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A CLASS OF SYMMETRIC-HYPERBOLIC PDES MODELLING FLUID AND SOLID CONTINUA *, **

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Abstract. We generalize a new symmetric-hyperbolic system of PDEs proposed in [ESAIM:M2AN 55 (2021) 807-831] for Maxwell fluids to a *class* of systems that define unequivocally multi-dimensional visco-elastic flows.
Precisely, within a general setting for continuum mechanics, we specify constitutive assumptions i) that ensure the unequivocal definition of motions satisfying widely-admitted physical principles, and ii) that contain [ESAIM:M2AN 55 (2021) 807-831] as one particular realization of those assumptions.
The new class can capture the mechanics of various materials, from solids to viscous fluids, possibly with temperature dependence and heat conduction.

Résumé. Nous généralisons un nouveau système d'EDPs hyperbolique-symétrique proposé dans [ESAIM:M2AN 55 (2021) 807-831] pour des fluides de Maxwell, en une *classe* de systèmes qui définissent de manière non-équivoque des écoulements visco-élastiques multi-dimensionels.

Précisément, au sein d'un formalisme général pour la mécanique des milieux continus, nous spécifions des hypothèse constitutives i) qui assurent la définition non-équivoque de mouvements satisfaisant des principes physiques largement admis, et ii) qui contiennent [ESAIM:M2AN 55 (2021) 807-831] comme réalisation particulière de ces hypothèses.

La nouvelle classe peut représenter la mécanique de matériaux variés, des solides aux fluides visqueux, éventuellement avec dépendance en la température et conduction de chaleur.

1. INTRODUCTION

Continuum mechanics has kept evolving since pioneering works of Bernoullis, D'Alembert, Euler, Cauchy etc., in particular with a view to describing motions of various materials more and more realistically. A tenet of continuum mechanics is the description of the motions of infinitely-many "glued" particles usually termed *bodies* through functions continuous in time and space, with directional derivatives (like $\boldsymbol{\phi}_t^{-1}$ defined in Section 2). The functions are defined as solutions to Boundary Value Problems (BVPs) using Partial Differential Equations (PDEs). To define motions of various materials, the PDEs vary: two bodies with identical initial positions move differently depending on the modelled materials.

The variety of PDEs considered by continuum mechanics reflects the variety of *constitutive assumptions* that can be postulated to complement the general continuum mechanics theory and specify one *model* for the motions of some particular materials (the motion of a body is unequivocally defined as a PDE solution). Despite many efforts toward

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systematization, choosing constitutive assumptions for one particular materials remains an art despite some guiding principles and a number of existing constitutive assumptions [44].

In the present work, we discuss constitutive assumptions that lead to a large class of (symmetric-hyperbolic) PDEs covering many materials between elastic solids and viscous fluids. What is exactly meant by those material behaviours needs precising, of course: this belongs to our discussion below. We also show particular examples of constitutive laws that realize our proposed constitutive assumptions.

So far, continuum mechanics and its PDEs mainly split into solid mechanics on the one hand, and fluid mechanics on the other hand. Solid mechanics usually postulates a time-independent "stress-free" reference state. The stress-free state is then mapped to all current states through a time-parametrized *diffeomorphism* – the inverse of the so-called deformation field – supposedly continuous in time. On the contrary, fluid mechanics take vortices into account, as well as viscous friction. Such energy-dissipative phenomenas generally hinder the use of a single reference state. But solid mechanics can also consider energy-dissipative phenomenas on the one hand such as plasticity, see e.g. [24, 43]. And fluid mechanics can also consider non-Newtonian elastic fluids "with memory" [2]. So the main difference between what is usually meant by "solid" and "fluid" behaviours is in fact different dynamics in e.g. relaxation to an equilibrium after external forcing. Solids tend to remember their past i.e. a stress-free configuration, while fluids tend to forget and adapt to current configuration. Now, a number of materials behave sometimes as solids and sometimes as fluids [16]. Constitutive assumptions that encompass standard solid and fluid materials behaviours are therefore useful in continuum mechanics…and existing propositions still need consolidating.

Maxwell proposed in 1867 a seminal 1D *visco-elastic fluid* model with a relaxation time, which formally describes elastic solids when the relaxation time is infinite and Newtonian fluids when the relaxation time is zero [30]. Inspiring to many rheologists, numerous extensions of the model have been proposed for various visco-elastic materials and geometries, see e.g. [16]. But the model has proved difficult to extend soundfully to multi-dimensional flows, unless diffusion that prevents waves is added to the model. The extension of Maxwell ideas for application to realistic flows remains an active research topics [2, 23].

In [ESAIM:M2AN 55 (2021) 807-831] i.e. [4], we proposed a mathematically-sound multi-dimensional formulation for flows of Maxwell fluids that captures finite-speed waves. That proposition consists in a symmetric-hyperbolic system of balance laws compatible with elastodynamics for hyperelastic materials in the infinite relaxation-time asymptotics $\lambda \to \infty$. (Recall a symmetric-hyperbolic formulation is essential for well-posed Cauchy problems with quasilinear systems, and typically provided by a strictly convex *mathematical entropy*.) In comparison with standard formulations of the Maxwell's model for compressible fluids it essentially uses additional variables to model time-evolving material properties. The proposition copes well with established extensions of Maxwell's model; we believe it offers a sound framework for physical extensions that aim at unifying solids with various (complex, non-Newtonian) fluids, using additional variables to model various "imperfections".

Here, in this work, we extend the proposition we made in [4] to general constitutive assumptions. In Section 6, we precisely state those general constitutive assumptions. Our general constitutive assumptions are new in the sense that they complement usual constitutive assumptions (ensuring widely-admitted physical principles like thermodynamics, frame-indifference) to cover *a large class of models rigorously unifying unequivocal solid and fluid motions at finite-wave speed* i.e. in a hyperbolic (PDE) framework. The assumptions cover, in particular, standard motions that have long been proposed for elastic solids, our formulation of visco-elastic motions for fluids of Maxwell-type [4] possibly non-isothermal [6],as well as motions that are new to our knowledge (see Section 6). We believe such new general constitutive assumptions are needed to model well a number of real motions, geophysical flows like ladnslide in particular. The article is organized as follows.

In Section 2, we give a short introduction to continuum mechanics, see e.g. [15, 24, 29, 43] for more.
In Section 3, we recall how one defines *some classical* isentropic motions of hyperelastic solid materials, given a stored energy functional.
In Section 4, we propose *a class of stored energy functionals* that allow one to define many isentropic motions of hyperelastic solid materials with good mathematical and physical properties on following the classical approach above.
In Section 5, we recall how one standardly defines fluid motions, not isentropic.

In Section 6, we propose our new class of *visco-hyperelastic* fluids, which bases on the admissible and reasonable constitutive assumptions of Section 4 and which takes full advantage of our previous ideas in [4] for extension to fluids. The new class of materials can depend on temperature; it covers standard and new motions. In Section 7, we conclude about the new class and its perspectives. We also state one possible set of constitutive assumptions for a class of rigid heat-conductors (following seminal ideas of Cattaneo) that naturally couples with our constitutive assumptions of Section 6 for solid/fluid thermo-mechanics without heat conduction.

Note that the stored energy functionals that allow one to define isentropic motions of hyperelastic solid in Section 4 can be found in most textbooks [15,24,29,43]. But we are not aware of a similar study, that carefully constructs stored energy functionals achieving the (standard) constitutive assumptions for isentropic motions of hyperelastic solids, as from our proposed class. In particular, to satisfy constitutive assumptions, we propose to require stored energy functionals satisfying **(H2)**, and show in Proposition 1 that this is actually achievable, see Section 4.

2. Continuum Mechanics setting

In this work, we use Einstein convention for repeated indices. We denote t the time, and x^i, $i = 1 \dots d$ the coordinates in a Cartesian basis $\boldsymbol{e}_i$ of the Euclidean space $\mathbb{R}^d = \{\boldsymbol{x} = x^i \boldsymbol{e}_i, x_i \in \mathbb{R}\}$ for $d = 2$ or 3.

We consider (continuous) bodies that fill $\mathbb{R}^d$ for $t \in [0,T)$, such that there exists a diffeomorphism $\boldsymbol{\phi}_t^{-1}(\boldsymbol{x}) = \boldsymbol{a} \in \mathbb{R}^d$ onto a stress-free configuration also equipped with another Cartesian coordinate system $\{a^\alpha, \alpha = 1 \dots 3\}$. (The terminology "stress-free" will be clarified in Section 3 below.)

We identify $\boldsymbol{\phi}_t^{-1}(\boldsymbol{x}) = \boldsymbol{a}$ with a *particle* of mass density $\hat{\rho}$ given any $\boldsymbol{x} \in \mathbb{R}^d$, and we assume $\hat{\rho}$ constant i.e. considered materials are *homogeneous* in mass density.

We assume the *back-to-label map* $\boldsymbol{\phi}_t^{-1}$ smooth with respect to $\boldsymbol{x} \in \mathbb{R}^d$ and t, so one can first establish *isentropic* (i.e. time-reversible) deformations $\boldsymbol{\phi}_t$ of one time-independent stress-free configuration diffeomorphic to $\mathbb{R}^d$ such as "purely elastic solid" motions of hyperelastic materials, see Sections 3 and 4. Non-isentropic motions with dissipation (viscous vortices e.g. in fluids) are standardly considered next only. The essence of the present work is to propose in Section 6 a new class of fluids that are characterized by a relaxation time like in [30], and that allow one to define unequivocally *multi-dimensional* non-isentropic motions like in [4].

Denoting σ_{ijk} Levi-Civita's symbol for $i,j,k \in \{1,2,3\}$, $\boldsymbol{u} := \partial_t \boldsymbol{\phi}_t$ the *velocity*, $\boldsymbol{F} = F^i_\alpha \boldsymbol{e}_i \otimes \boldsymbol{e}^\alpha$ the *deformation gradient* where $F^i_\alpha := \partial_\alpha \phi^i$, $|\boldsymbol{F}|$ its determinant and $\hat{\boldsymbol{F}}$ its cofactor (equiv. transpose adjugate), the following conservation laws

$$\partial_t F^i_\alpha - \partial_\alpha u^i = 0 \tag{1}$$

$$\partial_t |\boldsymbol{F}| - \partial_\alpha \left(\hat{F}^i_\alpha \, u^i \right) = 0 \tag{2}$$

$$\partial_t \hat{F}^i_\alpha + \sigma_{ijk} \sigma_{\alpha\beta\gamma} \partial_\beta \left(F^j_\gamma \, u^k \right) = 0 \tag{3}$$

are established using classical derivatives on $\mathbb{R} \times \mathbb{R}^d \ni t, \boldsymbol{a}$ and Piola's identities [46]

$$\sigma_{\alpha\beta\gamma} \partial_\beta F^i_\gamma = 0 = \partial_\alpha \hat{F}^i_\alpha \quad \forall i \,. \tag{4}$$

Equivalent identities also hold in *Eulerian description* i.e. using *spatial coordinates*:

$$\partial_t \left(\rho F^i_\alpha \right) + \partial_j \left(\rho F^i_\alpha u^j - \rho u^i F^j_\alpha \right) = 0 \tag{5}$$

$$\partial_t \rho + \partial_j \left(\rho u^j \right) = 0 \tag{6}$$

$$\partial_t G^i_\alpha + \partial_i \left(G^j_\alpha u^j \right) = 0 \tag{7}$$

having defined $\rho := |\boldsymbol{F}|^{-1}\hat{\rho}$, $\boldsymbol{G} = \boldsymbol{F}^{-T}$ (the transpose of the matrix inverse $\boldsymbol{F}^{-1}$), denoting similarly functions of $\boldsymbol{x}$ or $\boldsymbol{a}$ depending on the context, and rewriting Piola's identities in spatial coordinates

$$\partial_j(\rho F^j_\alpha) = 0 = \sigma_{ijk}\partial_j(\hat{\rho}G^k_\alpha) \quad \forall\alpha\,. \tag{8}$$

In tensor notation one could write, in spatial coordinates

$$\partial_t\left(\rho\boldsymbol{F}^T\right) - \boldsymbol{\nabla}\times\left(\rho\boldsymbol{F}^T\times\boldsymbol{u}\right) = \boldsymbol{0} \tag{9}$$

$$\partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = 0 \tag{10}$$

$$\partial_t\left(\boldsymbol{G}^T\right) + \boldsymbol{\nabla}\left(\boldsymbol{G}^T\cdot\boldsymbol{u}\right) = \boldsymbol{0} \tag{11}$$

where $\boldsymbol{G}^T$ is the dual (matrix transpose) of $\boldsymbol{G}$, using Piola's identities (8)

$$\operatorname{div}(\rho\boldsymbol{F}^T) = \boldsymbol{0} = \boldsymbol{\nabla}\times(\hat{\rho}\boldsymbol{G}^T)\,, \tag{12}$$

or in material coordinates, for the so-called *Lagrangian description*

$$\partial_t\boldsymbol{F}^T = \boldsymbol{\nabla}_{\boldsymbol{a}}\boldsymbol{u} \tag{13}$$

$$\partial_t|\boldsymbol{F}| = \operatorname{div}_{\boldsymbol{a}}\left(\hat{\boldsymbol{F}}^T\cdot\boldsymbol{u}\right) \tag{14}$$

$$\partial_t\hat{\boldsymbol{F}}^T + \boldsymbol{\nabla}_{\boldsymbol{a}}\times\left(\boldsymbol{F}^T\times\boldsymbol{u}\right) = \boldsymbol{0} \tag{15}$$

which makes use of the Piola's identities (4)

$$\operatorname{div}_{\boldsymbol{a}}\hat{\boldsymbol{F}}^T = \boldsymbol{0} = \boldsymbol{\nabla}_{\boldsymbol{a}}\times\boldsymbol{F}^T\,. \tag{16}$$

Note that when $d = 2$, (15) are redundant with (13) and (11) with (9).

Now, to unequivocally define motions, i.e. $\boldsymbol{u}(\boldsymbol{x},t)$ for all $\boldsymbol{x}$ and small times $t\in[0,T)$ at least, continuum mechanics complements the above "kinematical" PDEs with physics principles like energy conservation and Galilean invariance.

3. Isentropic motions of hyperelastic materials

In Lagrangian description, the general balance of energy

$$\hat{\rho}\partial_t\left(\frac{|\boldsymbol{u}|^2}{2} + e\right) = \partial_\alpha\left(S^{i\alpha}u^i - Q^\alpha\right) + \hat{\rho}(u^i f^i + r) \tag{17}$$

(thermodynamics'first principle) is required for a body with entropy η, temperature

$$\theta = \partial_\eta e \tag{18}$$

supposedly non-negative, given force and heat sources $\boldsymbol{f}, r$. The *strain* or *stored energy* $e(\boldsymbol{F},\eta,\boldsymbol{p})$ determines the first Piola-Kirchoff stress tensor $\boldsymbol{S}$ in (17) through

$$S^{i\alpha} = \hat{\rho}\partial_{F^i_\alpha}e \tag{19}$$

as well as the heat flux $\boldsymbol{Q}\equiv Q^\alpha\boldsymbol{e}_\alpha$ when e actually depend on a state variable $\boldsymbol{p}$ to account for heat transfer by conduction – see Section 7.1. But to start with, as already said, we consider only isentropic motions

$$\partial_t\eta = 0 \tag{20}$$

where $r = 0 = Q^\alpha$ and the balance of energy (17) reduces to

$$\hat{\rho}\partial_t\left(\frac{|\boldsymbol{u}|^2}{2} + e\right) = \partial_\alpha\left(S^{i\alpha}u^i\right) + \hat{\rho}u^i f^i \tag{21}$$

or in spatial coordinates and tensor notation

$$\partial_t\left(\frac{\rho}{2}|\boldsymbol{u}|^2 + \rho e\right) + \operatorname{div}\left(\left(\frac{\rho}{2}|\boldsymbol{u}|^2 + \rho e\right)\boldsymbol{u} - \boldsymbol{\sigma}\cdot\boldsymbol{u}\right) = \rho\boldsymbol{f}\cdot\boldsymbol{u} \tag{22}$$

where $\sigma^{ij} := |\boldsymbol{F}|^{-1}S^{i\alpha}F^j_\alpha$ is Cauchy stress.

Requiring (21) and Galilean invariance leads to the linear momentum balance

$$\hat{\rho}\partial_t u^i = \partial_\alpha S^{i\alpha} + \hat{\rho}f^i \tag{23}$$

in material coordinates, or equiv. in spatial coordinates

$$\partial_t\left(\rho u^i\right) + \partial_j\left(\rho u^i u^j - \sigma^{ij}\right) = \rho f^i \tag{24}$$

see e.g. [45]. In tensor notation, (23) rewrites

$$\hat{\rho}\partial_t\boldsymbol{u} = \operatorname{div}_{\boldsymbol{a}}\boldsymbol{S} + \hat{\rho}\boldsymbol{f} \tag{25}$$

and the equivalent Eulerian balance (24) rewrites

$$\partial_t\left(\rho\boldsymbol{u}\right) + \operatorname{div}\left(\rho\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{\sigma}\right) = \rho\boldsymbol{f}\,. \tag{26}$$

So, at this stage, one can expect isentropic motions to be computable, well-defined solutions $\boldsymbol{u}(t,\boldsymbol{x})$ for $(t,\boldsymbol{x}) \in [0,T)\times\mathbb{R}^3$ to (25) or (26) complemented by i) kinematic PDEs established in Section 2, ii) initial conditions (for $\boldsymbol{u}$, $\boldsymbol{F}$ ...) and iii) a functional $e(\boldsymbol{F})$ that actually allows one to formulate a well-posed Cauchy (initial-value) problem either with the Lagrangian description or with the Eulerian description.

But various isentropic motions should be computable if the models have to cover various real materials. To precisely compute the motions of one particular materials, modelers would choose only *one* particular functional $e(\boldsymbol{F})$ (one *constitutive law*) among mathematical expressions that yield a well-posed BVP, so that the functional embodies physical specificities characterizing the motions of that particular materials.

The purpose of *constitutive assumptions* for isentropic motions is exactly to guide the choice of a functional $e(\boldsymbol{F})$, so that it is not only *mathematically reasonable* (in the sense: unequivocal solutions to Cauchy problems can be defined on small times at least), but also *physically admissible* in the sense: the mathematically-defined motions actually allow modellers to understand real motions.

In Section 6, we propose new constitutive assumptions **(H3)** and **(H4)** for non-isentropic motions, to cover both fluids and solids. To encompass standard assumptions for fluids, see Section 5, our new constitutive assumptions **(H3)** and **(H4)** contain visco-elastic fluid models, as well as isentropic motions of solids which are usually covered by more restrictive constitutive assumptions like **(H1)** and **(H2)** recalled below in Section 4. Note that in this work, the classical constitutive assumptions **(H1)**, which characterize stored energy functionals $e(\boldsymbol{F})$ for isentropic motions of solids, is precised as **(H2)** to define unequivocally the motions.

4. An admissible & reasonable class of solid materials

For physical admissibility, it is usual to require *material frame indifference* thus:
(H1) a *reduced stored energy function* $\hat{e}$ exists such that $e(\boldsymbol{F}) \equiv \hat{e}(\boldsymbol{C})$ depends on $\boldsymbol{F}$ through the right Cauchy-Green

strain $\boldsymbol{C} := \boldsymbol{F}^T \cdot \boldsymbol{F}$, see [29, Th. 2.10 of CH.3] or [43, (2.19)]. It implies that Cauchy stress tensor reads $\boldsymbol{\sigma} \equiv 2\rho \boldsymbol{F} \cdot \partial_{\boldsymbol{C}} \hat{e} \cdot \boldsymbol{F}^T$

$$\sigma^{ij} \equiv 2\hat{\rho} |\boldsymbol{F}|^{-1} F^i_\alpha \, F^j_\beta \, \partial_{C^{\alpha\beta}} \hat{e} \,. \tag{27}$$

Next, following seminal ideas by Noll [31, 32], it is usual to classify functionals $e(\boldsymbol{F}) \equiv \hat{e}(\boldsymbol{C})$ depending on their *material symmetry group* see e.g. [43]

$$\mathcal{G} := \{\boldsymbol{R} \,|\, \hat{e}(\boldsymbol{R}^T \cdot \boldsymbol{C} \cdot \boldsymbol{R}) = \hat{e}(\boldsymbol{C}) , \forall \boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}\}$$

a subset of the unimodular group $\mathbb{U} := \{\boldsymbol{R} \in \mathbb{R}^{d\times d} \,,\ |\boldsymbol{R}| = 1\}$. Much freedom apparently remains for an expression $e(\boldsymbol{F})$ at that stage. In fact, few physically-motivated expressions $e(\boldsymbol{F})$ are mathematically reasonable (i.e. yield a well-posed BVP and unequivocal motions) with the PDE model (1–2–3–4–19–23). For instance, if $e(\boldsymbol{F})$ is strictly convex in $\boldsymbol{F}$ and reads e.g. using a constant $c_1^2 > 0$

$$\frac{c_1^2}{2} \operatorname{tr}(\boldsymbol{C} - \boldsymbol{I}) \equiv \frac{c_1^2}{2} (F^k_\alpha F^k_\alpha - d) \,, \tag{28}$$

with the special orthogonal group as the material symmetry group (usually classified as suitable for "isotropic solids"), then the model is mathematically reasonable: isentropic motions are well-defined by unequivocal solutions to the Lagrangian system (1–19–23). Indeed, (1–19–23) has a *symmetric-hyperbolic* formulation thanks to (21) and Godunov-Mock theorem [22], so one can define unique time-continuous solutions, see Prop. 2 below[3]. But requiring the (strict) convexity of $e(\boldsymbol{F})$ in $\boldsymbol{F}$ seems contradictory with many observed motions [1, Section 2.7], especially with a view to unifying solids with fluids ("ultimately" in some asymptotics, see Section 5) hence to capturing deformations that are mostly determined by a *spheric* contribution $-p\boldsymbol{I}$ to Cauchy stress, with a pressure $p := -\hat{\rho}\partial_{|\boldsymbol{F}|} e_0$ resulting from a (major contribution to the) strain energy $e \approx e_0(|\boldsymbol{F}|)$ that only depends on the determinant. Recall fluids are usually modelled from the "perfect" case with maximal *material symmetry* $e(\boldsymbol{F}) = e(\boldsymbol{F}\boldsymbol{R})$ for the whole unimodular group $\mathbb{U} \ni \boldsymbol{R}$ [43], and the determinant $|\boldsymbol{F}|$ is not convex in $\boldsymbol{F}$!

Requiring $e(\boldsymbol{F})$ *polyconvex* in $\boldsymbol{F}$ can ensure the well-posedness of Cauchy problems and allow dependence of e on $|\boldsymbol{F}|$. Indeed, assuming $e(\boldsymbol{F})$ *polyconvex*, an additional conservation law can hold for a function strictly convex in the conserved variables, which ensures that a system of conservation laws like (1–2–3–4–19 –23) has i) a symmetric-hyperbolic formulation by Godunov-Mock theorem [22], hence ii) unequivocal small-time solutions to Cauchy problems [15]. Therefore, in this work, we require (strict) polyconvexity in constitutive assumptions:

(H2) There exists $\bar{e}(\boldsymbol{A}, \boldsymbol{B}, c)$ defined for symmetric positive matrices $(\boldsymbol{A}, \boldsymbol{B}) \in \boldsymbol{S}^d_+ \times \boldsymbol{S}^d_+$ and $c \in \mathbb{R}$ such that the function $(\boldsymbol{F}, \hat{\boldsymbol{F}}, |\boldsymbol{F}|) \to \bar{e}(\boldsymbol{F}^T \cdot \boldsymbol{F}, \hat{\boldsymbol{F}}^T \cdot \hat{\boldsymbol{F}}, |\boldsymbol{F}|)$ is stricly convex on $\mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \times \mathbb{R}$ and defines a stored energy function by $e(\boldsymbol{F}) \equiv \bar{e}(\boldsymbol{F}^T \cdot \boldsymbol{F}, \hat{\boldsymbol{F}}^T \cdot \hat{\boldsymbol{F}}, |\boldsymbol{F}|)$.

Note that well-studied functions yield **(H2)** – thus **(H1)** –, see e.g. [41]:

Proposition 1. **(H2)** *is fulfilled as soon as $\bar{e}$ is matrix-monotone i.e.*

$$A_1 \geq A_2 \;\&\; B_1 \geq B_2 \Rightarrow e(A_1, B_1, c) \geq e(A_2, B_2, c) \quad \forall A_1, A_2, B_1, B_2 \in \boldsymbol{S}^d_+ \,,$$

and convex in each argument, strictly convex in c.

Proof. Recall for any $\boldsymbol{F}_1, \boldsymbol{F}_2 \in \mathbb{R}^{d\times d}$, $\theta \in (0,1)$

$$\theta \boldsymbol{F}_1 \boldsymbol{F}_1^T + (1-\theta)\boldsymbol{F}_2\boldsymbol{F}_2^T - (\theta\boldsymbol{F}_1 + (1-\theta)\boldsymbol{F}_2)^T (\theta\boldsymbol{F}_1 + (1-\theta)\boldsymbol{F}_2) \\ = \theta(1-\theta)(\boldsymbol{F}_1 - \boldsymbol{F}_2)(\boldsymbol{F}_1 - \boldsymbol{F}_2)^T \in \boldsymbol{S}^d_+ \,. \tag{29}$$

[3] This is a particular case of Prop. 2 where the PDE system (1–19–23) is *linear* so smooth solutions can in fact be defined *globally* in time $\forall t \geq 0$ c.f. e.g. [15].

Then one can show that **(H2)** holds on composing the result above i.e.

$$\theta \boldsymbol{F}_1 \boldsymbol{F}_1^T + (1-\theta)\boldsymbol{F}_2 \boldsymbol{F}_2^T \geq (\theta \boldsymbol{F}_1 + (1-\theta)\boldsymbol{F}_2)^T (\theta \boldsymbol{F}_1 + (1-\theta)\boldsymbol{F}_2)$$

with the matrix-monotonicity and convexity of $\bar{e}$, similarly to [38, Th 5.1] for monotone convex functions on $\mathbb{R}$. Recall the determinant is matrix-monotone [41]. Then, for all $\theta \in [0,1]$, it first holds

$$\begin{aligned} e(\theta \boldsymbol{F}_1 + (1-\theta)\boldsymbol{F}_2) \equiv \bar{e}((\theta \boldsymbol{F}_1 + (1-\theta)\boldsymbol{F}_2)^T(\theta \boldsymbol{F}_1 + (1-\theta)\boldsymbol{F}_2), \ldots \\ \ldots (\theta \hat{\boldsymbol{F}}_1 + (1-\theta)\hat{\boldsymbol{F}}_2)^T(\theta \hat{\boldsymbol{F}}_1 + (1-\theta)\hat{\boldsymbol{F}}_2), |\theta \boldsymbol{F}_1 + (1-\theta)\boldsymbol{F}_2|) \\ \leq \bar{e}\left(\theta \boldsymbol{F}_1^T \boldsymbol{F}_1 + (1-\theta)\boldsymbol{F}_2^T \boldsymbol{F}_2, \theta \hat{\boldsymbol{F}}_1^T \hat{\boldsymbol{F}}_1 + (1-\theta)\hat{\boldsymbol{F}}_2^T \hat{\boldsymbol{F}}_2, \theta|\boldsymbol{F}_1| + (1-\theta)|\boldsymbol{F}_2|\right) \end{aligned} \tag{30}$$

by matrix-monotonicity of $\bar{e}$ and one concludes

$$e(\theta \boldsymbol{F}_1 + (1-\theta)\boldsymbol{F}_2) \leq \theta e(\boldsymbol{F}_1) + (1-\theta) e(\boldsymbol{F}_2)$$

by (standard) convexity in each argument. □

Using **(H2)**, (21), Godunov-Mock theorem [22] and the available theory for symmetric-hyperbolic quasilinear systems of PDEs [15], the following holds:

Proposition 2. *The system (1–2–3–4–19–23) has a* symmetric-hyperbolic *formulation. As a consequence, when complemented by smooth initial conditions compatible with* (4)*, it unequivocally defines smooth motions through solutions*

$$(u^i, F^i_\alpha) \in C^0_t\left([0,T), H^s(\mathbb{R}^d)^d \times H^s(\mathbb{R}^d)^{d\times d}\right)$$

i.e. time-continuous solutions with values in Sobolev spaces $H^s(\mathbb{R}^d)$, $s > \frac{d}{2}$.

The constitutive assumption **(H2)** has often been used with (1–2–3–4–19–23) to compute realistic *isentropic* motions of purely elastic solids. Many computations use a stored energy $e(\boldsymbol{F})$ of type

$$e_0(|\boldsymbol{F}|) + \hat{e}_1(\boldsymbol{C}) + \hat{e}_2(\hat{\boldsymbol{C}}) \tag{31}$$

where $\hat{\boldsymbol{F}}$ is the transpose adjugate of $\boldsymbol{F}$ so $\hat{\boldsymbol{F}}^T \cdot \hat{\boldsymbol{F}} \equiv |\boldsymbol{C}|\boldsymbol{C}^{-1} \equiv \hat{\boldsymbol{C}}$, and $e_0, \hat{e}_1, \hat{e}_2$ are three monotone convex functions (of order $1, d, d$ respectively), like the constitutive law of Ogden [33] famous for isotropic elastic solids referred to as rubbery materials (with only three scalar parameters to calibrate, see [12, Th. 4.9-2.] for a mathematical exposition and [14, 19] for physical justifications).

In [5], we explicited the symmetric-hyperbolic formulation for a particular 2D choice of type (31) without the last term $\hat{e}_2$: when $d = 2$, the components of $\boldsymbol{F}$ and $\hat{\boldsymbol{F}}$ only differ by sign so the argument $\hat{\boldsymbol{F}}^T \cdot \hat{\boldsymbol{F}}$ is superfluous then.

Now many applications need consider *non-isentropic* motions. Moreover, a full (dynamical) thermo-mechanics accounting for non-isentropic motions can rarely neglect heat conduction – with additional PDEs of "Cattaneo-type" [8] (equiv. a thermo-elastic theory "with second-sound" [37]) to preserve a hyperbolic viewpoint and ensure information travels at finite speed, then. But few works have tackled that direction [6, 27, 40]. To begin with, as in e.g. [9], let us consider mechanical waves where $e(\boldsymbol{F})$ additionally depends on entropy η (or on temperature θ) under the second thermodynamics principle but without heat conduction though, then we shall come back to heat conduction in Section 7.1.

Non-isentropic motions imply a production of entropy associated with the apparition of structural "defects" in the dynamics (i.e. microscopic phenomenas) as regards mechanics. In real solids, such defects lead to fluid-like behaviours: viscosity, plasticity. Recalling that one goal of the present work is to unify the isentropic elastic solid motions of hyperelastic materials with non-isentropic fluid motions (e.g. Newtonian like below), the next step is now to enlarge the setting of continuum mechanics for hyperelastic materials (variables and PDEs) to cover fluid-like motions. We propose to tackle the issue by following existing extensions of solid mechanics beyond pure elasticity (hyperelastic materials). We shall add structural variables to dynamically describe defects like e.g. [39,42] or [43, Chap. 13] as regards elasto-plastic solids, in the Eulerian description of elastic solids (heat-insulators to begin with). The new variable shall be useful to define visco-hyperelastic fluids and constitutive assumptions in Section 6, recalling from [46]:

Proposition 3. *The Lagrangian description (1–2–3–4–19–23) has a symmetric-hyperbolic formulation by virtue of Godunov-Mock theorem if, and only if, the Eulerian description (5–6–7–8–19–24) has a symmetric-hyperbolic formulation by virtue of Godunov-Mock theorem. Consequently, motions described by smooth solutions*

$$(u^i, F^i_\alpha) \in C^0_t\left([0,T), H^s(\mathbb{R}^d)^d \times H^s(\mathbb{R}^d)^{d\times d}\right),\ s > \frac{d}{2}$$

to the quasilinear system (1–2–3–4–19–23) in material coordinates are equivalently described in spatial coordinates by smooth solutions to (5–6–7–8–19–24).

But before let us first recall how one standardly treats fluid motions in the present continuum setting.

5. Some standard fluid motions

Some fluid motions are routinely described after introducing additional ingredients in the isentropic motions of hyperelastic materials, which can be viewed as a modelling of “defects” that necessarily occur in fluids, generally non-isentropic.

Let us recall that *perfect* isentropic fluid motions are unequivocally defined after reducing the Eulerian description (5–6–7–8–19–24) to the gas dynamics system:

$$\begin{aligned} &\partial_t \rho + \partial_i(u^i \rho) = 0 \\ &\rho\left(\partial_t u^i + u^j \partial_j u^i\right) + \partial_i\, p = \rho f^i \end{aligned} \tag{32}$$

with a spheric stress $\boldsymbol{\sigma} = -p\boldsymbol{\delta}$ of pressure $p := -\partial_{\rho^{-1}}\hat{e}_0 \equiv C_0\rho^\gamma$, $C_0 > 0$, assuming the stored energy invariant by the material symmetry group $\mathbb{U}$ therefore of type

$$\hat{e}_0(\rho) \equiv \frac{C_0}{\gamma - 1}\rho^{\gamma-1}\,. \tag{33}$$

The Eulerian system (32) is symmetric-hyperbolic when $\gamma > 1$, but it has *not* one unequivocal Lagrangian description [17]. Non-spheric “viscous” stress induced by defects are usually introduced in (32) on adding *extra-stress* $\boldsymbol{\tau}$ in Cauchy stress

$$\boldsymbol{\sigma} = -p\boldsymbol{\delta} + \boldsymbol{\tau} \tag{34}$$

such that it is frame indifferent, symmetric (to preserve angular momentum) and “dissipative” for the sake of the thermodynamics principles [13]. For instance, the second thermodynamics principle (35) is satisfied with *dissipation* $\mathcal{D} \equiv \tau^{ij}\partial_j u^i \geq 0$

$$\partial_t \eta + (u^j \partial_j)\eta = \mathcal{D}/\theta \tag{35}$$

in the case of a *Newtonian* extra-stress with two parameters $\ell, \dot{\mu} > 0$

$$\tau^{ij} = 2\dot{\mu} D(u)^{ij} + \ell\, D(u)^{kk}\,\delta_{ij} \tag{36}$$

where $D(u)^{ij} := \frac{1}{2}\left(\partial_i u^j + \partial_j u^i\right)$. The compatibility of (35) with (22) can be achieved on choosing e as the sum of (33) and another independant contribution (function of $\eta, \boldsymbol{Q}\ldots$) that handles $\mathcal{D}$ as heat, but such a compressible Newtonian model does not cope with our framework: (6–24–34–36) is not a hyperbolic system.

To model viscous fluid motions in a hyperbolic framework common with purely elastic solids, we propose to extend the hyperelastic framework of Section 3 and introduce structural defects to develop Maxwell's seminal concept of visco-elastic fluids [30]. First, we add time-dependent "structure tensors" to the Eulerian description of motions. Next, a time-evolution of those tensors can be postulated so that it implies a specific non-Newtonian fluid *constitutive law* for the Cauchy stress. For some constitutive laws, the compressible *ideal* Newtonian fluid motions can be recovered as asymptotic limits when the Cauchy stress converges to a Newtonian viscous stress, with a non-spheric contribution of the form (36). The idea was shown successful in previous contributions of ours [4–6].

In [4], introducing $\boldsymbol{A} := \boldsymbol{Y}^{-2} \in \boldsymbol{S}^d_{+,*}$ *positive definite* in the stored energy as

$$\frac{C_0}{\gamma - 1}\rho^{\gamma-1} + \frac{c_1^2}{2}\left(\boldsymbol{F}\cdot\boldsymbol{A} : \boldsymbol{F} - \log|\boldsymbol{F}\cdot\boldsymbol{A}\cdot\boldsymbol{F}^T|\right) \tag{37}$$

was indeed shown compatible with the following constitutive law

$$\lambda \overset{\blacktriangledown}{\tau} + \tau = 2\acute{\mu}\boldsymbol{D}(\boldsymbol{u}) \tag{38}$$

satisfied by the non-spheric contribution $\boldsymbol{\tau} \equiv \rho c_1^2(\boldsymbol{F}\cdot\boldsymbol{A}\cdot\boldsymbol{F}^T - \boldsymbol{I})$ to the Cauchy stress (34), on denoting the pressure $p \equiv C_0\rho^\gamma$ and the extra-stress time-rate

$$\overset{\blacktriangledown}{\boldsymbol{\tau}} := \partial_t\boldsymbol{\tau} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\tau} - \boldsymbol{\nabla}\boldsymbol{u}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\boldsymbol{\nabla}\boldsymbol{u}^T + (\operatorname{div}\boldsymbol{u})\,\boldsymbol{\tau} \tag{39}$$

which is frame indifferent i.e. "objective" so (38) is a physically admissible version of Maxwell's visco-elastic constitutive law. To that aim, it suffices to require $\acute{\mu} = \rho\lambda c_1^2$ and the following relaxation for the tensor $\boldsymbol{A}$ that makes e anisotropic

$$\lambda(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A} = \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}\,. \tag{40}$$

The constitutive law (38) is not only a physically admissible Upper-Convected version of Maxwell's 1D law [30] for compressible fluids [3]. It is also mathematically reasonable: (5–6–8–19–24) is a symmetric-hyperbolic system of quasilinear PDEs. Then, the ideal elastodynamics of elastic compressible neo-Hookean solids with shear modulus (equiv. Lamé second coefficient) c_1^2 is satisfied asymptotically by solutions when $\lambda \sim \acute{\mu} \to \infty$ [7], and (formally) unified with compressible Newtonian fluids of viscosity $\acute{\mu} = c_1^2\lambda$ by solutions in the (formal) asymptotic limit $\boldsymbol{\tau} \xrightarrow{\lambda\to 0} 2\acute{\mu}\boldsymbol{D}(\boldsymbol{u})$. Note that the second principle of thermodynamics reads (35) with $\rho\mathcal{D} \equiv \rho\frac{c_1^2}{2\lambda}(\boldsymbol{I} - \boldsymbol{c}^{-1}) : (\boldsymbol{c} - \boldsymbol{I}) > 0$ where we denoted $\boldsymbol{c} = \boldsymbol{F}\cdot\boldsymbol{A}\cdot\boldsymbol{F}^T \in \boldsymbol{S}^d_{+,*}$. The idea was extended in [6] to include heat-conduction and various *rheologies* i.e. various constitutive laws for a non-spheric contribution $\boldsymbol{\tau}$ to Cauchy stress.

In the present work, we strive to delineate the full scope of the ideas from [4]: we state new constitutive assumptions, physically admissible and mathematically reasonable, that allow to cover standard fluid and solid motions, on extending the set of variables and of PDEs as in [4]. We term those fluids visco-hyperelastic.

6. Visco-hyperelastic fluids

Let us now consider the non-isentropic motions of continuum materials with a stored energy that not only satisfies **(H2)** but also depends on symmetric positive-definite structural tensor variables $\boldsymbol{Y}_i \in \boldsymbol{S}^d_{+,*}$, $i = 1, 2$ with a view to expressing *defects* in fluid-like materials. We assume that the structural tensors $\boldsymbol{Y}_i$ satisfy

$$\lambda_i(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{Y}_i = \mathfrak{f}_i(q) \tag{41}$$

in Eulerian description, with $\lambda_i \geq 0$ and $\mathfrak{f}_i(q) \in \boldsymbol{S}^d_+$ smooth algebraic functions with symmetric-positive matrix values of the state variables $q = (\boldsymbol{u}, \boldsymbol{F}, \boldsymbol{Y}_i)$.

First, let us neglect thermal influences on mechanics. We propose the following constitutive assumptions for a physically admissible and mathematically reasonable Eulerian description of fluids that could be termed "visco-hyperelastic" (to precise the usual terminology "visco-elastic" that unifies viscous fluids and elastic solids):
(H3) There exists $\bar{e}(\boldsymbol{Y}_i, \boldsymbol{A}, \boldsymbol{B}, c)$ defined for $\boldsymbol{Y}_i, \boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{S}_+^d$, $c \in \mathbb{R}$ such that the function $(\boldsymbol{Y}_i, \boldsymbol{F}, \hat{\boldsymbol{F}}, |\boldsymbol{F}|) \to \bar{e}(\boldsymbol{Y}_i, \boldsymbol{F}^T \cdot \boldsymbol{F}, \hat{\boldsymbol{F}}^T \cdot \hat{\boldsymbol{F}}, |\boldsymbol{F}|)$ is stricly convex in $\boldsymbol{Y}_i \in \boldsymbol{S}_+^d$, $\boldsymbol{F} \in \mathbb{R}^{d\times d}$, $\hat{\boldsymbol{F}} \in \mathbb{R}^{d\times d}$, $|\boldsymbol{F}| \in \mathbb{R}$ jointly. There also exists $e_{\boldsymbol{Y}_i}(\boldsymbol{Y}_i)$ such that for any $e_\eta(\eta)$ convex the stored energy $e(\boldsymbol{F}) \equiv \bar{e}(\boldsymbol{Y}_i, \boldsymbol{F}^T \cdot \boldsymbol{F}, \hat{\boldsymbol{F}}^T \cdot \hat{\boldsymbol{F}}, |\boldsymbol{F}|) + e_{\boldsymbol{Y}_i}(\boldsymbol{Y}_i) + e_\eta(\eta)$ is compatible with all thermodynamics principles: (21) (energy balance in absence of heat transfer) and (35) (entropy increase). A non-spheric contribution to Cauchy stress $\boldsymbol{\sigma}$ models viscous stresses in fluids.

Requiring **(H3)** ensures a physically-admissible and mathematically-reasonable model, that satisfies **(H2)** when $\lambda_i \to \infty$ and the structural tensores $\boldsymbol{Y}_i$ assume constant values. Indeed, (smooth) motions are unequivocally defined on small times as solutions to (5–6–8–19–24–41), which is symmetric-hyperbolic insofar as without source terms (i.e. when $\boldsymbol{f} = \boldsymbol{0}$ in (24) and $\mathfrak{f}_i = 0$ in (41)) a conservation law holds for the functional $\bar{e}(\boldsymbol{Y}_i, \boldsymbol{F}^T \cdot \boldsymbol{F}, \hat{\boldsymbol{F}}^T \cdot \hat{\boldsymbol{F}}, |\boldsymbol{F}|) + \text{tr}(\boldsymbol{Y}_i^2)$ jointly convex in all the unknown variables of the PDE system (5–6–8–19–24–41).

In the sequel, we propose analytical expressions for $\bar{e}$ and $\mathfrak{f}_i$ that satisfy **(H3)** and cover the useful multi-dimensional motions of Maxwell-type fluids which we proposed in [4]. In addition, the analytical expressions also provide one with many multi-dimensional motions for visco-elastic fluids of "rate-type", see Prop. 4. To that aim, recalling $\boldsymbol{C} \equiv \boldsymbol{F}^T \cdot \boldsymbol{F}$, $\hat{\boldsymbol{C}} \equiv \hat{\boldsymbol{F}}^T \cdot \hat{\boldsymbol{F}}$, we construct stored energies like[4]

$$\hat{e}_0(\rho^{-1}, \boldsymbol{Y}_1, \boldsymbol{Y}_2) + \hat{e}_1(\boldsymbol{C}, \boldsymbol{Y}_1) + \hat{e}_2(\hat{\boldsymbol{C}}, \boldsymbol{Y}_2) + e_{\boldsymbol{Y}_i}(\boldsymbol{Y}_i) + e_\eta(\eta) \tag{42}$$

see Section 6.1. Next, we investigate source terms $\mathfrak{f}_i(q)$ in (41) that provide one with a second principle of thermodynamics i.e. dissipation in (35), and capture viscous fluid behaviour, see Section 6.2. Note that under **(H3)**, the entropy production $\mathcal{D}$ in (35) is a by-product of mechanics (thermal influences on mechanics were neglected); it is easily calculated (and the convexity of e_η is not necessary, in passing: monotonicity is enough, for the definition of temperature).

Last, to consider mechanics with heat transfers (i.e. thermo-mechanics, with entropy or temperature), we propose in Section 6.3 the improved assumptions:
(H4) There exists $\bar{e}(\boldsymbol{Y}_i, \boldsymbol{A}, \boldsymbol{B}, c, \eta)$ defined for $\boldsymbol{Y}_i, \boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{S}_+^d$, $c, \eta \in \mathbb{R}$ such that the function $(\boldsymbol{Y}_i, \boldsymbol{F}, \hat{\boldsymbol{F}}, |\boldsymbol{F}|, \eta) \to \bar{e}(\boldsymbol{Y}_i, \boldsymbol{F}^T \cdot \boldsymbol{F}, \hat{\boldsymbol{F}}^T \cdot \hat{\boldsymbol{F}}, |\boldsymbol{F}|, \eta)$ is stricly convex in $\boldsymbol{Y}_i \in \boldsymbol{S}_+^d$, $\boldsymbol{F} \in \mathbb{R}^{d\times d}$, $\hat{\boldsymbol{F}} \in \mathbb{R}^{d\times d}$, $|\boldsymbol{F}| \in \mathbb{R}$, $\eta \in \mathbb{R}$ (jointly), monotone increasing in η. There also exists $e_{\boldsymbol{Y}_i}(\boldsymbol{Y}_i)$ such that the stored energy $e(\boldsymbol{F}) \equiv \bar{e}(\boldsymbol{Y}_i, \boldsymbol{F}^T \cdot \boldsymbol{F}, \hat{\boldsymbol{F}}^T \cdot \hat{\boldsymbol{F}}, |\boldsymbol{F}|, \eta) + e_{\boldsymbol{Y}_i}(\boldsymbol{Y}_i)$ is compatible with all thermodynamics principles: (21) (energy balance in absence of heat transfer) and (35) (entropy increase). A non-spheric contribution to Cauchy stress $\boldsymbol{\sigma}$ models viscous stresses in fluids.

The constitutive assumptions **(H4)** are more general than **(H3)** insofar as they allow more general motions with heat transfers. They can be easily complemented further to incorporate heat conduction in our hyperbolic framework (i.e. $\boldsymbol{Q} \neq 0$ in (17) and thermal waves), as we explain in the final Section 7.1.

6.1. Stored energy functions using structural tensors

Analytical expressions for $\hat{e}_1$ and $\hat{e}_2$ in (42) that satisfy **(H3)** are suggested by [26, Corollary 2.1]:

$$\hat{e}_1 = \frac{1}{2} h_1 \left((\text{tr}\,(\boldsymbol{A}_1 \cdot \boldsymbol{C}))^{q_1} \right) \quad \text{and} \quad \hat{e}_2 = \frac{1}{2} h_2 \left(\left(\text{tr}\,(\boldsymbol{A}_2 \cdot \hat{\boldsymbol{C}})\right)^{q_2} \right) \tag{43}$$

where h_1, h_2 are monotone increasing convex functions, while $r_1 \in (0,1]$ is as small as necessary for $q_1 \geq (2-r_1)^{-1} > \frac{1}{2}$, $\boldsymbol{A}_1 = \boldsymbol{Y}_1^{-r_1}$, and $r_2 \in (0,1]$ is as small as necessary for $q_2 \geq (2-r_2)^{-1} > \frac{1}{2}$, $\boldsymbol{A}_2 = \boldsymbol{Y}_2^{-r_2}$.

[4] Note that the expression (31) does *not* coincide with other expressions that separate volumetric contributions to stress and other "distortional" contributions as in [11]. However, (31) is one standard expression, and it guarantees unequivocal *isentropic* motions (see Prop. 2). The PDE system (5–6–8–19–24) with (42) and *constant* $\boldsymbol{Y}_i \in \boldsymbol{S}_{+,*}^d$, which is obtained formally when $\lambda_i \to \infty$, has already been used for *anisotropic* solids [43].

Note that the various expressions that we have proposed so far in previous works [4, 6] are all covered [26, Corollary 2.1], which reads:

Theorem 1. *For all $r \in (0,1]$, $q \geq (2-r)^{-1}$ the following function*

$$(\boldsymbol{F},\boldsymbol{Y}) \in \mathbb{R}^{d\times d} \times \boldsymbol{S}^d_{+,*} \to \left(\operatorname{tr}\left(\boldsymbol{Y}^{-r}\cdot \boldsymbol{F}^T\cdot \boldsymbol{F}\right)\right)^q \tag{44}$$

is convex jointly in all its arguments.

Using Th. 1 as a building block to functional operations (like sums of convex functions and composition by monotone increasing convex, see e.g. [38, Chap. 3]) one could generate much more analytical expression convex in $|\boldsymbol{F}|, \boldsymbol{F}, \hat{\boldsymbol{F}}$ and the (symmetric positive definite) structural tensors $\boldsymbol{Y}_i$ than (42), although we are not aware of a general expression resulting from such operations. For instance, inspired by [20, 21], one could choose $x \to \frac{e^{b^2x}-1}{2b^2}$, $b^2>0$ for h_i, or $[0,b^2) \ni x \to -b^2\log\left(1-x/b^2\right)$ see also [6, Sec. 4.2]. (Note that solutions a priori preserve the bound $0 \geq (\operatorname{tr}(\boldsymbol{A}_i\cdot\boldsymbol{C}))^{q_i} < b^2$ on small times only, just like the orientation-preserving constraint $\rho\geq 0$ and the hyperbolicity domain $\boldsymbol{Y}_1,\boldsymbol{Y}_2 \in \boldsymbol{S}^d_{+,*}\times\boldsymbol{S}^d_{+,*}$, see e.g. [4, Corollary 2]).

Next, any volumetric term $\hat{e}_0$ in (42) that does not depend on $\boldsymbol{Y}_1,\boldsymbol{Y}_2$ and that is strictly convex in ρ^{-1} can also be chosen, independently of $\hat{e}_1$ and $\hat{e}_2$. In [4,6], we used various expressions from the literature for $\hat{e}_0$ see e.g. [6, Sec. 4.3].

Finally, given a stored energy like (42) using (43), the Cauchy stress reads

$$\boldsymbol{\sigma} = -p_0\boldsymbol{I} + \tau_1 + \tau_2 \tag{45}$$

where, on denoting for $i=1,2$

$$\mu_i(s) = q_i s^{q_i-1}h_i'(s^{q_i}) > 0\,, \tag{46}$$

the spheric contribution $p_0 := -\partial_{\rho^{-1}}\hat{e}_0$ is complemented by

$$\tau_1 := \mu_1\left(\operatorname{tr}\left(\boldsymbol{A}_1\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\right)\right)\rho\boldsymbol{F}\cdot\boldsymbol{A}_1\cdot\boldsymbol{F}^T\,, \tag{47}$$

$$\tau_2 := \mu_2\left(\operatorname{tr}\left(\boldsymbol{A}_2\cdot\hat{\boldsymbol{F}}^T\cdot\hat{\boldsymbol{F}}\right)\right)\rho\left(-\hat{\boldsymbol{F}}\cdot\boldsymbol{A}_2\cdot\hat{\boldsymbol{F}}^T + (\hat{\boldsymbol{C}}:\boldsymbol{A}_2)\boldsymbol{I}\right)\,. \tag{48}$$

Choosing one specific expression for the stored energy fully characterizes the mechanics (i.e. stress and waves speed) in the solid-motions limit $\lambda_i\to\infty$.

But when $\lambda_i>0$ is finite, the stress $\boldsymbol{\sigma}(t)$ at time $t>0$ is a priori not yet fully determined as a function of $\boldsymbol{F}(t)$ at the same time; it also depends on $\boldsymbol{Y}_i(t)$ which accounts for the past states $\boldsymbol{F}(s)$, $s\leq t$ through $\mathfrak{f}_i$ in (41).

6.2. **Structural tensors dynamics**

From the mathematical viewpoint, possible expressions for $\mathfrak{f}_i$ in (41) can be any smooth functions in the variable $q := (\boldsymbol{u},\boldsymbol{F},\boldsymbol{Y}_1,\boldsymbol{Y}_2)$. But from the physical viewpoint, the second-principle of thermodynamics should be satisfied, which means here that the source terms in the defects evolutions have to be chosen to induce dissipation $\mathcal{D}\geq 0$ in (35) with

$$\mathcal{D} \equiv -\frac{1}{\lambda_i}\partial_{\boldsymbol{Y}_i}e : \mathfrak{f}_i\,, \tag{49}$$

to produce entropy η. In particular, e and $\mathfrak{f}_i$ cannot be chosen independently !

Now, recall $\boldsymbol{A}_i := \boldsymbol{Y}_i^{-r_i}$ is uniquely defined by $\boldsymbol{Y}_i \in \boldsymbol{S}^d_{+,*}$, $r_i\in(0,1]$, so one can equivalently fix $\mathfrak{f}_i(q)$ in (41) or choose $\mathfrak{g}_i(q)\in\boldsymbol{S}^d$ in

$$(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{A}_i = \mathfrak{g}_i(q) \tag{50}$$

which yields practical rate-type differential constitutive relations for stress.

Proposition 4. *Given a stored energy like (42) using (43), the two contributions* τ_1 *and* τ_2 *in (45) satisfy*

$$\overset{\blacktriangledown}{\tau}_1 = \mu_1 \rho \boldsymbol{F} \cdot \mathfrak{g}_1 \cdot \boldsymbol{F}^T + \left(2 \frac{\tau_1 : \nabla \boldsymbol{u}}{\rho \mu_1} + \mathfrak{g}_1 : \boldsymbol{C} \right) \frac{\mu_1'}{\mu_1} \tau_1 \tag{51}$$

$$\overset{\blacktriangle}{\tau}_2 = -\mu_2 \rho \hat{\boldsymbol{F}} \cdot \mathfrak{g}_2 \cdot \hat{\boldsymbol{F}}^T + \left(2 \frac{\tau_2 : \bar{\boldsymbol{D}}}{\rho \mu_2} + \mathfrak{g}_2 : \hat{\boldsymbol{C}} \right) \left(\frac{\mu_2'}{\mu_2} \tau_2 + \mu_2 \rho \boldsymbol{I} \right) + 2 \frac{\operatorname{tr} \tau_2}{d-1} \bar{\boldsymbol{D}} \tag{52}$$

denoting $\bar{\boldsymbol{D}}(\boldsymbol{u}) := \boldsymbol{D}(\boldsymbol{u}) - (\operatorname{div} \boldsymbol{u}) \boldsymbol{I}$*, using the frame-indifferent time-rates (39) and*

$$\overset{\blacktriangle}{\tau} := \partial_t \tau + (\boldsymbol{u} \cdot \nabla) \tau + \tau \cdot \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T \cdot \tau - (\operatorname{div} \boldsymbol{u}) \tau \,. \tag{53}$$

If $\mathfrak{g}_i$ *are as in (59) using any* α_i, β_i, μ_i *then (51) and (52) respectively precise as*

$$\overset{\blacktriangledown}{\tau}_1 = \left(\left(2 \frac{\tau_1 : \nabla \boldsymbol{u}}{\rho} + \frac{\alpha_1 \operatorname{tr} \tau_1}{\rho} + d \beta_1 \right) \frac{\mu_1'}{\mu_1^2} + \alpha_1 \right) \tau_1 + (\beta_1 \rho) \boldsymbol{I} \tag{54}$$

$$\overset{\blacktriangle}{\tau}_2 = \left(\left(2 \frac{\tau_2 : \bar{\boldsymbol{D}}}{\rho} + \alpha_2 \frac{\operatorname{tr} \tau_2}{\rho(d-1)} + d \beta_2 \right) \frac{\mu_2'}{\mu_2^2} + \alpha_2 \right) \tau_2 + (\beta_2 \rho (d-1) + 2 \tau_2 : \bar{\boldsymbol{D}}) \boldsymbol{I} + 2 \frac{\operatorname{tr} \tau_2}{d-1} \bar{\boldsymbol{D}} \,. \tag{55}$$

Proof. It holds (46) on recalling h_i are assumed increasing. Formulas (47) and (48) in (45) result from lengthy but straightforward computations on recalling (27) and $\partial_{F^j_\beta} \hat{F}^i_\alpha = \sigma_{ijk} \sigma_{\alpha\beta\gamma} F^k_\gamma$. Then, recalling from (5–8)

$$\partial_t F^i_\alpha + (u^j \partial_j) F^i_\alpha = \partial_j u^i F^j_\alpha \tag{56}$$

$$\partial_t \hat{F}^i_\alpha + (u^j \partial_j) \hat{F}^i_\alpha = -\hat{F}^j_\alpha \partial_i u^j + \hat{F}^i_\alpha \partial_k u^k \tag{57}$$

(51) and (52) also follow straightforwardly. Last, assumptions (59) on $\mathfrak{g}_i$ imply

$$\boldsymbol{F} \cdot \mathfrak{g}_1 \cdot \boldsymbol{F}^T = \alpha_1 \boldsymbol{F} \cdot \boldsymbol{A}_1 \cdot \boldsymbol{F}^T + \frac{\beta_1}{\mu_1} \boldsymbol{I} = \alpha_1 \frac{\tau_1}{\mu_1 \rho} + \frac{\beta_1}{\mu_1} \boldsymbol{I}$$

$$\hat{\boldsymbol{F}} \cdot \mathfrak{g}_2 \cdot \hat{\boldsymbol{F}}^T = \alpha_2 \hat{\boldsymbol{F}} \cdot \boldsymbol{A}_2 \cdot \hat{\boldsymbol{F}}^T + \frac{\beta_2}{\mu_2} \boldsymbol{I} = -\alpha_2 \frac{\tau_2}{\mu_2 \rho} + \left(\frac{\beta_2}{\mu_2} + \alpha_2 \operatorname{tr}(\hat{\boldsymbol{F}} \cdot \boldsymbol{A}_2 \cdot \hat{\boldsymbol{F}}^T) \right) \boldsymbol{I}$$

hence $\operatorname{tr} \hat{\boldsymbol{F}} \cdot \mathfrak{g}_2 \cdot \hat{\boldsymbol{F}}^T = \alpha_2 \frac{\operatorname{tr} \tau_2}{\mu_2 \rho (d-1)} + d \frac{\beta_2}{\mu_2}$ on noting (48) implies $\operatorname{tr}(\hat{\boldsymbol{F}} \cdot \boldsymbol{A}_2 \cdot \hat{\boldsymbol{F}}^T) = \frac{\operatorname{tr} \tau_2}{\mu_2 \rho (d-1)}$, and (54) (55) after lengthy but straightforward computations. □

Some particular choices $\mathfrak{g}_i$ can guarantee the second thermodynamics principle.

Proposition 5. *Assume* **(H3)** *with a stored energy e of type (42), with* $\hat{e}_i$ *as in (43) i.e. independent of* $\boldsymbol{Y}_i \equiv \boldsymbol{A}_i^{\frac{1}{r_i}}$ *as well as* $\hat{e}_0$*. If one chooses, in e of type (42),*

$$e_{\boldsymbol{Y}_i} = \frac{1}{2} \frac{\beta_i}{\alpha_i} \log |\boldsymbol{A}_i| \equiv -r_i \frac{1}{2} \frac{\beta_i}{\alpha_i} \log |\boldsymbol{Y}_i| \tag{58}$$

with $\beta_i > 0$ *and constant* $\frac{\beta_i}{\alpha_i} < 0$*, then the source terms*

$$\mathfrak{g}_1 = \alpha_1 \boldsymbol{A}_1 + \frac{\beta_1}{\mu_1} \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T} \,, \quad \mathfrak{g}_2 = \alpha_2 \boldsymbol{A}_2 + \frac{\beta_2}{\mu_2} \hat{\boldsymbol{F}}^{-1} \cdot \hat{\boldsymbol{F}}^{-T} \tag{59}$$

in (50) are dissipative i.e. they induce dissipation following (49).

Proof. The dissipation condition (49) reads $\partial_{\boldsymbol{Y}_i} e : \mathfrak{g}_i \leq 0$, i.e. with our hypotheses

$$\left(\frac{\beta_1}{\alpha_1}\boldsymbol{A}_1^{-1} + \mu_1 \boldsymbol{C}\right) : \left(\alpha_1 \boldsymbol{A}_1 + \beta_1 \mu_1^{-1} \boldsymbol{C}^{-1}\right) + \left(\frac{\beta_2}{\alpha_2}\boldsymbol{A}_2^{-1} + \mu_2 \hat{\boldsymbol{C}}\right) : \left(\alpha_2 \boldsymbol{A}_2 + \beta_2 \mu_2^{-1} \hat{\boldsymbol{C}}^{-1}\right) = \beta_1 \left(\mu_1 \frac{\alpha_1}{\beta_1}(\boldsymbol{A}_1 : \boldsymbol{C}) - 2d + \frac{1}{\mu_1}\frac{\beta_1}{\alpha_1}(\boldsymbol{C}^{-1} : \boldsymbol{A}_1^{-1})\right) + \beta_2 \left(\mu_2 \frac{\alpha_2}{\beta_2}(\boldsymbol{A}_2 : \hat{\boldsymbol{C}}) - 2d + \frac{1}{\mu_2}\frac{\beta_2}{\alpha_2}(\hat{\boldsymbol{C}}^{-1} : \boldsymbol{A}_2^{-1})\right) \leq 0 \quad (60)$$

which is satisfied on recalling $x - 2 + \frac{1}{x} \geq 0$ holds for all $x > 0$ as well as in trace sense for all symmetric positive definite matrices like $\boldsymbol{F} \cdot \boldsymbol{A}_1 \cdot \boldsymbol{F}^T$ and $\hat{\boldsymbol{F}} \cdot \boldsymbol{A}_2 \cdot \hat{\boldsymbol{F}}^T$ in (60) – recall $\boldsymbol{A}_1 : \boldsymbol{C} = \operatorname{tr}(\boldsymbol{F} \cdot \boldsymbol{A}_1 \cdot \boldsymbol{F}^T)$, $\boldsymbol{A}_2 : \hat{\boldsymbol{C}} = \operatorname{tr}(\hat{\boldsymbol{F}} \cdot \boldsymbol{A}_2 \cdot \hat{\boldsymbol{F}}^T)$ etc. ☐

The symmetric source terms $\mathfrak{g}_i$ in (59) have no sign though. To ensure $\boldsymbol{A}_i \in \boldsymbol{S}^d_{+,*}$, it is thus convenient to choose $e_{\boldsymbol{Y}_i}$ as (58) and $\mathfrak{g}_i$ as in (59) with *constants* $\beta_i > 0$, $\alpha_i < 0$, $\mu_i > 0$ (i.e. $q_i = 1$ and constants $h_i' > 0$) so the solution to (50) reads

$$\boldsymbol{A}_1(t) = \boldsymbol{A}_1(0)e^{\alpha_1 t} + \int_0^t \left(\frac{\beta_1}{\mu_1}\boldsymbol{C}^{-1}(s)e^{\alpha_1(t-s)}\right) ds \,, \quad \boldsymbol{A}_2(t) = \boldsymbol{A}_2(0)e^{\alpha_2 t} + \int_0^t \left(\frac{\beta_2}{\mu_2}\hat{\boldsymbol{C}}^{-1}(s)e^{\alpha_2(t-s)}\right) ds \,, \quad \forall s < t \,, \quad (61)$$

in *Lagrangian coordinates*, which guarantees $\boldsymbol{A}_i \in \boldsymbol{S}^d_{+,*}$ as long as the Lagrangian description holds. Another benefit of the latter *integral* formula for the stress as a function of $\boldsymbol{F}$, well known in rheology as K-BKZ formula (assuming that $\boldsymbol{A}_i$ remain bounded as $t \to -\infty$, see [4] for more explanations), is to show how (visco-hyperelastic) fluid motions "have memory", with $\boldsymbol{\sigma}(t)$ a function of $\{\boldsymbol{F}(s), s < t\}$.

Choosing $e_{\boldsymbol{Y}_i}$ as (58) and $\mathfrak{g}_i$ as in (59) with constants $\mu_i > 0$, $\beta_i = \frac{\mu_i}{\lambda_i} > 0$, $\alpha_i = -\frac{1}{\lambda_i} < 0$ precisely yields the motions we proposed for the first time in [4] for Maxwell fluids, see Proposition 4. Then, when $\mu_i' = 0$ i.e. $q_i = 1$ and h_i' is constant, the differential "rate-type" constitutive relations (54) and (55) simplify to

$$\overset{\blacktriangledown}{\boldsymbol{\tau}}_1 = \alpha_1 \boldsymbol{\tau}_1 + \beta_1 \rho \boldsymbol{I} \quad (62)$$

$$\overset{\blacksquare}{\boldsymbol{\tau}}_2 = \alpha_2 \boldsymbol{\tau}_2 + \beta_2 \rho (d-1) \boldsymbol{I} \quad (63)$$

where we have introduced a new objective derivative using $\bar{\boldsymbol{D}}(\boldsymbol{u}) := \boldsymbol{D}(\boldsymbol{u}) - (\operatorname{div} \boldsymbol{u})\boldsymbol{I}$

$$\overset{\blacksquare}{\boldsymbol{\tau}}_2 := \overset{\blacktriangle}{\boldsymbol{\tau}}_2 - 2(\boldsymbol{\tau}_2 : \bar{\boldsymbol{D}})\boldsymbol{I} - 2\frac{\operatorname{tr} \boldsymbol{\tau}_2}{d-1}\bar{\boldsymbol{D}} \,.$$

With $\frac{\beta_i}{\alpha_i}$ constant, (54)–(55) rewrite using $\boldsymbol{T}_1 := \boldsymbol{\tau}_1 + \frac{\beta_1}{\alpha_1}\rho \boldsymbol{I}$, $\boldsymbol{T}_2 := \boldsymbol{\tau}_2 - \frac{\beta_2}{\alpha_2}\rho(d-1)\boldsymbol{I}$

$$\overset{\blacktriangledown}{\boldsymbol{T}}_1 = \alpha_1 \boldsymbol{T}_1 - 2\frac{\beta_1}{\alpha_1}\rho \boldsymbol{D}(\boldsymbol{u}) \quad (64)$$

$$\overset{\blacksquare}{\boldsymbol{T}}_2 = \alpha_2 \boldsymbol{T}_2 - \frac{\beta_2}{\alpha_2}\rho(d-1)\left(\frac{2}{d-1}\boldsymbol{D}(\boldsymbol{u}) + (3 - \frac{2}{d-1})(\operatorname{div} \boldsymbol{u})\boldsymbol{I}\right) . \quad (65)$$

So one can expect our framework to capture Newtonian fluid behaviours through

$$\boldsymbol{T}_1 \approx 2\nu 1\rho \boldsymbol{D}(\boldsymbol{u}) \text{ as } \alpha_1 \to \infty\,,\ \nu_1 := -\frac{\beta_1}{\alpha_1^2} > 0 \text{ constant}$$

$$\boldsymbol{T}_2 \approx \nu_2\rho\left(\tfrac{2}{d-1}\boldsymbol{D}(\boldsymbol{u}) + (3-\tfrac{2}{d-1})(\operatorname{div}\boldsymbol{u})\boldsymbol{I}\right) \text{ as } \alpha_2 \to \infty\,,\ \nu_2 := -\frac{\beta_2}{\alpha_2^2} > 0 \text{ constant}$$

The constitutive laws (62) (for $\boldsymbol{\tau}_1$) and (64) (for $\boldsymbol{T}_1$) exactly coincide with the reformulation of Upper-Convected Maxwell (UCM) fluids that we proposed in [4]. The law (63) (for $\boldsymbol{\tau}_2$) and (65) (for $\boldsymbol{T}_2$) is exactly the extension to Lower-Convected Maxwell (LCM) fluids that we proposed in [6]. Note that (64), (65) hold when α_i, β_i are functions provided $\frac{\beta_i}{\alpha_i}$ remain constant, which bears the possibility of physically-motivated thermo-mechanical extensions, as already mentioned in [6] (see also Section 7.1 below to introduce heat transfer).

Many other (new) models can also easily be formulated from our constitutive assumptions, Prop. 5 and (59) with *non*-constant $h_i' \equiv \mu_i > 0$ in (43) (though they are possibly stable on very short times only, to ensure $\boldsymbol{A}_i \in \boldsymbol{S}^d_{+,*}$). For instance, we already mentionned the choice $h_i(s) = -b_i \log\left(1 - \frac{s}{b_i}\right)$, $b_i > 0$, $h_i'(s) = \frac{1}{1-\frac{s}{b_i}}$ for solids [21], which makes sense as long as the "conformation tensors"

$$\boldsymbol{s}_1 := \boldsymbol{F}\boldsymbol{A}_1\boldsymbol{F}^T \quad \boldsymbol{s}_2 := \hat{\boldsymbol{F}}\boldsymbol{A}_2\hat{\boldsymbol{F}}^T$$

remain symmetric positive definite *with eigenvalues in* $(0, b_i) \ni s$, recall (47–48), thus which relates to closures of so-called FENE (Finitely Extensible Nonlinear Elastic) fluids [25] already suggested in [6]. In Prop. 6, we precise the differential form of the latter model, which is a new closure of FENE-type to our knowledge.

Proposition 6. *Assume* $q_i = 1$*,* $h_i(s) = -b_i \log\left(1 - \frac{s}{b_i}\right)$*,* $b_i > 0$ *so it holds*

$$\mu_i(s) = \frac{1}{1 - \frac{s}{b_i}}\,. \tag{66}$$

Then, provided $\beta_i > 0$*, one can choose any constants* $\frac{\alpha_i}{\beta_i} < 0$ *to fulfill* **(H3)** *such that the true stress components in* (45) *read* $\boldsymbol{\tau}_1 \equiv \mu_1\rho\boldsymbol{s}_1$*,* $\boldsymbol{\tau}_2 \equiv \mu_2\rho(\boldsymbol{s}_2 - \boldsymbol{I}\operatorname{tr}\boldsymbol{s}_2)$ *where, recalling (47–48),* μ_i *(*$i = 1, 2$*) depend on the tensors* $\boldsymbol{s}_i$ *that satisfy*

$$\overset{\triangledown}{\boldsymbol{s}}_1 = \frac{\beta_1}{\mu_1}\left(\boldsymbol{I} + \frac{\alpha_1}{\beta_1}\mu_1\boldsymbol{s}_1\right) \tag{67}$$

$$\overset{\vartriangle}{\boldsymbol{s}}_2 = \frac{\beta_2}{\mu_2}\left(\boldsymbol{I} + \frac{\alpha_2}{\beta_2}\mu_2\boldsymbol{s}_2\right) \tag{68}$$

while we have used the standard Upper-Convected time-rate i.e.

$$\overset{\triangledown}{\boldsymbol{\tau}} := \partial_t\boldsymbol{\tau} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\tau} - \boldsymbol{\nabla}\boldsymbol{u}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\boldsymbol{\nabla}\boldsymbol{u}^T \tag{69}$$

in (67)*, and the following Lower-Convected time-rate in* (68)*:*

$$\overset{\vartriangle}{\boldsymbol{\tau}} := \partial_t\boldsymbol{\tau} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\tau} + \boldsymbol{\tau}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^T\cdot\boldsymbol{\tau} - 2(\operatorname{div}\boldsymbol{u})\boldsymbol{\tau}\,. \tag{70}$$

Proof. Proceed similarly to the proof of Prop. 4 using the known dynamics of $\boldsymbol{F}$, $\boldsymbol{A}_i$, $\hat{\boldsymbol{F}}$ in $\boldsymbol{s}_i$. □

The dynamics (67) of $\boldsymbol{s}_1$ coincides with the so-called FENE-P equation [36] when $\lambda_i = \frac{\beta_i}{\mu_i}$, $i = 1$. However, the initial FENE-P model was proposed for incompressible fluids. The dynamics (68) of $\boldsymbol{s}_2$ is a new FENE-type model, to our knowledge.

6.3. **Thermal influences on mechanics**

To explicitly use **(H4)** with some functional expression for $\bar{e}$, we follow an approach that has been used extensively to incorporate thermal influences into stored energies of hyperelastic materials, e.g. for the thermoelasticity of rubberlike materials [9,10,14,34].

Assuming e convex in η, we start with an "empirical" expression (i.e. compatible with observations) for the Legendre transform (i.e. the convex conjugate)

$$e^\star(\boldsymbol{F},\boldsymbol{Y}_i,\theta) := \sup_{\eta\geq 0}\left(e(\boldsymbol{F},\boldsymbol{Y}_i,\eta)-\theta\eta\right)$$

which is usually termed the Helmholtz free energy and denoted $\psi(\boldsymbol{F},\boldsymbol{Y}_i,\theta)$:

$$\hat{\psi}_0(\rho^{-1},\theta)+K_1(\theta)\hat{e}_1(\boldsymbol{C},\boldsymbol{Y}_1)+K_2(\theta)\hat{e}_2(\hat{\boldsymbol{C}},\boldsymbol{Y}_2)+k_i(\theta)e_{\boldsymbol{Y}_i}(\boldsymbol{Y}_i)\,. \tag{71}$$

Assuming $\hat{\psi}_0$ convex in $(\rho^{-1},\theta)\in\mathbb{R}^+\times\mathbb{R}^+$ and denoting its Legendre transform $\hat{e}_0(\rho^{-1},\eta) := \sup_{\theta\geq 0}\left(\hat{\psi}_0(\rho^{-1},\theta)-\theta\eta\right)$, one obtains an expression for a stored energy $e(\boldsymbol{F},\boldsymbol{Y}_i,\eta)$ Legendre transform of (71) provided $K_i(\theta)$, $k_i(\theta)$ are *affine* in θ:

$$\begin{aligned}\hat{e}_0\left(\rho^{-1},\eta+(\partial_\theta K_1)\hat{e}_1(\boldsymbol{C},\boldsymbol{Y}_1)+(\partial_\theta K_2)\hat{e}_2(\hat{\boldsymbol{C}},\boldsymbol{Y}_2)+(\partial_\theta k_i)e_{\boldsymbol{Y}_i}(\boldsymbol{Y}_i)\right)\\ +(K_1-\theta\partial_\theta K_1)\hat{e}_1(\boldsymbol{C},\boldsymbol{Y}_1)+(K_2-\theta\partial_\theta K_2)\hat{e}_2(\hat{\boldsymbol{C}},\boldsymbol{Y}_2)+(k_i-\theta\partial_\theta k_i)e_{\boldsymbol{Y}_i}(\boldsymbol{Y}_i)\end{aligned} \tag{72}$$

where $\partial_\theta K_i$, $\partial_\theta k_i$, $K_i-\theta\partial_\theta K_i$, $k_i-\theta\partial_\theta k_i\in\mathbb{R}^+$ are constant.

Proposition 7. *The stored energy (72) using matrix-monotone and convex functions $\hat{e}_1$, $\hat{e}_2$ like in (43), $\hat{e}_0(\rho^{-1},\eta)$ strictly convex in $(\rho^{-1},\eta)\in\mathbb{R}^+\times\mathbb{R}^+$, monotone increasing in η, and $\partial_\theta K_i$, $\partial_\theta k_i$, $K_i-\theta\partial_\theta K_i$, $k_i-\theta\partial_\theta k_i\in\mathbb{R}^+$ fulfills* **(H4)**.

Then, the Cauchy stress reads $\boldsymbol{\sigma}=-p_0\boldsymbol{I}+\boldsymbol{\tau}_1+\boldsymbol{\tau}_2$ as in (45) with the spheric contribution $p_0 := -\partial_{\rho^{-1}}\hat{e}_0$ as in Prop. 4, and extra-stresses $\boldsymbol{\tau}_i=K_i(\theta)\rho(\partial_{\boldsymbol{F}}\hat{e}_i\cdot\boldsymbol{F})$ $(i=1,2)$ where $\rho(\partial_{\boldsymbol{F}}\hat{e}_i\cdot\boldsymbol{F})$ reads as in (47)–(48).

Proof. Proving **(H4)** is immediate on noting $\eta+(\partial_\theta K_1)\hat{e}_1(\boldsymbol{C},\boldsymbol{Y}_1)+(\partial_\theta K_2)\hat{e}_2(\hat{\boldsymbol{C}},\boldsymbol{Y}_2)+(\partial_\theta k_i)e_{\boldsymbol{Y}_i}(\boldsymbol{Y}_i)$ is convex in $(\boldsymbol{Y}_i,\boldsymbol{C},\hat{\boldsymbol{C}},\eta)$. Formulas follow by computations. □

An expression (72) using $\hat{e}_1$, $\hat{e}_2$ as in (43) was already proposed in our former extension [6] of [4]. Various expressions for $\hat{e}_0(\rho^{-1},\eta)$ strictly convex in $(\rho^{-1},\theta)\in\mathbb{R}^+\times\mathbb{R}^+$, monotone increasing in η, were also proposed in [6], inspired by typical examples from the fluid literature. Note that such expressions for the Helmholtz free energy can be justified by an underlying statistical physics theory as in [18]. In any case, Prop. 7 captures only an *affine* dependence on temperature of the non-spheric contribution to Cauchy stress. So, although the total stress satisfies a non-trivial PDE on account of the time-dependency of

$$\theta=\partial_\eta\hat{e}_0\left(\rho^{-1},\eta+(\partial_\theta K_1)\hat{e}_1(\boldsymbol{C},\boldsymbol{Y}_1)+(\partial_\theta K_2)\hat{e}_2(\hat{\boldsymbol{C}},\boldsymbol{Y}_2)+(\partial_\theta k_i)e_{\boldsymbol{Y}_i}(\boldsymbol{Y}_i)\right)$$

the conformation tensors in $\rho(\partial_{\boldsymbol{F}}\hat{e}_i\cdot\boldsymbol{F})$ given by (47)–(48) are unchanged and follow the same differential rate-type constitutive relations as in the case where mechanics is temperature-independent.

7. Conclusion and Perpsectives

We proposed new constitutive assumptions to unequivocally define physically-sensible non-isentropic motions through symmetric-hyperbolic PDEs, covering both solids and viscous fluids. The new constitutive assumptions **(H3)**, in absence of thermal influences, generalize our symmetric-hyperbolic formulation of flows of Maxwell type [4] and allow one to propose new visco-elastic motions. The new constitutive assumptions **(H4)**, which take into account thermal influences on Cauchy stress (at most affine as regards non-spheric contributions), generalize our extension [6] of [4] and also allow one to propose new visco-elastic motions. For instance, a new FENE-type model has been easily constructed in Prop. 6 simply by working around standard choices through our constitutive assumptions. Questions remain.

On the mathematical side, one may want to rigorously investigate the structural stability of the model in the viscous fluid limit, as in the solid limit case [7].

On the physical side, one may want to help identify specific choices within constitutive assumptions from observations like chemical structure of the materials, and to incorporate more phenomenas like heat conduction.

We already proposed how to incorporate heat conduction in [4], see [6]. One can similarly incorporate heat conduction into the general class of fluids defined herein after [4]. The constitutive assumptions proposed above simply need complementing as in Prop. 8 below (for rigid heat-conductors) to incorporate heat conduction. But the effectivity of the diffusive regime expected as a consequence of those additional assumptions is then another remaining question…to be elucidated in future works, like the choice of specific constitutive relations from observations.

7.1. **Heat transfer with conduction**

Within real materials, heat is not only transfered by convection but also by conduction (and radiation etc.). Capturing accurately the full heat transfer dynamics can be key to describe the motions of material whose mechanical properties depend on the heat (i.e. on the temperature) [28]. Now, it is possible to introduce heat conduction in the hyperbolic modelling framework proposed above to unify fluid and solid motions. For the sake of clarity, let us simply explain how to model heat conduction in a hyperbolic framework *independently of the mechanics*, i.e. for rigid heast conductors with a stored energy that only depends on entropy and additional variables $\boldsymbol{p}$ measuring temperature variations. Precisely, when (23) holds with $\partial_{\boldsymbol{F}} e \equiv 0$ and $\boldsymbol{S} = 0$, (17) reduces to

$$\hat\rho \partial_t e + \partial_\alpha Q^\alpha = \hat\rho r \,. \tag{73}$$

For compatibility of (73) with the second principle in case of heat conduction

$$\partial_t \eta + \partial_\alpha q^\alpha = (r + \mathcal{D})/\theta \tag{74}$$

one can assume e.g. $e(\eta, \boldsymbol{p}) = e_s(\eta) + e_p(\boldsymbol{p})$, $Q^\alpha = \hat\rho\theta q^\alpha$, and

$$(\partial_{p^\alpha} e)\partial_t p^\alpha + q^\alpha \partial_\alpha \theta = \mathcal{D} \tag{75}$$

with $\mathcal{D} \geq 0$, which remains to be chosen for (formal) compatibility with experimental observations. Precisely, when $r = 0$, (74) implies for $\theta := \partial_\eta e$

$$\rho C_1(\partial_t + u^i \partial_i)\theta + \partial_i \left(\theta\rho F^i_\alpha q^\alpha\right) = \rho \left(\mathcal{D} + F^i_\alpha q^\alpha \partial_i \theta\right) \tag{76}$$

on denoting $C_1(\theta) := \partial_\theta e_s^\star$, which one can expect *asymptotically* identical to

$$\rho C_1(\partial_t + u^i \partial_i)\theta - \partial_i \left(\kappa_{ij}\partial_j \theta\right) = 0 \tag{77}$$

(i.e. heat diffusion) for some matrix $\boldsymbol{\kappa} \in \boldsymbol{S}^d_{+,*}$ when

$$\theta\rho F^i_\alpha q^\alpha \to -\kappa_{ij}\partial_j\theta \text{ equiv. } \theta\rho q^\alpha \to -\hat\kappa_{\alpha\beta}\partial_\beta\theta, \quad \hat{\boldsymbol{\kappa}} := \boldsymbol{F}^{-1}\cdot\boldsymbol{\kappa}\cdot\boldsymbol{F}^{-T}$$

hold simultaneously as (the so-called Fourier's law)

$$F^i_\alpha q^\alpha \partial_i \theta + \mathcal{D} \equiv q^\alpha \partial_\alpha \theta + \mathcal{D} \to 0 \text{ i.e. } \mathcal{D} \to \theta\rho(q^\alpha[\hat\kappa^{-1}]_{\alpha\beta} q^\beta) > 0.$$

Note that such a limit regime implies $\boldsymbol{p}$ stationary i.e. $(\partial_{p^\alpha} e)\partial_t p^\alpha \to 0$ by (75). Following previous propositions in the literature see e.g. [6, 35], we therefore propose to assume $e_p(\boldsymbol{p}) = \tau|\boldsymbol{p}|^2/2$ and a relaxation process for the state variable $\boldsymbol{p} = p^\alpha \boldsymbol{e}_\alpha$

$$\tau \partial_t p^\alpha + \partial_\alpha \zeta(\theta) = -\rho\theta |\zeta'(\theta)|^2 [\hat{\kappa}^{-1}]_{\alpha\beta} p^\beta \tag{78}$$

so that (75) is a consequence when $\boldsymbol{q} := \zeta'(\theta)\boldsymbol{p}$, $\zeta'(\theta) \lessgtr 0$, $\mathcal{D} := \theta\rho(q^\alpha [\hat{\kappa}^{-1}]_{\alpha\beta} q^\beta) > 0$. Compatibility with experimental observations (heat diffusion with Fourier's law) can then be formally expected when $\tau \to 0$ and (78) yields $\rho \boldsymbol{F} \boldsymbol{p} \to -(\boldsymbol{\kappa}\nabla\theta)/\theta\zeta'(\theta)$.

A conservation law like (78) with a view to defining the heat and entropy fluxes $\boldsymbol{Q}$, $\boldsymbol{q}$ in (73)–(74) seems to have first been postulated by Cattaneo [8]. It still often bears his name despite many various "hyperbolic" formulations proposed since to capture the "second-sound" phenomenon observed in experiments [6, 35, 37].

Proposition 8. *Given $e(\eta, \boldsymbol{p}) = e_s(\eta) + \frac{\tau}{2}|\boldsymbol{p}|^2/2$, $\tau > 0$, a strictly convex function $e_s \in C^2$, a C^1 matrix-valued function $\hat{\boldsymbol{\kappa}}(\eta, p) \in \boldsymbol{S}^d_{+,*}$ and a stricly monotone function $\zeta \in C^1(\mathbb{R}^+)$, the quasilinear system (74)–(78) for $(\eta, \boldsymbol{p})$, with $\theta = \partial_\eta e$, $\boldsymbol{q} = \zeta'(\theta)\boldsymbol{p}$, $\mathcal{D} = \theta\rho(q^\alpha [\hat{\kappa}^{-1}]_{\alpha\beta} q^\beta)$ and source terms ρ, r, is symmetric-hyperbolic.*

Proof. This is a straightforward consequence of Godunov-Mock theorem insofar as (74)–(78) has been constructed in order to satisfy the additional conservation law (73) where $\boldsymbol{Q} = \hat{\rho}\theta\boldsymbol{q}$ and e is strictly (jointly) convex in $(\eta, \boldsymbol{p})$. □

To complement the constitutive assumptions of the previous sections and incorporate heat conduction, note that one can simply add $+\tau|\boldsymbol{p}|^2/2$ in the stored energies and postulate (78) with some scalar function $\zeta'(\theta) \lessgtr 0$.

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