathcal{E}_h^0}\int_E\epsilon^{\frac12}\big(\{\!\{\mathbf{r}\}\!\}-\mathbf{C}_{12}[\![\mathbf{r}]\!]\big)[\![z_h]\!]\,ds+\sum_{E\in\mathcal{E}_h^\partial}\int_E\epsilon^{\frac12}z_h\mathbf{r}\,ds.\end{aligned} \tag{37}$$

Analogously, we apply the duality argument to find an estimate for $\|z[u_h]-z_h\|_{0,\Omega}$. It is easy see that (ϕ,ψ) satisfies

$$\mathcal{A}(\phi,\psi;v,\mathbf{r})=\sum_{K\in\mathcal{T}_h}\int_K\big(z[u_h]-z_h\big)v\,dx. \tag{38}$$

Setting $v=z[u_h]-z_h$ and $\mathbf{r}=\mathbf{p}[u_h]-\mathbf{p}_h$ in (38), and using the operators defined in (12) and the definition of $\mathcal{A}^*(\cdot,\cdot)$, we obtain

$$\begin{aligned}\|z[u_h]-z_h\|_{0,\Omega}^2 &= \mathcal{A}(\phi,\psi;z[u_h]-z_h,\mathbf{p}[u_h]-\mathbf{p}_h)=\mathcal{A}^*(z[u_h]-z_h,\mathbf{p}[u_h]-\mathbf{p}_h;\phi,\psi) \\ &= \mathcal{A}^*(z[u_h]-z_h,\mathbf{p}[u_h]-\mathbf{p}_h;\phi-i_h\phi,\psi-\mathbf{I}_h\psi)+(y[u_h]-y_h,i_h\phi).\end{aligned} \tag{39}$$

Using the residual (39) with the estimates (13), regularity estimate in (14), and Young's inequality, we get

$$\begin{aligned}\|z[u_h]-z_h\|_{0,\Omega}^2 \;\leq\; & C\Bigg(\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\epsilon}\|y_h-y^d-\nabla\cdot(\epsilon^{\frac12}\mathbf{p}_h-\beta z_h\|_{0,K}^2+\sum_{K\in\mathcal{T}_h}\frac{h_k^2}{\epsilon}\|\epsilon^{\frac12}\nabla z_h+\mathbf{p}_h\|_{0,K}^2 \\ & +\sum_{E\in\mathcal{E}_h^0}\left(\frac{h_E}{2}\|[\![\mathbf{p}_h]\!]\|_{0,E}^2+\left(\frac{h_E}{2\epsilon}+\frac{\epsilon}{h_E}\right)\|[\![z_h]\!]\|_{0,E}^2\right) \\ & +\sum_{E\in\mathcal{E}_h^\partial}\left(\frac{h_E}{\epsilon}+\frac{\epsilon}{h_E}\right)\|z_h\|_{0,E}^2\Bigg)+C\|y[u_h]-y_h\|_{0,\Omega}^2 \\ \leq\; & C\eta_z^2+C\|y[u_h]-y_h\|_{0,\Omega}^2\leq C\big(\eta_z^2+\eta_y^2\big).\end{aligned} \tag{40}$$

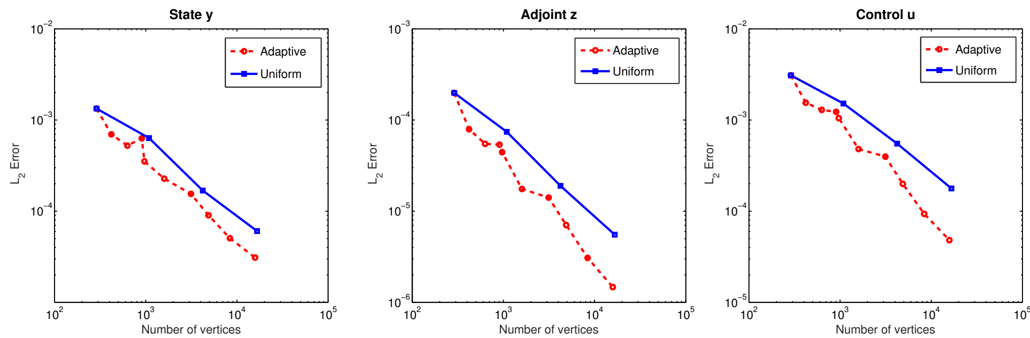

FIGURE 1. Global errors of the state, adjoint, and control in L^2–norm on adaptively and uniformly refined meshes with $\epsilon = 10^{-2}$ and $\theta = 0.6$ in the bulk criterion.

Combination of (16) and (40) yields

$$\|z - z_h\|_{0,\Omega}^2 \leq C\left(\eta_y^2 + \eta_z^2\right) + C\|u - u_h\|_{0,\Gamma}^2. \tag{41}$$

In conclusion, the desired result is obtained by combining (28), (36), and (41). □

4. Numerical Experiments

An adaptive procedure consists of successive loops of the following sequence:

$$\textbf{SOLVE} \rightarrow \textbf{ESTIMATE} \rightarrow \textbf{MARK} \rightarrow \textbf{REFINE}. \tag{42}$$

The **SOLVE** step is the numerical solution of the optimal control problem with respect to the given triangulation $\mathcal{T}_h$ using the LDG discretization with upwind for the convection. The **ESTIMATE** step requires the computation of the residual–based a posteriori error estimators. We use a bulk criterion in the **MARK** step to specify the elements in $\mathcal{T}_h$ by using the a posteriori error estimator and by choosing subsets $\mathcal{M}_K \subset \mathcal{T}_h$ such that the bulk criterion is satisfied for a given marking parameter Θ with $0 < \Theta < 1$: $\Theta \sum\limits_{K \in \mathcal{T}_h} \eta_K \leq \sum\limits_{K \in \mathcal{M}_K} \eta_K$, where η_K is the a posteriori error estimator derived in Section 3. Finally, in the **REFINE** step, the marked elements are refined by longest edge bisection, whereas the elements of the marked edges are refined by bisection strategy. The adaptive procedure repeats until a given complexity #vertices, i.e., the number of vertices, is satisfied.

Now, we test the performance of the residual-based a posteriori error estimators (18) for the following data:

$$\Omega = [0,1]^2, \quad \beta = [1,1]^T, \quad \alpha = 2, \quad \omega = 0.1.$$

The source function f and the desired state y^d are computed by using the following analytical solutions

$$y(x_1, x_2) = -\epsilon\big(x_1(1 - x_1) + x_2(1 - x_2)\big), \qquad z(x_1, x_2) = \omega\big(x_1 x_2(1 - x_1)(1 - x_2)\big).$$

Figure 1 illustrates the performance of the error estimator (18) in terms of the number of vertices for the marking parameter $\theta = 0.60$ and the diffusion parameter $\epsilon = 10^{-2}$ on adaptively and uniformly refined meshes. In Table 1, we display global errors with $\epsilon = 10^{-3}$ on adaptively and uniformly refined meshes. Due to oscillation on the inflow boundary, we do not obtain stable solutions for the control on uniformly refined meshes. However, more stable solutions are obtained on adaptively refined meshes, see Figure 2. In conclusion, the obtained results show that adaptive refinements lead to better approximate solutions than uniform refinements.

TABLE 1. Global errors of the state, adjoint, and control in L^2–norm on adaptively and uniformly refined meshes with $\epsilon = 10^{-3}$ and $\theta = 0.5$ in the bulk criterion.

Uniform				Adaptive			
DoF	$\|y - y_h\|$	$\|z - z_h\|$	$\|u - u_h\|$	DoF	$\|y - y_h\|$	$\|z - z_h\|$	$\|u - u_h\|$
1089	2.68e-04	4.84e-05	2.03e-04	2589	1.58e-04	2.78e-05	3.58e-04
4225	2.54e-04	4.58e-05	3.51e-04	4144	9.22e-05	1.49e-05	2.38e-04
16641	1.60e-04	2.84e-05	3.63e-04	5355	7.06e-05	1.06e-05	1.88e-04

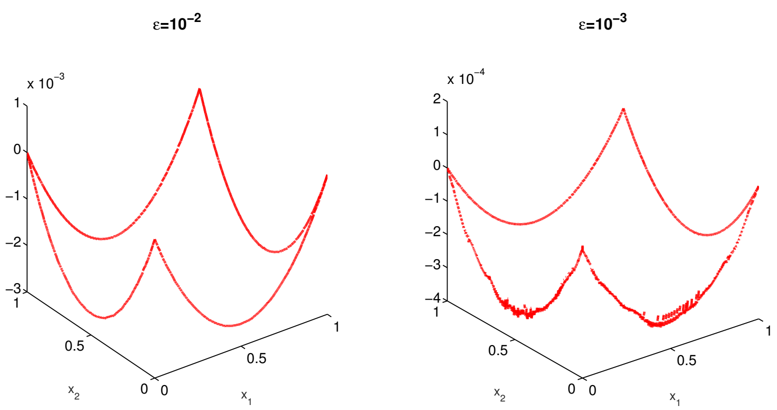

FIGURE 2. Computed control on adaptive meshes for $\epsilon = 10^{-2}$ (left) and $\epsilon = 10^{-3}$ (right).

REFERENCES

[1] D. N. Arnold, F. Brezzi, B. Cockburn, L. D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal. 39 (5) (2002) 1749–1779.

[2] I. Babuška, W. C. Rheinboldt, Error estimates for adaptive finite element computations, SIAM J. Numer. Anal. 15 (1978) 736–754.

[3] R. Becker, H. Kapp, R. Rannacher, Adaptive finite element methods for optimal control of partial differential equations: Basic concepts, SIAM J. Control and Optimization 39 (2000) 113–132.

[4] F. Belgacem, H. E. Fekih, H. Metoui, Singular perturbation for the Dirichlet boundary control of elliptic problems, M2AN Math. Model. Numer. Anal. 37 (2003) 883–850.

[5] O. Benedix, B. Vexler, A posteriori error estimation and adaptivity for elliptic optimal control problems with state constraints, Comput. Optim. App. 44 (1) (2009) 3–25.

[6] P. Benner, H. Yücel, Adaptive symmetric interior penalty Galerkin method for boundary control problems, SIAM J. Numer. Anal. 55 (2) (2017) 1101–1133.

[7] M. Berggren, Approximations of very weak solutions to boundary-value problems, SIAM J. Numer. Anal. 42 (2) (2004) 860–877.

[8] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Computational Mathematics, Vol. 15, Springer, Berlin, 1991.

[9] E. Casas, J.-P. Raymond, Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations, SIAM J. Control Optim. 45 (5) (2006) 1586–1611.

[10] L. Chang, W. Gong, N. Yan, Weak boundary penalization for Dirichlet boundary control problems governed by elliptic equations, J. Math. Anal. Appl. 453 (1) (2017) 529–557.

[11] S. Chowdhury, T. Gudi, A. K. Nandakumaran, Error bounds for a Dirichlet boundary control problem based on energy spaces, Math. Comput. 86 (2017) 1103–1126.

[12] B. Cockburn, C.-W. Shu, The local discontinuous Galerkin method for time–dependent convection–diffusion systems, SIAM J. Numer. Anal. 35 (6) (1998), 2440-2463.

[13] K. Deckelnick, A. Günther, M. Hinze, Finite element approximation of Dirichlet boundary control for elliptic PDEs on two–and three–dimensional curved domains, SIAM J. Control Optim. 48 (2009) 2798–2819.

[14] W. Gong, W. Liu, Z. Tan, N. Yan, A convergent adaptive finite element method for elliptic Dirichlet boundary control problems, IMA J. Numer. Anal. 39 (4) (2019), 1985-2015.

[15] W. Gong, N. Yan, Mixed finite element method for Dirichlet boundary control problem governed by elliptic PDEs, SIAM J. Control Optim. 49 (3) (2011) 984–1014.

[16] M. Hintermüller, R. H. W. Hoppe, Goal-oriented adaptivity in control constrained optimal control of partial differential equations, SIAM J. Control Optim. 47 (4) (2008) 1721–1743.

[17] M. Hinze, N. Yan, Z. Zhou, Variational discretization for optimal control governed by convection dominated diffusion equations, J. Comp. Math. 27 (2-3) (2009) 237–253.

[18] R. H. W. Hoppe, M. Kieweg, Adaptive finite element methods for mixed control-state constrained optimal control problems for elliptic boundary value problems, Comput. Optim. Appl. 46 (3) (2010) 511–533.

[19] K. Kohls, A. Rösch, and K. Siebert, A posteriori error analysis of optimal control problems with control constraints, SIAM J. Control. Optim., 52(3) (2014) 1832–1861.

[20] D. Leykekhman, M. Heinkenschloss, Local error analysis of discontinuous Galerkin methods for advection-dominated elliptic linear-quadratic optimal control problems, SIAM J. Numer. Anal. 50 (4) (2012) 2012–2038.

[21] J.-L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer, Berlin, 1971.

[22] R. Li, W. Liu, H. Ma, T. Tang, Adaptive finite element approximation for distributed elliptic optimal control problems, SIAM J. Control Optim. 41 (5) (2002) 1321–1349.

[23] W. Liu, N. Yan, A posteriori error estimates for distributed convex optimal control problems, Adv. Comp. Math. 15 (2001) 285–309.

[24] B. Rivière, Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations. Theory and Implementation, Frontiers Appl. Math., SIAM, Philadelphia, 2008.

[25] H. G. Roos, M. Stynes, and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations*, Computational Mathematics, Vol. 24, Springer, Berlin, second ed., 2008.

[26] F. Tröltzsch, Optimal Control of Partial Differential Equations: Theory, Methods and Applications, Vol. 112 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2010.

[27] B. Vexler, W. Wollner, Adaptive finite elements for elliptic optimization problems with control constraints, SIAM J. Control Optim. 47 (2008) 509–534.

[28] H. Yücel, Goal–oriented a posteriori error estimation for Dirichlet boundary control problems, J. Comput. Appl. Math. 381 (2021) 113012.

[29] H. Yücel, P. Benner, Adaptive discontinuous Galerkin methods for state constrained optimal control problems governed by convection diffusion equations, Comput. Optim. Appl. 62 (2015) 291–321.

[30] H. Yücel, B. Karasözen, Adaptive symmetric interior penalty Galerkin (SIPG) method for optimal control of convection diffusion equations with control constraints, Optimization 63 (2014) 145–166.

[31] Z. Zhou, X. Yu, N. Yan, The local discontinuous Galerkin approximation of convection-dominated diffusion optimal control problems with control constraints, Numer. Methods Partial Differ. Equ. 30 (1) (2014) 339–360.