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BAER-NUNZIATO TYPE MODELS FOR ISOTHERMAL AND ISENTROPIC FLOWS

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Abstract. Multi-component Baer-Nunziato-type models for isothermal and isentropic flows are investigated where each component is characterized by an equation of state for the pressure assuming constant temperature and entropy, respectively. These models are given by balance equations for volume fractions, density and momentum for each component accounting for the relaxation to equilibrium by means of relaxation terms. Mathematical properties of the models are derived for the larger class of barotropic flows such as hyperbolicity and symmetrization. The fields are characterized and corresponding Riemann invariants are determined. Appropriate entropy-entropy flux pairs are derived taking into account the phasic energy equations including the heat flux. Physically meaningful constraints on the interfacial velocity and the interfacial pressures are presented that ensure the entropy inequality to hold.

Introduction

Compressible multi-component flows where the fluid is a mixture of several components all of which may be present in different aggregate states have a wide range of applications, for instance a mixture of reacting gases or a mixture of a liquid and a gas. They have been successfully modeled by Baer-Nunziato-type models. Originally, these models have been introduced by Baer and Nunziato [1] proposing a two-fluid model for detonation waves in granular explosives. This model falls into the class of ensemble averaged models introduced by Drew and Passman [6, 7] to provide some gross features of the flow. In contrast to volume averaging, time averaging or statistical averaging, the ensemble averaging allows the interpretation of flow phenomena in terms of repeatability of multi-component flows.

Baer-Nunziato-type models are full non-equilibrium models where each component has its own pressure, velocity and temperature and is governed by its own set of fluid equations. In contrast to other two-fluid models separating the fluids, here the fluids are assumed to coexist in each location due to an averaging procedure. The presence of more than one component in the mixture is modeled by the concept of volume fractions α_k describing the ratio of volume of component k to the total volume in a ball when the volume of the ball tends to zero, see [7].

To model the relaxation process between the components relaxation terms have been introduced. For instance, Saurel and Abgrall [51] included relaxation terms for the pressure and the velocities of the components. Further relaxation procedures can be used to drive the temperatures and the Gibbs free energies into equilibrium, see

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Abgrall et al. [43, 53] or Zein et al. [57]. By instantaneous relaxation equilibrium values for the pressure, the velocity, the temperature and the chemical potentials can be found.

Since mechanical relaxation, thermal relaxation and relaxation of the chemical potentials proceed on different time scales, cf. [40, 52, 53], reduced models have been derived assuming zero relaxation times for some of the non-equilibrium quantities relaxing much faster than the remaining quantities, see [21, 31]. Thus, the stiffness inherent in the non-equilibrium model is avoided that allows for a much faster numerical simulation of the reduced model. Usually reduced models suffer from some short-comings. For instance, conservation of energy might be violated, the system loses its hyperbolicity or the reduced model is not the correct asymptotic limit of the nonlinear model. For more details we refer the interested reader to [56] and the references cited therein.

In [9, 34] a hierarchy of two-fluid models derived from the Baer-Nunziato model [1] is investigated where mechanical, thermal and chemical relaxation is assumed to proceed in different order. This hierarchy splits into two branches of models distinguished by the assumption of dynamical velocity equilibrium and local velocity equilibrium, where either both fluids have the *same* velocity or the fluids have *different* velocities that coincide for a particular state.

Since Baer-Nunziato-type models are averaged models, the resulting balance laws are underdetermined. To compensate for the loss of information *closure conditions* have to be imposed additionally. There exists a rich literature on Baer-Nunziato-type models, see for instance [5, 10, 20, 24, 38, 49] investigating their properties, in particular, *hyperbolicity* and *thermodynamical consistency*. These properties are essential from an analytical, numerical and physical point of view. For instance, hyperbolicity is needed in the analysis of the Riemann problem and the construction of (approximate) Riemann solvers whereas a non-negative entropy production ensures consistency with the 2nd law of thermodynamics and also might ensure well-posedness, although not yet mathematically rigorously verified. The analysis helps to identify physically admissible closure conditions for the interfacial pressure and the interfacial velocity that occur as model parameters and cannot be closed due to the averaging procedure. However, a unique closure does not exist. Typically, the interfacial velocity is modeled as a convex combination of the phasic velocities to ensure Galilean invariance. Then the interfacial pressures can be uniquely determined assuming that the induced entropy production term due to interfacial velocity and interfacial pressures vanishes, cf. [24, 38].

In recent years, Baer-Nunziato-type models have been applied to different types of fluids. Originally, the components of the fluids are assumed to be immiscible. In [27,30] hybrid mixtures consisting of both immiscible and miscible components are investigated. A mixture of weakly compressible fluids is considered in [41]. Our main interest in the present work is on isothermal and isentropic fluids characterized by a pressure law depending on density and constant temperature or entropy, respectively. These fall in the larger class of barotropic fluids, where the pressure law depends only on density. Note that in the literature sometimes barotropic fluids are identified with isentropic fluids. For a one-dimensional model Hèrard [25] investigates some properties of a K-component Baer-Nunziato-type model for barotropic fluids. Choosing the interfacial velocity as a convex combination of the phasic velocities and assuming that the entropy production due to the interfacial pressures and the interfacial velocity vanishes there uniquely exist interfacial pressures for $K = 2, 3$ and 4. In case of barotropic fluids a mathematical entropy inequality for multi-phase fluids is verified. For a special class of pressure-velocity relaxation the induced entropy production is consistent with the entropy inequality. This also holds when taking into account mass transfer. Also for a one-dimensional model Saleh and Seguin [50] have proven weak hyperbolicity by determining eigenvalues and eigenvectors, the existence of a convex mathematical entropy in case of barotropic fluids and the existence of a symmetric form. The analysis is confined to a particular choice of the interfacial velocity and interfacial pressures. In contrast to [25, 50] a multi-dimensional three-component model is considered in [2]. A mathematical entropy inequality is proven. Moreover, the corresponding quasi-1D model is verified to be weakly hyperbolic and Riemann invariants corresponding to the linearly degenerated field associated with the interfacial velocity are given. Again, the results are subject to a specific choice for the interfacial velocity and interfacial pressures. In a recent work a conservative compressible two phase mixture model has been discussed analytically in [55] and numerically in [42]. This model is the isothermal/isentropic version of a model studied in [46, 47] and belongs to the class of Symmetric Hyperbolic

and Thermodynamically Compatible (SHTC) systems. An overview on the theory of SHTC systems can be found in [13–15, 39, 44, 45] and we refer to the aforementioned literature for further references. In [55] the instantaneous relaxation to the Kapila model limit is studied. Further it is shown in [55] that for smooth solutions the conservative isothermal/isentropic two fluid SHTC model is equivalent to the isothermal/isentropic Baer-Nunziato system for a particular choice of interface quantities and numerical studies are provided. Therefore these results necessitate a further thorough study of the isothermal/isentropic Baer-Nunziato system to provide a sound basis for further comparative studies, also including isothermal sharp interface models studied in [18,23]. For further literature on symmetric hyperbolic systems for mixtures we also refer to [48].

The main objective of the current work is (i) to extend results in the literature to multi-component and multi-dimensional flows with general interfacial velocity and interfacial pressures and (ii) to investigate the special case of isothermal fluids where, in particular, we derive another entropy-entropy flux pair. In detail, the issues we are addressing in our work are as follows: (i) eigenvalues and eigenvectors for the quasi-1D model are determined, hyperbolicity away from resonance states is verified, the corresponding characteristic fields are investigated and a complete set of linearly independent Riemann invariants is given for all fields; note that the analysis is not restricted to isothermal and isentropic fluids but applies to general barotropic fluids; (ii)mathematical entropy inequalities according to Lax' concept of entropy-entropy flux pair are verified based on either thermodynamical energy or mathematical energy; (iii) unique interfacial pressures are determined ensuring that the entropy production induced by the interfacial states is thermodynamically consistent; the interfacial pressures depend on the interfacial velocity chosen as a convex combination of the phasic velocities; (iv) the entropy production due to mechanical relaxation and relaxation of chemical potentials ($K = 2$ (liquid-vapor), $K = 3$ (liquid-vapor-gas), isothermal fluids) is verified to be thermodynamically consistent; (v) the quasi-1D model is symmetrized. We emphasize that all investigations are performed for mixtures with an arbitrary number of components K except for the relaxation of the chemical potentials. Furthermore we want to remark that the system is Galilean invariant and the proof is in complete analogy to the one for the full system discussed in [38].

Thus, the paper is organized as follows: In Sect. 1 we present the model. Mathematical properties are discussed in Sect. 2. An entropy-entropy-flux pair is derived in Sect. 3. In particular, we present a thermodynamically consistent closure for the interfacial pressures and the interfacial velocity. Finally, in Sect. 4 we summarize our findings and give an outlook on future work.

1. Baer-Nunziato-type models with barotropic equation of state

First of all, we summarize the balance laws for the multi-component Baer-Nunziato-type model where the number of components is arbitrary. The phasic fluids are assumed to be barotropic, i.e., each of the single fluids is governed by a pressure only depending on the density of the fluid, see Section 1.1. To close the model we have to provide appropriate equations of state (EoS) for the phasic pressures. Starting from fundamental relations of thermodynamics we provide the stiffened gas EoS for both isothermal fluids and isentropic fluids, see Section 1.2. Finally, in Section 1.3 we discuss appropriate relaxation models.

1.1. Evolution equations

It will be convenient to first recall the full Baer-Nunziato model, cf. [38]. This model is described by a non-equilibrium model where all components are present in each point of the space-time continuum. Each component $k = 1, \dots, K$ has its own density ρ_k, velocity $\boldsymbol{v}_k$ and internal energy e_k. Neglecting viscosity but accounting for heat exchange the full model reads

$$\partial_t \alpha_k + \boldsymbol{V}_I \cdot \nabla \alpha_k = S_{\alpha,k}, \tag{1.1a}$$

$$\partial_t (\alpha_k \, \rho_k) + \nabla \cdot (\alpha_k \, \rho_k \, \boldsymbol{v}_k) = S_{\alpha\rho,k}, \tag{1.1b}$$

$$\partial_t (\alpha_k \, \rho_k \, \boldsymbol{v}_k) + \nabla \cdot (\alpha_k \, \rho_k \, \boldsymbol{v}_k \, \boldsymbol{v}_k^T + \alpha_k \, p_k \, \boldsymbol{I}) + \sum\nolimits_{l=1,\neq k}^{K} P_{k,l} \, \nabla \alpha_l = \boldsymbol{S}_{\alpha\rho\boldsymbol{v},k}, \tag{1.1c}$$

$$\partial_t(\alpha_k\,\rho_k\,E_k) + \nabla\cdot(\alpha_k\,\rho_k\,\boldsymbol{v}_k\,(E_k + p_k/\rho_k)) + \sum\nolimits_{l=1,\neq k}^{K} P_{k,l}\,\boldsymbol{V}_I\cdot\nabla\alpha_l = -\nabla\cdot(\alpha_k\,\boldsymbol{q}_k) + S_{\alpha\rho E,k} \tag{1.1d}$$

with the heat flux $\boldsymbol{q}_k$ and total $E_k = e_k + \boldsymbol{v}_k^2/2$ determined by the internal energy e_k and kinetic energy $u_k = \boldsymbol{v}_k^2/2$. The amount of each component is determined by its volume fraction α_k. The volume fractions are related by the *saturation constraint*

$$\sum\nolimits_{k=1}^{K}\alpha_k = 1, \quad \alpha_k \in (0,1). \tag{1.2}$$

Due to the saturation condition (1.2) we only need $K-1$ evolution equations (1.1a) for the volume fractions. Without loss of generality we express α_K by the other volume fractions, i.e.,

$$\alpha_K = 1 - \sum\nolimits_{k=1}^{K-1}\alpha_k,\ \nabla\alpha_K = -\sum\nolimits_{k=1}^{K-1}\nabla\alpha_k,\ S_{\alpha,K} = -\sum\nolimits_{k=1}^{K-1} S_{\alpha,k}. \tag{1.3}$$

Typically, the fluid equations are supplemented by an equation of state for the phasic pressures p_k depending on two out of four other phasic thermodynamical quantities density ρ_k, internal energy e_k, temperature T_k and entropy s_k, see Section 1.2 for more details, i.e.,

$$p_k = p_k(\rho_k, e_k) \quad \text{or} \quad p_k = p_k(\rho_k, T_k) \quad \text{or} \quad p_k = p_k(\rho_k, s_k). \tag{1.4}$$

In the present work we are interested in isothermal and isentropic flows where we assume either *isothermal* fluids with constant temperatures, i.e.,

$$T_k = T_{ref} = const. \quad \forall k = 1,\dots,K \tag{1.5}$$

or *isentropic* fluids with constant entropies, i.e.,

$$s_k = const. \quad \forall k = 1,\dots,K. \tag{1.6}$$

In contrast to (1.4), the phasic pressures are now only a function of the phasic densities, i.e.,

$$p_k = p_k(\rho_k) \quad \text{resp.} \quad \rho_k = \rho_k(p_k) \tag{1.7}$$

known as barotropic fluids. Then the evolution equations for the volume fractions (1.1a), the phasic mass (1.1b), the phasic momentum (1.1c) decouples from the phasic evolution equations for the energy (1.1d). Thus, the Baer-Nunziato-type model for barotropic flows reads

$$\partial_t\alpha_k + \boldsymbol{V}_I\cdot\nabla\alpha_k = S_{\alpha,k}, \tag{1.8a}$$

$$\partial_t(\alpha_k\,\rho_k) + \nabla\cdot(\alpha_k\,\rho_k\,\boldsymbol{v}_k) = S_{\alpha\rho,k}, \tag{1.8b}$$

$$\partial_t(\alpha_k\,\rho_k\,\boldsymbol{v}_k) + \nabla\cdot(\alpha_k\,\rho_k\,\boldsymbol{v}_k\,\boldsymbol{v}_k^T + \alpha_k\,p_k\,\boldsymbol{I}) + \sum\nolimits_{l=1,\neq k}^{K} P_{k,l}\,\nabla\alpha_l = \boldsymbol{S}_{\alpha\rho\boldsymbol{v},k}. \tag{1.8c}$$

We emphasize that in case of an isothermal fluid the energy equation (1.1d) becomes a conditional equation for the heat flux, i.e.,

$$\nabla\cdot(\alpha_k\,\boldsymbol{q}_k) = -\partial_t(\alpha_k\,\rho_k\,E_k) - \nabla\cdot(\alpha_k\,\rho_k\,\boldsymbol{v}_k\,(E_k + p_k/\rho_k)) - \sum\nolimits_{l=1,\neq k}^{K} P_{k,l}\,\boldsymbol{V}_I\cdot\nabla\alpha_l + S_{\alpha\rho E,k}. \tag{1.9}$$

This is employed in the investigation of the entropy production, see Section 3.1. For isentropic single-component flows the energy equation becomes redundant. Note that this is not the case for multi-component or multi-phase flows. Nevertheless, energy balance equations typically are not taken into account when considering isentropic systems.

The equations (1.8) and (1.9) need further closure. The gradient terms in (1.8a), (1.8c) and (1.9) account for the interaction between different components. In particular, the term $P_{k,l}$ accounts for different pressures at the phase interface between phase k and l. The interfacial velocity is denoted by $\boldsymbol{V}_I$.

The source terms $S_{\alpha,k}$, $S_{\alpha\rho,k}$, $\boldsymbol{S}_{\alpha\rho\boldsymbol{v},k}$ and $S_{\alpha\rho E,k}$ on the right-hand sides of (1.8) and (1.9) describe the interaction of the components corresponding to mass, momentum and energy transfer via different relaxation types, e.g., due to velocity, pressure and chemical potentials $\xi \in \{v, p, \mu\}$

$$S_{\alpha,k} := \sum\nolimits_{\xi} S^{\xi}_{\alpha,k},\ S_{\alpha\rho,k} := \sum\nolimits_{\xi} S^{\xi}_{\alpha\rho,k},\ \boldsymbol{S}_{\alpha\rho\boldsymbol{v},k} := \sum\nolimits_{\xi} \boldsymbol{S}^{\xi}_{\alpha\rho\boldsymbol{v},k},\ S_{\alpha\rho E,k} := \sum\nolimits_{\xi} S^{\xi}_{\alpha\rho E,k}.$$

These depend on the specific components at hand that will be discussed in Section 1.3.

Obviously, the evolution equations are not in conservative form. To respect the physical principles of conservation of mass, momentum and energy we impose the following *conservation constraints* on the source terms

$$\sum\nolimits_{k=1}^{K} S^{\xi}_{\alpha,k} = 0, \quad \sum\nolimits_{k=1}^{K} S^{\xi}_{\alpha\rho,k} = 0, \quad \sum\nolimits_{k=1}^{K} \boldsymbol{S}^{\xi}_{\alpha\rho\boldsymbol{v},k} = \boldsymbol{0}, \quad \sum\nolimits_{k=1}^{K} S^{\xi}_{\alpha\rho E,k} = 0 \tag{1.10}$$

for each relaxation type as well as the interfacial pressures

$$\sum\nolimits_{k=1}^{K} \sum\nolimits_{l=1,\neq k}^{K} P_{k,l} \nabla \alpha_l = \sum\nolimits_{l=1}^{K-1} \left(\sum\nolimits_{k=1,\neq l}^{K} P_{k,l} - \sum\nolimits_{k=1}^{K-1} P_{k,K} \right) \nabla \alpha_l = 0. \tag{1.11}$$

Since by the saturation constraint (1.2) the gradients $\nabla \alpha_l$, $l = 1, \ldots, K-1$, are linearly independent, the conservation constraint (1.11) is equivalent to

$$\sum\nolimits_{k=1,\neq l}^{K} P_{k,l} = \sum\nolimits_{k=1}^{K-1} P_{k,K} =: P_I \quad \forall l = 1, \ldots, K-1. \tag{1.12}$$

Then the evolution equations for the mixture quantities

$$\rho := \sum\nolimits_{k=1}^{K} \alpha_k \rho_k,\ \rho \boldsymbol{v} := \sum\nolimits_{k=1}^{K} \alpha_k \rho_k \boldsymbol{v}_k \tag{1.13}$$

obtained by summation of the single-component fluid equations (1.8b), (1.8c) and employing the saturation constraint (1.2) as well as the conservation constraints (1.10) and (1.12) are in conservative form. See [38] for details.

So far, the model is not yet closed. For this purpose, we have to find closing conditions for the interfacial pressures $P_{k,l}$, the interfacial velocity $\boldsymbol{V}_I$ and the relaxation terms $S_{\alpha,k}$, $S_{\alpha\rho,k}$, $\boldsymbol{S}_{\alpha\rho\boldsymbol{v},k}$, $S_{\alpha\rho E,k}$. In the following sections we will derive appropriate constraints. However, these will not specify a unique model but some options are still remaining for the choice of the interfacial velocity and the relaxation terms.

Remark 1.1. *For a given barotropic pressure law* (1.7) *we always can introduce an "internal energy" function* $\overline{e}_k = \overline{e}_k(\rho_k)$ *as solution of the ODE* $\overline{e}_k'(\rho_k) = p_k/\rho_k^2$. *Defining the " total energy"* $\overline{E}_k = \overline{e}_k + \boldsymbol{v}_k^2/2$ *an "energy" equation can be derived from* (1.8)

$$\partial_t(\alpha_k \rho_k \overline{E}_k) + \nabla \cdot (\alpha_k \rho_k \boldsymbol{v}_k (\overline{E}_k + p_k/\rho_k)) + \sum\nolimits_{l=1,\neq k}^{K} P_{k,l} \boldsymbol{v}_k \cdot \nabla \alpha_l = p_k(\boldsymbol{V}_I - \boldsymbol{v}_k) \cdot \nabla \alpha_k + \overline{S}_{\alpha\rho\overline{E},k}. \tag{1.14}$$

This balance law is different to (1.1d). *In particular,* $\overline{S}_{\alpha\rho\overline{E},k}$ *differs from* $S_{\alpha\rho E,k}$. *In case of an isentropic fluid the "internal and total energies"* $\overline{e}_k$ *and* $\overline{E}_k$ *coincide with the energies* e_k *and* E_k, *respectively. In Section 3.2 for isentropic fluids we use* $\sum_{k=1}^{K} \alpha_k \rho_k \overline{E}_k$ *for a mathematical entropy in Lax' concept of entropy-entropy flux pairs.*

1.2. **Some remarks on thermodynamics**

In order to investigate thermodynamical properties of the model (1.8), we assume that the equation of state (1.7) is in agreement with thermodynamical principles. For this purpose, we recall from equilibrium thermodynamics that for each component k the internal energy e_k is a function of the entropy and the volume τ_k satisfying

$$de_k = T_k ds_k - p_k d\tau_k. \tag{1.15}$$

Thus, the pressure and the temperature are the partial derivatives of $e_k(\tau_k, s_k)$ that are assumed to be positive, i.e.,

$$p_k(\tau_k, s_k) := -\frac{\partial e_k}{\partial \tau_k}(\tau_k, s_k) \geq 0, \quad T_k(\tau_k, s_k) := \frac{\partial e_k}{\partial s_k}(\tau_k, s_k) \geq 0. \tag{1.16}$$

Furthermore, to ensure thermodynamical stability we assume that the Hessian of e_k is a convex function with respect to τ_k and s_k, i.e.,

$$\frac{\partial^2 e_k}{\partial^2 \tau_k}(\tau_k, s_k) > 0, \ \frac{\partial^2 e_k}{\partial^2 s_k}(\tau_k, s_k) > 0, \ \frac{\partial^2 e_k}{\partial^2 \tau_k}(\tau_k, s_k)\frac{\partial^2 e_k}{\partial^2 s_k}(\tau_k, s_k) > \left(\frac{\partial^2 e_k}{\partial \tau_k \partial s_k}(\tau_k, s_k)\right)^2. \tag{1.17}$$

Finally, in agreement with the third law of thermodynamics we assume

$$\tau_k \geq 0, \quad s_k \geq 0.$$

Alternatively, assuming that p_k and T_k are strictly positive, then e_k becomes a monotone function in τ_k and s_k and we may change variables, i.e., $s_k = s_k(\tau_k, e_k)$ satisfying

$$T_k ds_k = de_k + p_k d\tau_k \tag{1.18}$$

with partial derivatives

$$\frac{\partial s_k}{\partial \tau_k}(\tau_k, e_k) = \frac{\check{p}_k(\tau_k, e_k)}{\check{T}_k(\tau_k, e_k)} > 0, \quad \frac{\partial s_k}{\partial e_k}(\tau_k, e_k) = \frac{1}{\check{T}_k(\tau_k, e_k)} > 0. \tag{1.19}$$

and

$$\check{p}_k(\tau_k, e_k) := p_k(\tau_k, s_k(\tau_k, e_k)), \quad \check{T}_k(\tau_k, e_k) := T_k(\tau_k, s_k(\tau_k, e_k)).$$

It is well-known that $s_k = s_k(\tau_k, e_k)$ is a concave function, i.e., the Hessian is negative-definite

$$\frac{\partial^2 s_k}{\partial^2 \tau_k}(\tau_k, e_k) < 0, \ \frac{\partial^2 s_k}{\partial^2 e_k}(\tau_k, e_k) < 0, \ \frac{\partial^2 s_k}{\partial^2 \tau_k}(\tau_k, e_k)\frac{\partial^2 s_k}{\partial^2 e_k}(\tau_k, e_k) > \left(\frac{\partial^2 s_k}{\partial \tau_k \partial e_k}(\tau_k, e_k)\right)^2, \tag{1.20}$$

if and only if $e_k(\tau_k, s_k)$ is a convex function, i.e., thermodynamic stability holds.

Finally, following [54], p. 21 and 23, we introduce the speed of sound c_k and the fundamental derivative of thermodynamics $\mathcal{G}_k$ defined by means of (1.16) as

$$c_k^2(\tau_k, s_k) := -\tau_k^2 \frac{\partial p_k}{\partial \tau_k}(\tau_k, s_k), \quad \mathcal{G}_k(\tau_k, s_k) := \frac{1}{2}\tau_k^3 \frac{\frac{\partial^2 p_k}{\partial \tau_k^2}(\tau_k, s_k)}{c_k^2(\tau_k, s_k)} \tag{1.21}$$

or, equivalently,

$$c_k^2(\rho_k, s_k) = \frac{\partial p_k}{\partial \rho_k}(\rho_k, s_k), \quad \mathcal{G}_k(\rho_k, s_k) = \frac{1}{c_k(\rho_k, s_k)} \frac{\partial(\rho_k c_k)}{\partial \rho_k}(\rho_k, s_k), \tag{1.22}$$

or, equivalently,

$$c_k^2(\rho_k, T_k) = \frac{\partial p_k}{\partial \rho_k}(\rho_k, T_k) + \frac{T_k}{\rho_k^2 c_{v,k}} \left(\frac{\partial p_k}{\partial T_k}(\rho_k, T_k) \right)^2, \quad \mathcal{G}_k(\rho_k, T_k) = 1 + \frac{\rho_k}{2\, c_k(\rho_k, T_k)} \frac{\partial c_k}{\partial \rho_k}(\rho_k, T_k). \tag{1.23}$$

Example 1.1 (Stiffened gas EoS)**.** *Let the constant material parameters $c_{v,k}$, γ_k, π_k and q_k denoting the specific heat capacity at constant volume, the adiabatic exponent, the minimal pressure and the heat of formation of component k, respectively, chosen such that $c_{v,k} > 0$, $\pi_k \geq 0$, q_k and $\gamma_k > 1$. Let further $\rho_{0,k}$ be some reference density at reference temperature $T_{0,k}$. By means of these coefficients the internal energy, the pressure and the entropy are determined by*

$$e_k = c_{v,k} T_k + \pi_k / \rho_k + q_k \,, \tag{1.24a}$$

$$p_k = \rho_k (\gamma_k - 1) c_{v,k} T_k - \pi_k \,, \tag{1.24b}$$

$$s_k = c_{v,k} + c_{v,k} \ln \frac{T_k}{T_{0,k}} + q_k \frac{1}{T_{0,k}} - (\gamma_k - 1) c_{v,k} \ln \frac{\rho_k}{\rho_{0,k}} - \frac{\pi_k}{\rho_{0,k}} \frac{1}{T_{0,k}} \,. \tag{1.24c}$$

Then the speed of sound and the fundamental derivative of gas dynamics are determined by (1.23) as

$$c_k = \sqrt{\gamma_k T_k \, c_{v,k} \, (\gamma_k - 1)}, \quad \mathcal{G}_k = \frac{\gamma_k + 1}{2}. \tag{1.25}$$

For a derivation we refer to [19].

In the present work we are interested in isothermal and isentropic fluids where the pressure is only a function of the density, i.e.,

$$p_k(\rho_k) = p_k(\rho_k, s_k = const) \quad \text{or} \quad p_k(\rho_k) = p_k(\rho_k, T_k = const) \tag{1.26}$$

in (1.22) and (1.30). Thus, these fluids fall in the class of barotropic fluids. In the following we assume for barotropic fluids that

$$p_k'(\rho_k) > 0, \tag{1.27a}$$

$$\mathcal{G}_k(\rho_k) = 1 + \frac{\rho_k}{2\bar{c}_k^2(\rho_k)} p_k''(\rho_k) > 0. \tag{1.27b}$$

This ensures positivity of the barotropic speed of sound

$$\bar{c}_k(\rho_k) = \sqrt{p_k'(\rho_k)} > 0 \tag{1.28}$$

and convexity of the isentropes or isotherms in the p-τ-plane, in the special case of an isentropic or isothermal fluid, respectively.
In case of an isothermal fluid the isothermal speed of sound and the fundamental derivative of thermodynamics are defined as

$$\tilde{c}_k^2(\tau_k) := -\tau_k^2 \frac{\partial p_k}{\partial \tau_k}(\tau_k, T_{ref}), \quad \overline{\mathcal{G}}(\tau_k) := \frac{1}{2} \tau_k^3 \frac{\frac{\partial^2 p_k}{\partial \tau_k^2}(\tau_k, T_{ref})}{\tilde{c}_k^2(\tau_k, T_{ref})} \tag{1.29}$$

or, equivalently,

$$\tilde{c}_k^2(\rho_k) = \frac{\partial p_k}{\partial \rho_k}(\rho_k, T_{ref}), \quad \overline{\mathcal{G}}_k(\tau_k) = 1 + \frac{\rho_k}{\tilde{c}_k(\rho_k, Tref)} \frac{\partial \tilde{c}_k}{\partial \rho_k}(\rho_k, T_{ref}). \tag{1.30}$$

Note that the barotropic speed of sound $\bar{c}_k$, see (1.28), is identical to the thermodynamical speed of sound $c_k(\rho_k, s_k = const)$, see (1.22), for an isentropic fluid.

Since we are interested in isothermal and isentropic fluids, we summarize the isothermal stiffened gas EoS and the isentropic stiffened gas EoS which have been verified to be thermodynamically consistent, see [19].

Example 1.2 (Isothermal stiffened gas EoS)**.** *Starting from the fixed temperature T_{ref}, i.e., $T_k = T_{ref}$ for all k and the corresponding saturation pressures $p_{ref,k}$, we choose the ratio of specific heats $\gamma_k > 1$, the minimal pressure $\pi_k \geq 0$, the specific heat at constant volume $c_{v,k}$, and the heat of formation q_k for every component at the reference state. By means of these coefficients the pressure and the entropy are determined by*

$$p_k(\rho_k) = \tilde{c}_k^2 \rho_k - \pi_k, \tag{1.31}$$

$$s_k(\rho_k) = -(\gamma_k - 1)c_{v,k} \ln\left(\frac{p_k + \pi_k}{p_{ref,k} + \pi_k}\right) + c_{v,k} + \frac{q_k}{T_{ref}}. \tag{1.32}$$

In this particular case, the isothermal speed of sound and the isothermal fundamental derivative of gas dynamics become constants that are determined by (1.30) as

$$\tilde{c}_k = \sqrt{T_{ref}\, c_{v,k}\, (\gamma_k - 1)}, \quad \overline{\mathcal{G}}_k = 1. \tag{1.33}$$

We emphasize that the speed of sound c_k and the isothermal speed of sound $\tilde{c}_k$ defined by (1.23) *and* (1.30)*, respectively, are related by*

$$c_k^2 = \gamma_k \tilde{c}_k^2. \tag{1.34}$$

The fundamental derivatives $\mathcal{G}_k$ and $\overline{\mathcal{G}}_k$ have the same sign. For later use we also give the internal energies and the Gibbs free energies

$$e_k(\rho_k) = c_{v,k} T_{ref} + \frac{\pi_k}{\rho_k} + e_{0,k}, \quad e_{0,k} := q_k - (\gamma_k - 1)c_{v,k} T_{ref}, \tag{1.35}$$

$$g_k(\rho_k) = (\gamma_k - 1)c_{v,k} T_{ref} \ln\left(\frac{p_k(\rho_k) + \pi_k}{p_{ref,k} + \pi_k}\right). \tag{1.36}$$

Example 1.3 (Isentropic stiffened gas EoS)**.** *Again, starting from the uniform reference temperature T_{ref} and the corresponding saturation pressures $p_{ref,k}$, we choose the ratio of specific heats $\gamma_k > 1$, the minimal pressure $\pi_k \geq 0$, the specific heat at constant volume $c_{v,k}$, and the heat of formation q_k for every component at the reference state. By means of these coefficients the pressure and the temperature are determined by*

$$p_k(\rho_k) = A_k \rho_k^{\gamma_k} - \pi_k, \tag{1.37}$$

$$T_k(\rho_k) = B_k \rho_k^{\gamma_k - 1} \tag{1.38}$$

with

$$A_k = \frac{T_{0,k}^{\gamma_k}}{(p_{0,k} + \pi_k)^{\gamma_k - 1}} \left(c_{v,k}(\gamma_k - 1)\right)^{\gamma_k}, \quad B_k = \frac{T_{0,k}^{\gamma_k}}{(p_{0,k} + \pi_k)^{\gamma_k - 1}} \left(c_{v,k}(\gamma_k - 1)\right)^{\gamma_k - 1}. \tag{1.39}$$

For sake of completeness, we give the constant entropy at the reference state as well as the internal energy

$$s_{0,k} = c_{v,k} + \frac{q_k}{T_{ref}} \tag{1.40}$$

$$e_k(\rho_k) = c_{v,k} T_k(\rho_k) + \frac{\pi_k}{\rho_k} + e_{0,k}, \quad e_{0,k} := q_k - (\gamma_k - 1)c_{v,k} T_{ref}. \tag{1.41}$$

The speed of sound and the fundamental derivative of gas dynamics are determined by (1.22) as

$$c_k(\rho_k) = \sqrt{\gamma_k A_k \rho_k^{\gamma_k - 1}}, \quad \mathcal{G}_k = \frac{\gamma_k + 1}{2}. \tag{1.42}$$

So far we have discussed the individual components. However, since we are considering a mixture of K components the specific internal energy e of the mixture is given by the *Gibbs-Duhem-relation*, cf. [37],

$$de = Tds - pd\frac{1}{\rho} + \sum\nolimits_{k=1}^{K} \mu_k dN_k \tag{1.43}$$

where e, s, p denote the mixture quantities of the energy, the entropy and the pressure defined analogously to (1.13) as

$$\rho e = \sum\nolimits_{i=1}^{K} \alpha_k \rho_k e_k, \ \rho s = \sum\nolimits_{i=1}^{K} \alpha_k \rho_k s_k, \ p = \sum\nolimits_{i=1}^{K} \alpha_k p_k. \tag{1.44}$$

We do not comment on the mixture temperature T here in detail, since we are considering an isothermal process. However, in the case of different temperatures for each component we refer to [48, Def. 28.1] for a definition of the mixture temperature. There are also cases when a single temperature for all components is considered. In this case the temperature is obtained as usual as the derivative of the mixture energy with respect to the mixture entropy. The quantity μ_k denotes the chemical potential of component k and N_k is the mole number of the respective component. The chemical potential μ_k quantifies the change of the kth component due to phase change. For later use we want to specify the chemical potential for the mixture of ideal gases.

Example 1.4 (Chemical potential for a mixture of two isothermal ideal gases)**.** *Assume, components k and j form a mixture of two isothermal ideal gases, then from (1.43) we obtain*

$$\mu_k(\alpha_k, \rho_k, \alpha_j, \rho_j) = g_k(\rho_k) + (\gamma_k - 1) c_{v,k} T_{ref} \ln\left(\frac{\alpha_k p_k(\rho_k)}{\alpha_k p_k(\rho_k) + \alpha_j p_j(\rho_j)}\right). \tag{1.45}$$

In particular, if component j vanishes, then we have the pure substance k and $\mu_k(\rho_k) = g_k(\rho_k)$.

This example becomes relevant in Sect. 3.1.2 where we discuss the entropy production due to relaxation of chemical potentials in the context of a 3-component mixture consisting of two isothermal ideal gases (water vapor ($k = 1$) and some inert gas ($k = 3$)) and liquid water ($k = 2$) modeled by an isothermal stiffened gas.

1.3. **Relaxation Model**

The non-equilibrium model without relaxation terms allows for different values for velocities, pressures, temperatures as well as chemical potentials at the same point. The relaxation mechanism described by the source terms drives all these quantities into equilibrium. Typically it is distinguished between mechanical and thermal relaxation processes that relax either pressures and velocities or temperatures and chemical potentials to equilibrium. In the following we give the relaxation terms for the full Baer-Nunziato model (1.1) containing the energy equations. The energy relaxation will be needed in case of isothermal fluids, see Section 3.1, whereas it does not enter the relaxation process of the isentropic model.

Mechanical relaxation. The *pressure relaxation* implies volume variations that induce energy variations due to the interfacial pressure work. Following previous work in [38,51,56] we model the pressure relaxation in case of barotropic fluids by

$$S^p_{\alpha,k} = \theta_p \, \alpha_k \, (p_k - p), \ S^p_{\alpha\rho,k} = 0, \ \boldsymbol{S}^p_{\alpha\rho\boldsymbol{v},k} = \mathbf{0}, \ S^p_{\alpha\rho E,k} = \theta_p \, \alpha_k \, p(p - p_k) \, . \tag{1.46}$$

Here θ_p is the pressure relaxation parameter and p the mixture pressure defined in Eqn. (1.44) Similarly we apply for the *velocity relaxation*

$$S^v_{\alpha,k} = 0, \ S^v_{\alpha\rho,k} = 0, \ \boldsymbol{S}^v_{\alpha\rho\boldsymbol{v},k} = \theta_v \, \alpha_k \, \rho_k \, (\boldsymbol{v} - \boldsymbol{v}_k) \, , S^v_{\alpha\rho E,k} = \theta_v \, \alpha_k \, \rho_k \, \boldsymbol{v} \cdot (\boldsymbol{v} - \boldsymbol{v}_k) \tag{1.47}$$

with the velocity relaxation parameter θ_v and the mixture velocity $\boldsymbol{v}$ determined by the conserved mixture quantities (1.13). Performing velocity and pressure relaxation the fluid mixture is in mechanical equilibrium, i.e., $p_k = p^\infty$ and $\boldsymbol{v}_k = \boldsymbol{v}^\infty$, $k = 1, \dots, K$. Obviously, the mechanical relaxation terms satisfy the conservation constraints (1.10) due to the definition of the mixture quantities (1.13) and (1.44).

We like to mention that in [25] for barotropic fluids a general class of models for mechanical relaxation is considered:

$$S^p_{\alpha,k} = \theta_p \sum\nolimits_{l=1}^{K} d_{k,l}(p_k - p_l),\ S^p_{\alpha\rho,k} = 0,\ \boldsymbol{S}^p_{\alpha\rho\boldsymbol{v},k} = \boldsymbol{0} \tag{1.48a}$$

$$S^v_{\alpha,k} = S^v_{\alpha\rho,k} = 0,\ \boldsymbol{S}^v_{\alpha\rho\boldsymbol{v},k} = \theta_v \sum\nolimits_{l=1}^{K} e_{k,l}(\boldsymbol{v}_l - \boldsymbol{v}_k)\,, \tag{1.48b}$$

with state-dependent coefficients $d_{k,l} > 0$ and $e_{k,l} \geq 0$. In particular, choosing $d_{k,l} = \alpha_k \alpha_l$ and $e_{k,l} = \alpha_k \rho_k \alpha_l \rho_l / \sum_{j=1}^{K} \alpha_j \rho_j$ we note that the mechanical relaxation model (1.46) and (1.47) falls into the class of models (1.48).

Thermal relaxation. According to (1.5) for isothermal fluids the temperatures are all in equilibrium. For isentropic fluids the temperatures cannot be in equilibrium because all entropies are constant resulting in an overdetermined system. Thus, there exists no thermal relaxation process.

Chemical potential relaxation. In the following we consider only isothermal fluids because there is already no thermal relaxation for isentropic fluids. Mass transfer between different phases of the same substance occurs, whenever these phases are not in chemical equilibrium, i.e., when their chemical potentials are not the same. This physical fact is the key idea to model the mass transfer by relaxation of the chemical potentials. It is obvious, that from now on it is necessary to identify the components. In particular, we are interested in the two-phase model, i.e., $K = 2$, with water vapor ($k = 1$), liquid water ($k = 2$), and in the three-component model, i.e., $K = 3$, with water vapor ($k = 1$), liquid water ($k = 2$) and inert gas ($k = 3$).

Two-component mixtures. For a two-component mixture ($K = 2$) the relaxation of chemical potentials is modeled according to [56] by

$$S^\mu_{\alpha,1} = \theta_\mu \frac{\dot{m}}{\varrho},\ S^\mu_{\alpha\rho,1} = \theta_\mu \dot{m},\ \boldsymbol{S}^\mu_{\alpha\rho\boldsymbol{v},1} = \theta_\mu \dot{m}\, \hat{\boldsymbol{v}},\ S^\mu_{\alpha\rho E,1} = \theta_\mu \dot{m} \left(\epsilon + \frac{\hat{\boldsymbol{v}}^2}{2} \right), \tag{1.49a}$$

$$S^\mu_{\alpha,2} = -S^\mu_{\alpha,1},\ S^\mu_{\alpha\rho,2} = -S^\mu_{\alpha\rho,1},\ \boldsymbol{S}^\mu_{\alpha\rho\boldsymbol{v},2} = -\boldsymbol{S}^\mu_{\alpha\rho\boldsymbol{v},1}, S^\mu_{\alpha\rho E,2} = -S^\mu_{\alpha\rho E,1} \tag{1.49b}$$

with the relaxation parameter θ_μ. For the velocity $\hat{\boldsymbol{v}}$ we choose a convex combination

$$\hat{\boldsymbol{v}} = \sum\nolimits_{k=1}^{K} \beta^v_k \boldsymbol{v}_k, \quad \beta^v_k \in [0,1], \quad \sum\nolimits_{k=1}^{K} \beta^v_k = 1 \tag{1.50}$$

in order to guarantee Galilean invariance. Formulas for the parameters ϱ and ϵ can be found in [56] where they are determined such that pressure and thermal equilibrium is maintained during the relaxation. Obviously, the conservation constraints (1.10) are satisfied. Note that due to these constraints we are not allowed to introduce ϵ_k and ϱ_k differently for each component $k = 1, 2$. In [17] another choice for these parameters has been introduced that ensures non-negativity of the entropy production due to chemical relaxation.

Since for a two-component mixture the Gibbs free energy coincides with the chemical potential, chemical equilibrium is achieved, if the Gibbs free energies of the two components coincide, i.e., $g_1 = g_2$.

Three-component mixtures. Here we consider three components, i.e., $K = 3$, with water vapor ($k = 1$), liquid water ($k = 2$) and inert gas ($k = 3$), where the gas phase is assumed to be a mixture of water vapor and some inert gas. The relaxation terms of chemical potentials for isothermal fluids are given by

$$S^\mu_{\alpha,1} = \theta_\mu \frac{\dot{m}}{\varrho_1},\ S^\mu_{\alpha\rho,1} = \theta_\mu \dot{m},\ \boldsymbol{S}^\mu_{\alpha\rho\boldsymbol{v},1} = \theta_\mu \dot{m}\, \hat{\boldsymbol{v}},\ S^\mu_{\rho E,1} = \theta_\mu \dot{m} \left(\epsilon_1 + \frac{\hat{\boldsymbol{v}}^2}{2} \right), \tag{1.51a}$$

$$S^{\mu}_{\alpha,2} = \theta_\mu \frac{\dot{m}}{\varrho_2},\ S^{\mu}_{\alpha\rho,2} = -\theta_\mu \dot{m},\ \boldsymbol{S}^{\mu}_{\alpha\rho\boldsymbol{v},2} = -\theta_\mu \dot{m}\,\hat{\boldsymbol{v}},\ S^{\mu}_{\alpha\rho E,2} = -\theta_\mu \dot{m}\left(\epsilon_2 + \frac{\hat{\boldsymbol{v}}^2}{2}\right), \tag{1.51b}$$

$$S^{\mu}_{\alpha,3} = -\theta_\mu \dot{m}\left(\frac{1}{\varrho_1} + \frac{1}{\varrho_2}\right),\ S^{\mu}_{\alpha\rho,3} = 0,\ \boldsymbol{S}^{\mu}_{\alpha\rho\boldsymbol{v},3} = \boldsymbol{0}\,, S^{\mu}_{\alpha\rho E,3} = \theta_\mu \dot{m}\,(\epsilon_2 - \epsilon_1) \tag{1.51c}$$

with the relaxation parameter θ_μ. Formulas for the parameters ϱ_1, ϱ_2 can be found in [56, 58]. For details on the physics see the book of Müller and Müller [36].

Chemical equilibrium is achieved, if the chemical potential of the water vapor phase equals the Gibbs free energy of the liquid water phase. In the limit case of no inert gas the expression of the chemical potential of the water vapor phase reduces to the Gibbs free energy. The mass flux $\dot{m}$ between the liquid and the vapor phase is driven by the difference of their chemical potentials, i.e., $\dot{m} = \dot{m}(\mu_1 - \mu_2)$. In particular, the mass flux vanishes if the difference of the chemical potentials is zero, i.e., $\mu_1 = \mu_2$ or if one of the phases is completely extinct.

Note that the relaxation terms (1.49) resp. (1.51) differ from the terms given in [17] due to the different form of the chemical potentials in the isothermal case.

The relaxation terms (1.46), (1.47), (1.49), (1.51) are of major importance when dealing with interface problems, see for instance Saurel and Abgrall [51] or Lallemand and Saurel [33] for mechanical relaxation terms. Typically, it is assumed that pressure and velocity relax instantaneously, see [51], whereas the thermal relaxation and the relaxation of chemical potentials are much slower, see Zein [56]. Note that at equilibrium the equilibrium state does not depend on the order of relaxation, the relaxation parameters θ_ξ, $\xi \in \{p, v, \mu\}$, drop out and have not to be known explicitly. The same is true for the parameters ϱ_k, $k = 1, 2$. At non-equilibrium these parameters have a strong effect on the transient relaxation behavior, cf. [28, 29].

Furthermore we point out, that our modeling of the mass transfer is physically correct because we take into account chemical potentials instead of Gibbs free energies. In [56, 58] the Gibbs free energy is relaxed, which neglects the effect of mixture entropy in cases of impure substances.

We would like to mention that in [25] for barotropic fluids a general class of models for chemical relaxation (mass transfer) is considered:

$$S^{\mu}_{\alpha,k} = 0,\ S^{\mu}_{\alpha\rho,k} = \theta_\mu \sum\nolimits_{l=1,\neq k}^{K} \Gamma_{k,l},\ \boldsymbol{S}^{\mu}_{\alpha\rho\boldsymbol{v},k} = \theta_\mu \sum\nolimits_{l=1,\neq k}^{K} \boldsymbol{V}_{k,l}\Gamma_{k,l} \tag{1.52}$$

with interfacial mass transfer $\Gamma_{k,l}$ and interfacial velocities $\boldsymbol{V}_{k,l} = \beta_{k,l}\boldsymbol{v}_k + (1 - \beta_{k,l})\boldsymbol{v}_l$ with state-dependent coefficients $\beta_{k,l} \in [0, 1]$. To ensure the conservation constraint (1.10) it is assumed that $\Gamma_{k,l} = -\Gamma_{l,k}$ and $\boldsymbol{V}_{k,l} = \boldsymbol{V}_{l,k}$. In particular, it is suggested to use $\beta_{k,l} = 1/2$ and $\Gamma_{k,l} = f_{k,l}\left((p_l/\rho_l + e_l(\rho_l)) - (p_k/\rho_k + e_k(\rho_k))\right)$ for an entropy-consistent closure. Since in the relaxation models (1.49) and (1.51) the change of volume is accounted for in the evolution equations of the volume fractions, i.e., $S^{\mu}_{\alpha,k} \neq 0$, these models do not fit into the class (1.52). From a physical point of view, the volume will change when the mass of a component changes due to evaporation or condensation. This has to be accounted for in the relaxation terms $S^{\mu}_{\alpha,k}$. Furthermore, in [25] the existence of an equilibrium to (1.52) is not investigated.

Finally, we would like to mention that a physically meaningful relaxation model has to be Galilean invariant. For the models (1.46), (1.47), (1.49) and (1.51) this was investigated in [38]. In case of barotropic fluids Hérard's relaxation model determined by (1.48) and (1.52) can also be verified to be Galilean invariant.

Remark 1.2. *For the closure of the barotropic model* (1.8a), (1.3) *and*(1.8) *with barotropic pressure law* (1.7) *there is no need to give the relaxation terms* $S^{\xi}_{\alpha\rho E,k}$, $\xi \in \{p, v, \mu\}$. *These relaxation terms are only needed in the entropy investigation in the isothermal case in Sect. 3.1. Thus, Hérard's relaxation models* (1.48) *and* (1.52) *cannot be considered in this context.*

2. Mathematical Properties

In this section we investigate some mathematical properties of the barotropic Baer-Nunziato model. First of all, we derive in Sec. 2.1 the eigenvalues and eigenvectors for the transport part of the model and verify that it is hyperbolic if the non-resonance condition holds. Then we consider the corresponding characteristic fields, see Sec. 2.2, and derive for each field Riemann invariants, see Sec. 2.3. Finally, we derive a symmetrizer in Sec. 2.4.

2.1. **Hyperbolicity: eigenvectors and eigenvalues**

Neglecting the relaxation processes in the evolution equations (1.8) the model reduces to a homogeneous first order system describing transport effects only. Therefore this system should be hyperbolic to ensure that all wave speeds are finite and the system may be locally decoupled. From a mathematical point of view, this property is helpful in the construction of Riemann solvers. Therefore we determine here the eigenvalues and eigenvectors. Since the corresponding characteristic fields are essential in the construction of a Riemann solver we investigate these fields in more detail. This will also provide a physically reasonable closure for the interfacial velocity $\boldsymbol{V}_I$. Note that the following analysis holds for a general barotropic equation of state (1.7) satisfying the constraints (1.27) and (1.28). In particular, it is applicable to both isothermal and isentropic fluids.

To investigate hyperbolicity we may consider the evolution equations for the primitive variables α_k, $\boldsymbol{v}_k$, p_k instead of the variables α_k, $\alpha_k\rho_k$, $\alpha_k\rho_k\boldsymbol{v}_k$. This is justified because the change of variables is a bijective mapping under which the eigenvalues are invariant and the eigenvectors are related by the Jacobian of the transformation. Using the equation of state (1.7) we derive from (1.2) and (1.8) the system in primitive variables.

$$\partial_t \alpha_k + \sum\nolimits_{i=1}^{d} V_{I,i}\,\partial_{x_i}\alpha_k = S_{\alpha,k} \tag{2.1a}$$

$$\partial_t \boldsymbol{v}_k + \sum\nolimits_{i=1}^{d}\left(v_{k,i}\,\partial_{x_i}\boldsymbol{v}_k + \sum\nolimits_{l=1,l\neq k}^{K} \frac{1}{\alpha_k\,\rho_k}(P_{k,l}-p_k)\,\boldsymbol{e}_{d,i}\,\partial_{x_i}\alpha_l + \frac{1}{\rho_k}\,\boldsymbol{e}_{d,i}\,\partial_{x_i}p_k\right) = \boldsymbol{S}_{\boldsymbol{v},k}, \tag{2.1b}$$

$$\partial_t p_k + \sum\nolimits_{i=1}^{d}\left(v_{k,i}\,\partial_{x_i}p_k - \sum\nolimits_{l=1,l\neq k}^{K} \frac{\rho_k}{\alpha_k}\,\overline{c}_k^2\,(v_{k,i}-V_{I,i})\,\partial_{x_i}\alpha_l + \rho_k\,\overline{c}_k^2\,\partial_{x_i}v_{k,i}\right) = S_{p,k}, \tag{2.1c}$$

where $\boldsymbol{e}_{d,i}\in\mathbb{R}^d$ denotes the unit vector in the ith spatial direction. The relaxation terms are determined by

$$S_{p,k} := \frac{\overline{c}_k^2}{\alpha_k}(S_{\alpha\rho,k} - \rho_k S_{\alpha,k}), \quad \boldsymbol{S}_{\boldsymbol{v},k} := \frac{1}{\alpha_k\rho_k}\left(\boldsymbol{S}_{\alpha\rho\boldsymbol{v},k} - \boldsymbol{v}_k S_{\alpha\rho,k}\right). \tag{2.2}$$

Neglecting the relaxation terms, we now consider the projection of the homogeneous transport system onto normal direction $\xi := \boldsymbol{x}\cdot\boldsymbol{n}$ for arbitrary unit direction $\boldsymbol{n}\in\mathbb{R}^d$.

Remark 2.1. *For a general barotropic fluid the speed of sound $\overline{c}_k$ is determined by* (1.28)*. We emphasize that it may differ in general from the thermodynamical speed of sound* (1.21)*. In case of the isothermal stiffened gas it is identical to the isothermal speed of sound* (1.29)*, see* (1.31)*,* (1.33) *and* (1.34)*.*

Introducing the vector of primitive variables

$$\boldsymbol{w} = (\alpha_1,\ldots,\alpha_{K-1},\boldsymbol{w}_1^t,\ldots,\boldsymbol{w}_K^t)^t,\ \boldsymbol{w}_k = (\boldsymbol{v}_k^t,p_k)^t \tag{2.3}$$

the projected system can be written in quasi-conservative form as

$$\partial_t\boldsymbol{w} + \boldsymbol{B}_n(\boldsymbol{w})\,\partial_\xi\boldsymbol{w} = \boldsymbol{0}. \tag{2.4}$$

The block matrix $\boldsymbol{B}_n$ is given as

$$\boldsymbol{B}_n := \sum_{i=1}^{d} \boldsymbol{B}_i \, n_i = \begin{pmatrix} V_{I,n}\, \boldsymbol{I}_{K-1} & & & \\ \boldsymbol{A}_{1,n} & \boldsymbol{B}_{1,n} & & \\ \vdots & & \ddots & \\ \boldsymbol{A}_{K,n} & & & \boldsymbol{B}_{K,n} \end{pmatrix} \tag{2.5}$$

with the blocks defined as

$$\boldsymbol{A}_{k,n} := \begin{pmatrix} \frac{1}{\alpha_k} \boldsymbol{n} \boldsymbol{\beta}_k^t \\ \frac{\rho_k}{\alpha_k} \left(v_{k,n} - V_{I,n}\right) \boldsymbol{\gamma}_k^t \end{pmatrix}, \quad \boldsymbol{B}_{k,n} := \begin{pmatrix} v_{k,n}\, \boldsymbol{I}_d & \frac{1}{\rho_k} \boldsymbol{n} \\ \rho_k\, \overline{c}_k^2 \boldsymbol{n}^t & v_{k,n} \end{pmatrix}. \tag{2.6}$$

Here $\boldsymbol{I}_d$ and $\boldsymbol{I}_{K-1}$ are the unit matrices in $\mathbb{R}^{d\times d}$ and $\mathbb{R}^{(K-1)\times(K-1)}$, respectively, and $\boldsymbol{0}_d$ and $\boldsymbol{1}_{K-1}$ are vectors in $\mathbb{R}^d$ and $\mathbb{R}^{K-1}$ with value 0 or 1, respectively. The vectors $\boldsymbol{\beta}_k$ and $\boldsymbol{\gamma}_k$ are defined by their components $l = 1, \ldots, K-1$ as

$$\beta_{k,l} := \frac{1}{\rho_k} \left((P_{k,l} - p_k)(1 - \delta_{k,l}) - (P_{k,K} - p_k)(1 - \delta_{k,K}) \right), \tag{2.7}$$

$$\gamma_{k,l} := \overline{c}_k^2 (\delta_{k,l} - \delta_{k,K}) \tag{2.8}$$

with $\delta_{k,l}$ the Kronecker symbol. The normal components of the velocities and the interfacial velocity are defined as

$$v_{k,n} := \boldsymbol{v}_k \cdot \boldsymbol{n}, \quad V_{I,n} := \boldsymbol{V}_I \cdot \boldsymbol{n}. \tag{2.9}$$

Obviously, the eigenvalues of the matrix (2.5) can now be explicitly determined.

Theorem 2.1. *(Eigenvalues) Let be $\boldsymbol{n} \in \mathbb{R}^d$ with $|\boldsymbol{n}|$=1. Then the eigenvalues of the matrix $\boldsymbol{B}_n$ read*

$$\lambda_{I,k} = V_{I,n}, \ k = 1, \ldots, K-1, \tag{2.10a}$$

$$\lambda_{k,i} = v_{k,n}, \ k = 1, \ldots, K, \ i = 1, \ldots, d-1, \tag{2.10b}$$

$$\lambda_{k,\pm} = v_{k,n} \pm \overline{c}_k, \ k = 1, \ldots, K. \tag{2.10c}$$

Proof. Employing the block structure the characteristic polynomial reads

$$\det(\boldsymbol{B}_n - \lambda\, \boldsymbol{I}) = \det(V_{I,n}\, \boldsymbol{I}_{K-1} - \lambda\, \boldsymbol{I}_{K-1}) \prod_{k=1}^{K} \det(\boldsymbol{B}_{k,n} - \lambda\, \boldsymbol{I}_{d+1}) = 0. \tag{2.11}$$

The eigenvalues of the matrices $\boldsymbol{B}_{k,n}$ are determined by

$$\det(\boldsymbol{B}_{k,n} - \lambda\, \boldsymbol{I}_{d+1}) = (v_{k,n} - \lambda)^{d-1} \left((v_{k,n} - \lambda)^2 - \overline{c}_k^2 \right). \tag{2.12}$$

□

Next we need to verify that the left and right eigenvectors corresponding to these eigenvalues are linearly independent. The derivation of the eigenvectors is in complete analogy to the Baer-Nunziato model for general fluids, see [38]. For this purpose, we introduce the notation

$$\delta_k^n := v_{k,n} - V_{I,n}, \ \sigma_k := (\delta_k^n)^2 - \overline{c}_k^2 \tag{2.13}$$

for $k = 1, \ldots, K$. Note that δ_k^n is referred to as drift velocity in the literature. Then we can prove the following

Theorem 2.2. *(Hyperbolicity) Let be $\boldsymbol{n} \in \mathbb{R}^d$ with $|\boldsymbol{n}| = 1$. Let the interfacial velocity $\boldsymbol{V}_I$ be chosen such that the normal interfacial velocity $V_{I,n}$ does not coincide with the phasic acoustic wave speeds $v_{k,n} \pm \overline{c}_k$, i.e., the non-resonance condition*

$$V_{I,n} \neq v_{k,n} \pm \overline{c}_k, \qquad k = 1, \dots, K, \tag{2.14}$$

is satisfied. Then the homogeneous first order system (2.1) is hyperbolic, i.e., (i) the eigenvalues of the matrix $\boldsymbol{B}_n$ are all real but not necessarily distinct and (ii) there exist bases of corresponding left and right eigenvectors that are orthonormal.

Proof. Motivated by the block structure of the matrix $\boldsymbol{B}_n$ we make the following ansatz for computing the corresponding left and right eigenvectors

$$\boldsymbol{R}_n := \begin{pmatrix} \boldsymbol{I}_{K-1} & & & \\ \boldsymbol{R}^1_{I,n} & \boldsymbol{R}_{1,n} & & \\ \vdots & & \ddots & \\ \boldsymbol{R}^K_{I,n} & & & \boldsymbol{R}_{K,n} \end{pmatrix}, \quad \boldsymbol{L}_n := \begin{pmatrix} \boldsymbol{I}_{K-1} & & & \\ \boldsymbol{L}^1_{I,n} & \boldsymbol{L}_{1,n} & & \\ \vdots & & \ddots & \\ \boldsymbol{L}^K_{I,n} & & & \boldsymbol{L}_{K,n} \end{pmatrix} \tag{2.15}$$

with blocks $\boldsymbol{R}^k_{I,n}, \boldsymbol{L}^k_{I,n} \in \mathbb{R}^{(d+1)\times(K-1)}$ and $\boldsymbol{R}_{k,n}, \boldsymbol{L}_{k,n} \in \mathbb{R}^{(d+1)\times(d+1)}$. Here the matrices $\boldsymbol{R}_n$ and $\boldsymbol{L}_n$ are composed of the right and left eigenvectors in their columns and rows, respectively. To determine the blocks $\boldsymbol{R}_{k,n}$ we note that $\lambda_{k,i}$ and $\lambda_{k,\pm}$ are also eigenvalues of the matrix $\boldsymbol{B}_{k,n}$ and the eigenvalue problem for $\boldsymbol{B}_n$ decouples into eigenvalue problems for the matrices $\boldsymbol{B}_{k,n}$ corresponding to a single component. In a first step, we therefore compute the eigenvectors to these sub-problems where we first determine an orthonormal basis $\{\boldsymbol{n}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_{d-1}\}$ of $\mathbb{R}^d$ such that $\boldsymbol{t}_i \cdot \boldsymbol{t}_j = \delta_{i,j}$ and $\boldsymbol{t}_i \cdot \boldsymbol{n} = 0$. Then the right and left eigenvectors to the eigenvalues (2.10b) and (2.10c) are

$$\boldsymbol{r}_{k,i} = (\boldsymbol{t}_i^t, 0)^t, \ \boldsymbol{r}_{k,\pm} = (\pm\overline{c}_k/\rho_k \boldsymbol{n}^t, \overline{c}_k^2)^t, \tag{2.16}$$

$$\boldsymbol{l}_{k,i} = (\boldsymbol{t}_i^t, 0)^t, \ \boldsymbol{l}_{k,\pm} = \frac{1}{2}\overline{c}_k^{-2}(\pm\overline{c}_k\rho_k \boldsymbol{n}^t, 1)^t \tag{2.17}$$

for $i = 1, \dots, d-1$. Thus, there exists an eigenvalue decomposition of the matrix $\boldsymbol{B}_{k,n}$, i.e.,

$$\boldsymbol{L}_{k,n}\boldsymbol{B}_{k,n}\boldsymbol{R}_{k,n} = \boldsymbol{\Lambda}_{k,n}, \tag{2.18}$$

where $\boldsymbol{L}_{k,n}$ and $\boldsymbol{R}_{k,n}$ are defined by the left and right eigenvectors and $\boldsymbol{\Lambda}_{k,n}$ is a diagonal matrix with eigenvalues on the diagonal

$$\boldsymbol{R}_{k,n} := \begin{pmatrix} -\overline{c}_k/\rho_k \boldsymbol{n} & \boldsymbol{t}_1 & \dots & \boldsymbol{t}_{d-1} & \overline{c}_k/\rho_k \boldsymbol{n} \\ \overline{c}_k^2 & 0 & \dots & 0 & \overline{c}_k^2 \end{pmatrix}, \tag{2.19}$$

$$\boldsymbol{L}_{k,n} := \frac{1}{2\overline{c}_k^2}\begin{pmatrix} -\overline{c}_k\rho_k \boldsymbol{n} & 2\overline{c}_k^2 \boldsymbol{t}_1 & \dots & 2\overline{c}_k^2 \boldsymbol{t}_{d-1} & \overline{c}_k\rho_k \boldsymbol{n} \\ 1 & 0 & \dots & 0 & 1 \end{pmatrix}^t, \tag{2.20}$$

$$\boldsymbol{\Lambda}_{k,n} := \begin{pmatrix} v_{k,n} - \overline{c}_k & \boldsymbol{0}^t_{d-1} & 0 \\ \boldsymbol{0}_{d-1} & v_{k,n}\boldsymbol{I}_{d-1} & \boldsymbol{0}_{d-1} \\ 0 & \boldsymbol{0}^t_{d-1} & v_{k,n} + \overline{c}_k \end{pmatrix}. \tag{2.21}$$

To calculate the eigenvectors to the multiple eigenvalue $\lambda_{I,i}$ we employ the knowledge of the matrices $\boldsymbol{R}_{k,n}$. According to the block structure of the matrices $\boldsymbol{R}_n$ and $\boldsymbol{B}_n$ the matrices of corresponding right eigenvectors need to satisfy

$$(\boldsymbol{B}_{k,n} - \lambda_{I,i}\boldsymbol{I}_{d+1})\boldsymbol{R}^k_{I,n} = -\boldsymbol{A}_{k,n}\boldsymbol{R}^0_{I,n} = -\kappa_0 \boldsymbol{A}_{k,n}, \ k = 1, \dots, K.$$

Assuming that the eigenvalue $\lambda_{I,i}$ does not coincide with one of the eigenvalues $\lambda_{k,i}$ and $\lambda_{k,\pm}$, then $\boldsymbol{B}_{k,n} - \lambda_{I,i}\boldsymbol{I}_{d+1}$ is regular and there exists a unique solution for $\boldsymbol{R}^k_{I,n}$. With (2.13) these matrices are given by

$$\boldsymbol{R}^k_{I,n} := -\left(\boldsymbol{B}_{k,n} - V_{I,n}\boldsymbol{I}_{d+1}\right)^{-1}\boldsymbol{A}_{k,n} = \begin{pmatrix} \boldsymbol{n}(\boldsymbol{\gamma}^t_k - \boldsymbol{\beta}^t_k)\delta^n_k/(\alpha_k\sigma_k) \\ (\bar{c}^2_k\boldsymbol{\beta}^t_k - (\delta^n_k)^2\boldsymbol{\gamma}^t_k)\rho_k/(\alpha_k\sigma_k) \end{pmatrix}. \tag{2.22}$$

Since $\alpha_k \in (0,1)$ according to (1.2), the matrices $\boldsymbol{B}_{k,n} - \lambda_{I,i}\boldsymbol{I}_{d+1}$ are regular if and only if $\sigma_k \neq 0$ and $\delta^n_k \neq 0$ for $k = 1,\dots,K$. In particular, the inverse

$$\left(\boldsymbol{B}_{k,n} - V_{I,n}\boldsymbol{I}_{d+1}\right)^{-1} = \begin{pmatrix} (\delta^n_k)^{-1}\boldsymbol{I}_d + \bar{c}^2_k(\delta^n_k\sigma_k)^{-1}\boldsymbol{n}\boldsymbol{n}^t & -(\rho_k\sigma_k)^{-1}\boldsymbol{n} \\ -\rho_k\bar{c}^2_k\sigma_k^{-1}\boldsymbol{n}^t & \delta^n_k\sigma_k^{-1} \end{pmatrix}$$

exists. We emphasize that the right-hand side in (2.22) is well-defined also if $V_{I,n} = v_{k,n}$, i.e., $\delta^n_k = 0$. Thus, $\boldsymbol{R}^k_{I,n}$ is well-defined also if the non-resonance condition (2.14) holds.

Thus, the corresponding left eigenvectors are determined by the rows of the inverse of $\boldsymbol{R}_n$. Since $\boldsymbol{L}_n\boldsymbol{R}_n = \boldsymbol{I}$, the blocks turn out to be

$$\boldsymbol{L}_{k,n} = \boldsymbol{R}^{-1}_{k,n},\ k = 1,\dots,K, \tag{2.23}$$

$$\boldsymbol{L}^k_{I,n} = -\boldsymbol{L}_{k,n}\boldsymbol{R}^k_{I,n},\ k = 1,\dots,K, \tag{2.24}$$

and we obtain for the right eigenvectors

$$\boldsymbol{r}_{I,l} := \left((\overline{\boldsymbol{r}}^0_{I,l})^t, (\overline{\boldsymbol{r}}^1_{I,l})^t, \dots, (\overline{\boldsymbol{r}}^K_{I,l})^t\right)^t,\ l = 1,\dots,K-1. \tag{2.25}$$

where $\overline{\boldsymbol{r}}^0_{I,l} = \boldsymbol{e}_{K-1,l}$ and $\overline{\boldsymbol{r}}^k_{I,l} = (\overline{\boldsymbol{y}}^t_{k,l}, \overline{z}_{k,l})^t$ is determined by the components

$$\overline{\boldsymbol{y}}_{k,l} = \boldsymbol{n}\left(\gamma_{k,l} - \beta_{k,l}\right)\delta^n_k/(\alpha_k\sigma_k), \tag{2.26}$$

$$\overline{z}_{k,l} = \left(\bar{c}^2_k\beta_{k,l} - (\delta^n_k)^2\gamma_{k,l}\right)\rho_k/(\alpha_k\sigma_k) \tag{2.27}$$

Similar to (2.25) the columns of $\boldsymbol{L}_n$ are then given by

$$\boldsymbol{l}_{I,l} := \left((\boldsymbol{l}^0_{I,l})^t, (\boldsymbol{l}^1_{I,l})^t, \dots, (\boldsymbol{l}^K_{I,l})^t\right)^t,\ l = 1,\dots,K-1 \tag{2.28}$$

with

$$\boldsymbol{l}^0_{I,l} = \boldsymbol{e}_{l,K-1}, \quad \boldsymbol{l}^k_{I,l} = \frac{\rho_k}{2\bar{c}^2_k\alpha_k\sigma_k}(a^-_{k,l}, \boldsymbol{0}^t_{d-1}, a^+_{k,l})^t$$

and

$$a^\pm_{k,l} := \left(\delta^n_k\gamma_{k,l} \pm \bar{c}_k\beta_{k,l}\right)\left(\delta^n_k \mp \bar{c}_k\right). \tag{2.29}$$

After having determined the eigenvalues and the corresponding linearly independent right and left eigenvectors we finally end up with the eigenvalue decomposition of the matrix $\boldsymbol{B}_n$

$$\boldsymbol{L}_n\boldsymbol{B}_n\boldsymbol{R}_n = \boldsymbol{\Lambda}_n \tag{2.30}$$

with the block-diagonal matrix $\boldsymbol{\Lambda}_n = \mathrm{diag}(\boldsymbol{\Lambda}_{0,n}, \boldsymbol{\Lambda}_{1,n}, \dots, \boldsymbol{\Lambda}_{K,n})$ and $\boldsymbol{\Lambda}_{0,n} := V_{I,n}\boldsymbol{I}_{K-1}$. To verify this decomposition we make use of the identity $\boldsymbol{B}_{k,n}\boldsymbol{R}^k_{I,n} = \boldsymbol{R}^k_{I,n}\boldsymbol{\Lambda}_{0,n} - \boldsymbol{A}_{k,n}$.

□

For convenience we summarize the representation of the eigenvectors because we want to investigate the type of the characteristic fields.

Proposition 2.1. *(Eigenvectors) Let be $\boldsymbol{n} \in \mathbb{R}^d$ with $|\boldsymbol{n}| = 1$ and let be $\boldsymbol{t}_l \in \mathbb{R}^d$, $l = 1, \ldots, d-1$, such that $\{\boldsymbol{n}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_{d-1}\}$ is an orthonormal basis of $\mathbb{R}^d$, i.e., $\boldsymbol{t}_i \cdot \boldsymbol{t}_j = \delta_{i,j}$ and $\boldsymbol{t}_i \cdot \boldsymbol{n} = 0$.*

(1) (Transport waves) The right and left eigenvalues to the eigenvalues $\lambda_{k,i}$, $k = 1, \ldots, K$, $i = 1, \ldots, d-1$, are

$$\boldsymbol{r}_{k,i} = \boldsymbol{l}_{k,i} = (\mathbf{0}^t_{K-1}, \underbrace{\mathbf{0}^t_{d+1}, \ldots, \mathbf{0}^t_{d+1}}_{(k-1)-times}, (\boldsymbol{t}^t_i, 0), \underbrace{\mathbf{0}^T_{d+1}, \ldots, \mathbf{0}^T_{d+1}}_{(K-k)-times})^t. \tag{2.31}$$

(2) (Acoustic waves) The right and left eigenvalues to the eigenvalues $\lambda_{k,\pm}$, $k = 1, \ldots, K$, are

$$\boldsymbol{r}_{k,\pm} = (\mathbf{0}^t_{K-1}, \underbrace{\mathbf{0}^t_{d+1}, \ldots, \mathbf{0}^t_{d+1}}_{(k-1)-times}, \overline{\boldsymbol{r}}^t_{k,\pm}, \underbrace{\mathbf{0}^t_{d+1}, \ldots, \mathbf{0}^t_{d+1}}_{(K-k)-times})^t \tag{2.32}$$

$$\boldsymbol{l}_{k,\pm} = (\overline{\boldsymbol{l}}^t_{I,k,\pm}, \underbrace{\mathbf{0}^t_{d+1}, \ldots, \mathbf{0}^t_{d+1}}_{(k-1)-times}, \overline{\boldsymbol{l}}^t_{k,\pm}, \underbrace{\mathbf{0}^t_{d+1}, \ldots, \mathbf{0}^t_{d+1}}_{(K-k)-times})^t \tag{2.33}$$

with

$$\overline{\boldsymbol{r}}_{k,\pm} = (\pm \overline{c}_k \rho_k^{-1} \boldsymbol{n}^t, \overline{c}_k^2)^t, \tag{2.34}$$

$$\overline{\boldsymbol{l}}_{k,\pm} = \frac{1}{2} \overline{c}_k^{-2} (\pm \overline{c}_k \rho_k \boldsymbol{n}^t, 1)^t, \ \overline{\boldsymbol{l}}_{I,k,\pm} = \frac{1}{2} \rho_k \overline{c}_k^{-2} (\alpha_k \sigma_k)^{-1} \boldsymbol{a}^{\pm}_k \tag{2.35}$$

and the components of the vector $\boldsymbol{a}^{\pm}_k \in \mathbb{R}^{K-1}$ determined by (2.29).

(3) (Material waves) If the non-resonance condition (2.14) holds, then the right and left eigenvalues to the eigenvalues $\lambda_{I,k}$, $k = 1, \ldots, K-1$, are

$$\boldsymbol{r}^t_{I,k} = \left(\boldsymbol{e}^t_{K-1,k}, (\overline{\boldsymbol{r}}^1_{I,k})^t, \ldots, (\overline{\boldsymbol{r}}^K_{I,k})^t\right)^t, \tag{2.36}$$

$$\boldsymbol{l}_{I,k} = (\boldsymbol{e}^t_{K-1,k}, \underbrace{\mathbf{0}^t_{d+1}, \ldots, \mathbf{0}^t_{d+1}}_{K-times})^t, \tag{2.37}$$

where $\overline{\boldsymbol{r}}^l_{I,k} = (\overline{\boldsymbol{y}}^t_{l,k}, \overline{z}_{l,k})^t$ is determined by the components (2.26) and (2.27). Here $\boldsymbol{e}_{K-1,k}$ is the k^{th} unit vector in $\mathbb{R}^{K-1}$.

In particular, the left and right eigenvectors are orthonormal.

The eigenvectors are determined for the system (2.4) in primitive variables. Numerical simulations reported in [22] revealed that the matrices of right and left eigenvectors $\boldsymbol{R}_n(\boldsymbol{w})$ and $\boldsymbol{L}_n(\boldsymbol{w})$ may have very large condition numbers. Therefore it is recommended to use the eigensystem for the original variables. For convenience of the reader these systems are given in Appendix A.

2.2. **Characteristic fields**

By means of the eigenvalues and the right eigenvectors we can now characterize the characteristic fields.

Proposition 2.2. *(Characteristic fields)*

(1) The fields corresponding to the eigenvalue $\lambda_{k,i}$ are linearly degenerated.
(2) If the assumptions (1.27) hold, then the acoustic waves are genuinely nonlinear.
(3) If the interfacial velocity is a convex combination of the velocities $\boldsymbol{v}_k$, i.e.,

$$\boldsymbol{V}_I = \sum\nolimits_{k=1}^{K} \beta_k \boldsymbol{v}_k, \quad \beta_k \in [0,1], \quad \sum\nolimits_{k=1}^{K} \beta_k = 1 \tag{2.38}$$

with coefficients

$$\beta_k := \overline{\gamma}_k \alpha_k \rho_k / \hat{\rho}, \quad \hat{\rho} := \sum\nolimits_{k=1}^{K} \overline{\gamma}_k \alpha_k \rho_k, \tag{2.39}$$

for some arbitrary but fixed constants $\overline{\gamma}_k \in [0,1]$, *then the material waves are linearly degenerated.*

Proof. The characteristic fields corresponding to the different waves are characterized by the directional derivative $\nabla_{\boldsymbol{w}} \lambda(\boldsymbol{w}) \cdot \boldsymbol{r}(\boldsymbol{w})$.

Obviously, the derivatives of the eigenvalues $\lambda_{k,i} = v_{k,n}$, $k = 1, \dots, K$, $i = 1, \dots, d$, are

$$\partial_{\alpha_l} \lambda_{k,i} = 0, \ \partial_{v_{l,j}} \lambda_{k,i} = \delta_{k,l}\, n_j, \ \partial_{p_l} \lambda_{k,i} = 0,$$

and, thus, we conclude from the corresponding right eigenvectors (2.31)

$$\nabla_{\boldsymbol{w}} \lambda_{k,i}(\boldsymbol{w}) \cdot \boldsymbol{r}_{k,i}(\boldsymbol{w}) = 0,$$

i.e., these fields are linearly degenerated.

Next we consider the acoustic waves corresponding to the eigenvalues $\lambda_{k,\pm} = v_{k,n} \pm \overline{c}_k$. Obviously, it holds

$$\partial_{\alpha_l} v_{k,n} = 0, \ \partial_{v_{l,j}} \lambda_{k,i} = \delta_{k,l}\, n_j, \ \partial_{p_l} v_{k,n} = 0.$$

To determine the derivatives of the barotropic speed of sound $\overline{c}_k$ we perform a change of variables $\rho_k = \rho_k(p_k)$. Note that $p_k = p_k(\rho_k)$ is a monotone function according to the assumption (1.27a). Since it holds

$$\partial_* \overline{c}_k^2(\rho_k) = 2\overline{c}_k(\rho_k) \partial_* \overline{c}_k(\rho_k) = p_k''(\rho_k) \partial_* \rho_k,$$

the derivatives of c_k with respect to the primitive variables $\boldsymbol{w}$ read

$$\partial_{\alpha_l} \overline{c}_k = 0, \ \partial_{v_{l,j}} \overline{c}_k = 0, \ \partial_{p_l} \overline{c}_k = \delta_{k,l} \frac{1}{2\,\overline{c}_k^3(\rho_k)} p_k''(\rho_k)$$

and we obtain by the right eigenvectors (2.32)

$$\nabla_{\boldsymbol{w}} \lambda_{k,\pm}(\boldsymbol{w}) \cdot \boldsymbol{r}_{k,\pm}(\boldsymbol{w}) = \pm \frac{\overline{c}_k}{\rho_k} \mathcal{G}_k$$

with $\mathcal{G}_k$ the fundamental derivative of gas dynamics for a pure barotropic fluid corresponding to component k. According to the assumptions (1.27) both the barotropic speed of sound (1.28) and the fundamental derivative (1.27b) do not vanish, i.e., the acoustic waves are genuinely nonlinear.

Finally, we consider the characteristic fields corresponding to the eigenvalues $\lambda_{I,k} = V_{I,n}$. From the corresponding right eigenvectors (2.36) we conclude

$$\boldsymbol{r}_{I,i} \cdot \nabla_{\boldsymbol{w}} \lambda_{I,i} = \partial_{\alpha_i} V_{I,n} + \sum\nolimits_{k=1}^{K} \left(\overline{\boldsymbol{y}}_{k,i} \cdot \nabla_{\boldsymbol{v}_k} V_{I,n} + \overline{z}_{k,i} \partial_{p_k} V_{I,n} \right). \tag{2.40}$$

For the gradient of the interfacial velocity it holds

$$\partial_{\alpha_i} V_{I,n} = \frac{\beta_i}{\alpha_i} \delta_i^n - \frac{\beta_K}{\alpha_K} \delta_K^n, \ \nabla_{\boldsymbol{v}_k} V_{I,n} = \beta_k \boldsymbol{n}, \ \partial_{p_k} V_{I,n} = \frac{\beta_k}{\rho_k} \frac{1}{\overline{c}_k^2} \delta_k^n. \tag{2.41}$$

By a straightforward calculation incorporating the coefficients (2.26), (2.27) and using the convex combination (2.38) with coefficients (2.39) we can verify that the right-hand side in (2.40) vanishes, i.e., the material waves are linearly degenerated.

□

Remark 2.2. *If the material field is genuinely nonlinear field, then there is no way to cope with the non-conservative products in the phasic momentum, see Eqn. (1.8c). However, if fields associated to the non-conservative products only occur in linearly degenerated fields, then Riemann invariants of the associated field can be enforced and, thus, the exact solution of the Riemann problem exists as has been verified for Baer-Nunziato-type models in [11]. Therefore, Gallouët et al. [10] and Hérard [24] for a two-phase and a three-phase model, respectively, suggest to determine the interfacial velocity such that the fields associated with the eigenvalues $\lambda_{I,i}$, $i = 1, \dots, K-1$, are linearly degenerated. There the interfacial velocity is chosen as a convex combination of the velocities of the components, see Eqn. (2.38) with state-dependent coefficients β_k. Already in the thesis of Guillemaud [16] several options are discussed for the choice of these coefficients. Also coefficients that do not comply with a linearly degenerated characteristic field are investigated. For the coefficients $\overline{\gamma}_k$ the ansatz of Saleh [49], p. 195, Eqn. (4.3.40), in case of a two-phase mixture has been generalized providing (2.39). Finally, it is worthwhile mentioning that there exist other suitable but more complex closure laws in the literature, e.g. [4].*

2.3. **Riemann invariants**

In the following proposition we give for all characteristic fields a set of Riemann invariants whose gradients are linearly independent.

Proposition 2.3. *(Riemann invariants) The resonance condition (2.14) is assumed to hold. Introduce the functions*

$$\Psi_{j,l} := \boldsymbol{v}_j \cdot \boldsymbol{t}_l,\ j = 1, \dots, K,\ l = 1, \dots, d-1, \tag{2.42a}$$

$$\Psi_{I,j} := \alpha_j,\ j = 1, \dots, K-1, \tag{2.42b}$$

$$\Psi_{j,\pm} := \boldsymbol{v}_j \cdot \boldsymbol{n} \pm \int (\overline{c}_j \rho_j)^{-1} dp_j,\ j = 1, \dots, K. \tag{2.42c}$$

(1) For the transport waves corresponding to the right eigenvectors $\boldsymbol{r}_{k,i}$, $k = 1, \dots, K$, $i = 1, \dots, d-1$, the functions $\Psi_{j,l}$, $j = 1, \dots, K$, $j \neq k$, $l = 1, \dots, d-1$, $\Psi_{k,l}$, $l = 1, \dots, d-1$, $l \neq i$, $\Psi_{I,j}$, $j = 1, \dots, K-1$ and $\Psi_{j,\pm}$, $j = 1, \dots, K$ form a system of $K(d+2)-2$ Riemann invariants whose gradients are linearly independent. Other Riemann invariants are

$$\Psi = \boldsymbol{v}_k \cdot \boldsymbol{n} = v_{k,n} = \lambda_{k,i}, \tag{2.43a}$$

$$\Psi = v_{l,j},\ l = 1, \dots, K,\ l \neq k,\ j = 1, \dots, d, \tag{2.43b}$$

$$\Psi = p_l,\ l = 1, \dots, K. \tag{2.43c}$$

(2) For the acoustic waves corresponding to the right eigenvectors $\boldsymbol{r}_{k,\star}$, $k = 1, \dots, K$, $\star \in \{+, -\}$, the functions $\Psi_{j,l}$, $j = 1, \dots, K$, $l = 1, \dots, d-1$, $\Psi_{I,j}$, $j = 1, \dots, K-1$, and $\Psi_{j,\pm}$, $j = 1, \dots, K$, $(j, \pm) \neq (k, \star)$ form a system of $K(d+2)-2$ Riemann invariants whose gradients are linearly independent. Other Riemann invariants are

$$\Psi = v_{l,j},\ l = 1, \dots, K,\ l \neq k,\ j = 1, \dots, d, \tag{2.44a}$$

$$\Psi = p_l,\ l = 1, \dots, K,\ l \neq k. \tag{2.44b}$$

(3) For the material waves corresponding to the right eigenvectors $\boldsymbol{r}_{I,k}$, $k = 1, \dots, K-1$, the functions $\Psi_{j,l}$, $j = 1, \dots, K$, $l = 1, \dots, d-1$, $\Psi_{I,j}$, $j = 1, \dots, K-1$, $j \neq k$ and $\Psi_{j,\pm}$, $j = 1, \dots, K$ form a system of $K(d+2)-2$ Riemann invariants whose gradients are linearly independent.
Additionally assuming that the interfacial velocity is a convex combination of the velocities $\boldsymbol{v}_k$ according to (2.38) with coefficients (2.39) other Riemann invariants are

$$\Psi = V_{I,n} = \lambda_{I,k}, \tag{2.45a}$$

$$\Psi = \alpha_j \rho_j (\boldsymbol{v}_{j,n} - V_{I,n}),\ j = 1, \dots, K-1, \tag{2.45b}$$

$$\Psi = \sum\nolimits_{j=1}^{K} \left(\alpha_j p_j + \alpha_j \rho_j (\boldsymbol{v}_{j,n} - V_{I,n})^2 \right). \tag{2.45c}$$

Additionally assuming that the interfacial velocity and the interfacial pressures are $\boldsymbol{V}_I = \boldsymbol{v}_K$ *and* $P_{k,l} = \beta_k p_l + p_k(1-\beta_k)$, $k, l = 1, \dots, K$, $k \neq l$, *then for a general barotropic fluid* (1.7) *another family of Riemann invariants is given by*

$$\Psi = \overline{e}_j + \frac{p_j}{\rho_j} + \frac{1}{2}(\boldsymbol{v}_{j,n} - V_{I,n})^2,\ j = 1, \dots, K-1, \tag{2.45d}$$

where $\overline{e}_j$ *is the "internal energy" according to Remark 1.1.*

Proof. First of all, we determine the gradients of the functions defined in (2.42)

$$\nabla_{\boldsymbol{w}} \Psi_{j,i} = \boldsymbol{l}_{j,i},\ \nabla_{\boldsymbol{w}} \Psi_{I,j} = \boldsymbol{l}_{I,j},\ \nabla_{\boldsymbol{w}} \Psi_{j,\pm} = \pm \left(2\frac{\overline{c}_j}{\rho_j} \boldsymbol{l}_{j,\pm} - \frac{1}{\alpha_j \sigma_j \overline{c}_j} \sum\nolimits_{m=1}^{K} a^{\pm}_{j,m} \boldsymbol{l}_{I,m} \right),$$

where σ_j and $a^{\pm}_{j,m}$ are defined by (2.13) and (2.29). Since these gradients are linear combinations of the left eigenvectors it follows by the construction of the left and right eigenvectors, see Proposition 2.1,

$$\boldsymbol{r}_{k,i} \cdot \nabla_{\boldsymbol{w}} \Psi_{j,i} = 0,\ \boldsymbol{r}_{k,\star} \cdot \nabla_{\boldsymbol{w}} \Psi_{I,j} = 0,\ \boldsymbol{r}_{I,k} \cdot \nabla_{\boldsymbol{w}} \Psi_{j,\pm} = 0$$

for the functions specified in properties (1), (2) and (3), respectively. Therefore these functions are Riemann invariants. The hyperbolicity, see Proposition 2.1, ensures that the left eigenvectors and, thus, the gradients, are linearly independent.
Next we verify that (2.43) are also Riemann invariants for the material wave corresponding to the right eigenvector $\boldsymbol{r}_{k,i}$, i.e., the following needs to hold

$$\boldsymbol{r}_{k,i} \cdot \nabla_{\boldsymbol{w}} \Psi = \partial_{\boldsymbol{v}_k} \Psi \cdot \boldsymbol{t}_i = 0.$$

For the functions Ψ given in (2.43) this condition can be easily verified using that the vectors $\{\boldsymbol{n}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_{d-1}\}$ are orthonormal to each other.
To verify that (2.44) are also Riemann invariants for the acoustic wave corresponding to the right eigenvector $\boldsymbol{r}_{k,\star}$ we need

$$\boldsymbol{r}_{k,\pm} \cdot \nabla_{\boldsymbol{w}} \Psi = \pm \overline{c}_k \rho_k^{-1} \partial_{\boldsymbol{v}_k} \Psi \cdot \boldsymbol{n} + \overline{c}_k^2 \partial_{p_k} \Psi = 0.$$

For the functions Ψ given in (2.44) this condition can be easily verified using similar arguments as before.
Finally, we verify that (2.45) are also Riemann invariants to the material wave corresponding to the right eigenvector $\boldsymbol{r}_{I,k}$, i.e.,

$$\boldsymbol{r}_{I,k} \cdot \nabla_{\boldsymbol{w}} \Psi = \partial_{\alpha_i} \Psi + \sum\nolimits_{l=1}^{K} \left(\overline{\boldsymbol{y}}_{l,i} \cdot \nabla_{\boldsymbol{v}_l} \Psi + \overline{z}_{l,i} \partial_{p_l} \Psi \right) = 0.$$

For (2.45a) this holds true by construction of the coefficients (2.39).
For (2.45b) the gradient is determined by

$$\begin{aligned}
\partial_{\alpha_l} \Psi &= \delta_{l,j} \rho_j \left(v_{j,n} - V_{I,n} \right) - \alpha_j \rho_j \partial_{\alpha_l} V_{I,n}, \\
\nabla_{\boldsymbol{v}_l} \Psi &= \alpha_j \rho_j \delta_{l,j} \boldsymbol{n} - \alpha_j \rho_j \nabla_{\boldsymbol{v}_l} V_{I,n}, \\
\partial_{p_l} \Psi &= \alpha_j \rho_j'(p_j) \delta_{l,j} \left(v_{j,n} - V_{I,n} \right) - \alpha_j \rho_j \partial_{p_l} V_{I,n}.
\end{aligned}$$

Then we obtain

$$\boldsymbol{r}_{I,k}\cdot\nabla_{\boldsymbol{w}}\Psi=\delta_{j,k}\rho_j\left(v_{j,n}-V_{I,n}\right)+\alpha_j\rho_j\overline{\boldsymbol{y}}_{j,k}\cdot\boldsymbol{n}+\alpha_j\rho_j'(p_j)\overline{z}_{j,k}\left(v_{j,n}-V_{I,n}\right)-\alpha_j\rho_j\boldsymbol{r}_{I,k}\cdot\nabla_{\boldsymbol{w}}V_{I,n}.$$

The last term on the right-hand side vanishes because $\Psi=V_{I,n}$ is a Riemann invariant. The remaining terms vanish when plugging in (2.26), (2.27), (2.13) and using $\rho_j'(p_j)=1/p_j'(\rho_j)=1/\overline{c}_j^2$.
For (2.45c) the gradient is determined by

$$\begin{aligned}
\partial_{\alpha_l}\Psi&=p_l-p_K+\rho_l\left(v_{l,n}-V_{I,n}\right)^2-2\sum\nolimits_{j=1}^{K}\alpha_j\rho_j\left(v_{j,n}-V_{I,n}\right)\partial_{\alpha_l}V_{I,n},\\
\nabla_{\boldsymbol{v}_l}\Psi&=2\alpha_l\rho_l\left(v_{l,n}-V_{I,n}\right)\boldsymbol{n}-2\sum\nolimits_{j=1}^{K}\alpha_j\rho_j\left(v_{j,n}-V_{I,n}\right)\nabla_{\boldsymbol{v}_l}V_{I,n},\\
\partial_{p_l}\Psi&=\alpha_l\left(1+\rho_l'(p_l)\left(v_{l,n}-V_{I,n}\right)^2\right)-2\sum\nolimits_{j=1}^{K}\alpha_j\rho_j\left(v_{j,n}-V_{I,n}\right)\partial_{p_l}V_{I,n}.
\end{aligned}$$

This yields

$$\begin{aligned}
&\boldsymbol{r}_{I,k}\cdot\nabla_{\boldsymbol{w}}\Psi=\\
&p_k-p_K+\rho_i\left(v_{k,n}-V_{I,n}\right)^2+\sum\nolimits_{j=1}^{K}\alpha_j\left(2\rho_j\left(v_{j,n}-V_{I,n}\right)\overline{\boldsymbol{y}}_{j,k}\cdot\boldsymbol{n}+\left(1+\rho_j'(p_j)\left(v_{j,n}-V_{I,n}\right)^2\right)\overline{z}_{j,k}\right)\\
&-2\sum\nolimits_{j=1}^{K}\alpha_j\rho_j\left(v_{j,n}-V_{I,n}\right)\boldsymbol{r}_{I,k}\cdot\nabla_{\boldsymbol{w}}V_{I,n}.
\end{aligned}$$

The last term on the right-hand side vanishes because $\Psi=V_{I,n}$ is a Riemann invariant. Again, using $\rho_j'(p_j)=1/\overline{c}_j^2$ and employing (2.13), the remaining terms simplify to

$$\boldsymbol{r}_{I,k}\cdot\nabla_{\boldsymbol{w}}\Psi=p_k-p_K+\rho_k\left(\delta_k^n\right)^2-\sum\nolimits_{j=1}^{K}\rho_l\left(\gamma_{j,k}\left(\frac{\delta_j^n}{\overline{c}_j}\right)^2+\beta_{j,k}\right).$$

Plugging in (2.26) and (2.27) the right-hand side vanishes using (1.12).
Finally, we investigate (2.45d). Introducing the phasic specific enthalpies

$$\overline{h}_j(\rho_j):=\overline{e}_j(\rho_j)+p_j(\rho_j)/\rho_j$$

the gradient is determined by

$$\begin{aligned}
\partial_{\alpha_l}\Psi&=-\left(v_{j,n}-V_{I,n}\right)\partial_{\alpha_l}V_{I,n},\\
\nabla_{\boldsymbol{v}_l}\Psi&=\left(v_{j,n}-V_{I,n}\right)\delta_{j,l}\boldsymbol{n}-\left(v_{j,n}-V_{I,n}\right)\nabla_{\boldsymbol{v}_l}V_{I,n},\\
\partial_{p_l}\Psi&=h_j'(\rho_j)\rho_j'(p_j)\delta_{j,l}-\left(v_{j,n}-V_{I,n}\right)\partial_{p_l}V_{I,n}.
\end{aligned}$$

Then it holds

$$\boldsymbol{r}_{I,k}\cdot\nabla_{\boldsymbol{w}}\Psi=\sum\nolimits_{l=1}^{K}\left(\left(v_{j,n}-V_{I,n}\right)\delta_{j,l}\overline{\boldsymbol{y}}_{l,k}\cdot\boldsymbol{n}+h_j'(\rho_j)\rho_j'(p_j)\delta_{j,l}\overline{z}_{l,k}\right)-\left(v_{j,n}-V_{I,n}\right)\boldsymbol{r}_{I,k}\cdot\nabla_{\boldsymbol{w}}V_{I,n}.$$

Again, the last term on the right-hand side vanishes because $\Psi_I=V_{I,n}$ is a Riemann invariant. Using (2.26), (2.27) and (2.13) the remaining terms on the right-hand side become

$$\boldsymbol{r}_{I,k}\cdot\nabla_{\boldsymbol{w}}\Psi=\frac{1}{\alpha_j\sigma_j}\left(\left(\delta_j^n\right)^2\gamma_{j,k}\left(1-\rho_jh_j'(\rho_j)\rho_j'(p_j)\right)+\beta_{j,k}\left(\overline{c}_j^2\rho_jh_j'(\rho_j)\rho_j'(p_j)-\left(\delta_j^n\right)^2\right)\right).$$

From (1.7) and (1.28) we deduce $\rho_j'(p_j) = 1/\overline{c}_j^2$ implying $\overline{h}_j'(\rho_j) = \overline{c}_j^2/\rho_j$. Then we obtain with (2.13)

$$\boldsymbol{r}_{I,k} \cdot \nabla_{\boldsymbol{w}} \Psi = \frac{\beta_{j,k}}{\alpha_j \sigma_j} \left(\overline{c}_j^2 - \left(\delta_j^n\right)^2 \right) = -\frac{\beta_{j,k}}{\alpha_j}.$$

Due to the assumptions on the interfacial velocity and the interfacial pressures we obtain by definition (2.7) of the coefficients $\beta_{j,k}$ and the interfacial velocities (2.38) with coefficients (2.39)

$$\beta_{j,k} = \frac{\overline{\gamma}_j}{\alpha_j \hat{\rho}} (p_j - p_K)$$

for $j, k \in \{1, \ldots, K-1\}$. From the assumption $\boldsymbol{V}_I = \boldsymbol{v}_K$ we conclude $\beta_i = \overline{\gamma}_i = 0$, $i = 1, \ldots, K-1$, and $\beta_K = 1$. Thus, the right-hand side vanishes. This verifies the assertion.

□

Remark 2.3. *Finally, we would like to comment on existing results in the literature. In [50] the eigenstructure of the 1D homogeneous barotropic Baer-Nunziato model is investigated for the particular choice of the interfacial velocity $V_I = v_1$ and the interfacial pressures*

$$\begin{aligned} k = 1: &\ P_{1,l} = p_l, \quad l = 2, \ldots, K \\ k > 1: &\ P_{1,l} = p_l, \quad l = 1, \ldots, K,\ l \neq k, \end{aligned}$$

specifying the eigenvalues and the corresponding right eigenvectors for non-resonance states. For this particular setting the authors investigate the corresponding characteristic fields. In particular, they determine the Riemann invariants for the linearly degenerated fields corresponding to the eigenvalues $\lambda_{I,k}$, $k = 1, \ldots, K-1$. Note that the Riemann invariants (2.45) *coincide with those given in [50]. In contrast to this, we give the full eigenstructure and the corresponding characteristic fields for the quasi-1D model using a much larger class of interfacial velocity and interfacial pressures.*

For a three-component model, i.e., $K = 3$, the eigenvalues are also given in [2] for the quasi-1D model but without the corresponding eigenvectors. There, in addition, the Riemann invariants are derived for the 1D model.

We would like to point out that in [2, 50] the authors make explicit use of the "energies" $\overline{e}_k$ introduced in Remark 1.1 which coincide with e_k only for isentropic fluids.

2.4. **Symmetrization**

From Kato's theorem [32] it follows that there exists a local-in-time smooth solution to the Cauchy problem of the projected system (2.4), if the problem is symmetrizable, i.e., there exists a symmetric positive definite matrix $\boldsymbol{P}_n = \boldsymbol{P}_n(\boldsymbol{w})$ such that the matrix $\boldsymbol{P}_n \boldsymbol{B}_n$ is symmetric. It can be verified that the quasi-1D system (2.4) can be symmetrized.

Theorem 2.3. *(Symmetrization) If the non-resonance condition (2.14) holds, then the quasi-1D system (2.4) is symmetrizable.*

Proof. To construct a symmetrizer for $\boldsymbol{B}_n$ we first observe that the matrix

$$\boldsymbol{P}_{k,n} = \begin{pmatrix} \boldsymbol{T}_n \boldsymbol{D}_{k,n} \boldsymbol{T}_n^t & \boldsymbol{0} \\ \boldsymbol{0}^t & 1/(2\overline{c}_k^4) \end{pmatrix}$$

with $\boldsymbol{T}_n := (\boldsymbol{n}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_{d-1})$ an orthonormal matrix and $\boldsymbol{D}_{k,n} := \operatorname{diag}(\lambda_{n,k}, \lambda_1, \ldots, \lambda_{d-1})$ a diagonal matrix with entries $\lambda_{k,n} := \rho_k^2/(2\overline{c}_k^2)$ and $\lambda_i := 1$, $i = 1, \ldots, d-1$ is a symmetrizer of the phasic problem

$$\partial_t \boldsymbol{w}_k + \boldsymbol{B}_{k,n}(\boldsymbol{w}_k)\, \partial_\xi \boldsymbol{w}_k = \boldsymbol{0},$$

i.e., $\boldsymbol{P}_{k,n}$ is a symmetric positive definite matrix and the matrix $\boldsymbol{P}_{k,n}\boldsymbol{B}_{k,n}$ is symmetric. Then we make the following ansatz for a symmetrizer of (2.4):

$$\boldsymbol{P}_n = \begin{pmatrix} K\,P_{\alpha,n}\boldsymbol{I}_{K-1} & \boldsymbol{P}^t_{1,\alpha,n} & \cdots & \boldsymbol{P}^t_{K,\alpha,n} \\ \boldsymbol{P}_{1,\alpha,n} & \boldsymbol{P}_{1,n} & & \\ \vdots & & \ddots & \\ \boldsymbol{P}_{K,\alpha,n} & & & \boldsymbol{P}_{K,n} \end{pmatrix},$$

where $\boldsymbol{P}_{k,\alpha,n} = \boldsymbol{L}^t_{k,n}\left(\boldsymbol{\Lambda}_{k,n} - V_{I,n}\boldsymbol{I}_{d+1}\right)^{-1}\boldsymbol{R}^t_{k,n}\boldsymbol{P}_k\boldsymbol{A}_{k,n}$ with $\boldsymbol{A}_{k,n}$, $\boldsymbol{R}_{k,n}$, $\boldsymbol{L}_{k,n}$ and $\boldsymbol{\Lambda}_{k,n}$ defined by (2.6), (2.19), (2.20) and (2.21). Note that $\boldsymbol{P}_{k,\alpha,n}$ is well-defined if the non-resonance condition (2.14) holds true. Obviously, $\boldsymbol{P}_n$ is symmetric. It turns out that $\boldsymbol{P}_n\boldsymbol{B}_n$ is symmetric whenever $\boldsymbol{P}^t_{k,\alpha,n}\boldsymbol{A}_{k,n}$ is symmetric. The latter holds true provided that $\boldsymbol{P}_{k,n}\boldsymbol{R}_{k,n} = \boldsymbol{L}^t_{k,n}$. It remains to verify that $\boldsymbol{P}_n$ is positive definite. For this purpose we have to verify for any $\boldsymbol{a} = (\boldsymbol{a}^t_\alpha, \boldsymbol{a}^t_1, \ldots, \boldsymbol{a}^t_K)^t \neq \boldsymbol{0}$ with $\boldsymbol{a}_\alpha \in \mathbb{R}^{K-1}$, $\boldsymbol{a}_k \in \mathbb{R}^{d+1}$, $k = 1, \ldots, K$ that $\boldsymbol{a}^t\boldsymbol{P}_n\boldsymbol{a}$ is positive. A straightforward calculus yields

$$\boldsymbol{a}^t\boldsymbol{P}_n\boldsymbol{a} = P_{\alpha,n}\sum\nolimits_{k=1}^{K-1}\sum\nolimits_{i=1}^{K-1}\left(a_{\alpha,i} + (\boldsymbol{P}^t_{k,\alpha,n}\boldsymbol{a}_k)_i/P_{\alpha,n}\right)^2 + \sum\nolimits_{k=1}^{K} P^{-1}_{\alpha,n}\boldsymbol{a}^t_k\boldsymbol{Q}\boldsymbol{a}_k$$

with $\boldsymbol{Q} := P_{\alpha,n}\boldsymbol{P}_k - \boldsymbol{P}_{k,\alpha,n}\boldsymbol{P}^t_{k,\alpha,n}$. Since $\boldsymbol{P}_k$ is symmetric positive definite, the Cholesky decomposition $\boldsymbol{P}_k = \boldsymbol{C}_k\boldsymbol{C}^t_k$ exists. Furthermore, the matrix $\boldsymbol{E}_k := \boldsymbol{C}^{-1}_k\boldsymbol{P}^t_{k,\alpha,n}\boldsymbol{P}_{k,\alpha,n}\boldsymbol{C}^t_k$ is symmetric and, thus, there exists an orthogonal matrix $\boldsymbol{T}_k$ such that $\boldsymbol{T}_k\boldsymbol{E}_k\boldsymbol{T}^t_k = \boldsymbol{D}_k$ where $\boldsymbol{D}_k$ is a diagonal matrix with the eigenvalues μ^k_i of $\boldsymbol{E}_k$ as entries. Then we obtain

$$\boldsymbol{a}^t_k\boldsymbol{Q}\boldsymbol{a}_k = \boldsymbol{b}^t_k(P_{\alpha,n}\boldsymbol{I}_{d+1} - \boldsymbol{D}_k)\boldsymbol{b}_k = \sum\nolimits_{i=1}^{d+1} b^2_{k,i}\left(P_{\alpha,n} - \mu^k_i\right)$$

with $\boldsymbol{b}_k := \boldsymbol{T}^t_k\boldsymbol{C}^t_k\boldsymbol{a}_k$. Choosing $P_{\alpha,n} > \max_{i,k}\left\{|\mu^k_i|\right\} > 0$ the term $\boldsymbol{a}^t_k\boldsymbol{Q}\boldsymbol{a}_k$ is non-negative and is positive for $\boldsymbol{a}_k \neq \boldsymbol{0}$.

□

The idea of the proof is very similar to the proof in case of the quasi-1D full Baer-Nunziato model, see [38]. However, there the proof only works for the genuine 1D case, i.e., $d = 1$, because the symmetrizer for the corresponding phasic matrices $B_{k,n}$ was chosen too specific. It is worthwhile mentioning that modifying the symmetrizer $\boldsymbol{P}_k$ for the phasic model in [38], Section 4.3, replacing $0.5(\rho_k/\bar{c}_k)^2\,\boldsymbol{I}_d$ by $\boldsymbol{T}_n\boldsymbol{D}_{k,n}\boldsymbol{T}^t_n$, the proof also works for the quasi-1D case. Furthermore, we would like to mention that Theorem 2.3 generalizes the result for the genuinely 1D case in [50].

3. Mathematical Entropy

From a mathematical point of view, the concept of entropy-entropy flux pairs, cf. [12], has been introduced to characterize a unique weak solution of an initial (boundary) value problem of (inhomogeneous) conservation laws that in quasi-conservative form reads

$$\partial_t\boldsymbol{u} + \sum\nolimits_{i=1}^{d}\boldsymbol{A}_i(\boldsymbol{u})\partial_{x_i}\boldsymbol{u} = \boldsymbol{S}(\boldsymbol{u}) - \sum\nolimits_{i=1}^{d}\boldsymbol{B}_i(\boldsymbol{u})\partial_{x_i}\boldsymbol{u}, \quad \boldsymbol{A}_i(\boldsymbol{u}) := \partial_{\boldsymbol{u}}\boldsymbol{f}_i(\boldsymbol{u}) \tag{3.1}$$

where $\boldsymbol{u} : \mathbb{R}_+ \times \Omega \to D \subset \mathbb{R}^m$ with $\Omega \subset \mathbb{R}^d$, $\boldsymbol{f}_i : D \to \mathbb{R}^m$, $i = 1, \ldots, d$ and $\boldsymbol{S} : D \to \mathbb{R}^m$, denote the vector of m conserved quantities, the fluxes in the ith coordinate direction, $i = 1, \ldots, d$, and the source function, respectively. Here the last term on the right-hand side contains the non-conservative products in the original system (1.8). Motivated by thermodynamics, the entropy inequality

$$\partial_t U(\boldsymbol{u}) + \sum\nolimits_{i=1}^{d}\partial_{x_i}\boldsymbol{F}_i(\boldsymbol{u}) \leq 0 \tag{3.2}$$

has to hold in a weak sense for any convex function $U : D \to \mathbb{R}$ and functions $F_i : D \to \mathbb{R}$, $i = 1, \dots, d$, referred to as entropy and entropy flux, that satisfy the compatibility conditions

$$\nabla_{\boldsymbol{u}} U(\boldsymbol{u})^T \, \boldsymbol{A}_i(\boldsymbol{u}) = \nabla_{\boldsymbol{u}} F_i(\boldsymbol{u})^T, \ i = 1, \dots, d. \tag{3.3}$$

Due to these conditions we infer for smooth solutions of (3.1) the entropy equation

$$\partial_t U(\boldsymbol{u}) + \sum\nolimits_{i=1}^{d} \partial_{x_i} F_i(\boldsymbol{u}) = \nabla_{\boldsymbol{u}} U(\boldsymbol{u})^T \left(\boldsymbol{S}(\boldsymbol{u}) - \sum\nolimits_{i=1}^{d} \boldsymbol{B}_i(\boldsymbol{u}) \partial_{x_i} \boldsymbol{u} \right), \tag{3.4}$$

Obviously, the entropy inequality (3.2) holds if and only if the entropy production is non-positive, i.e.,

$$\nabla_{\boldsymbol{u}} U(\boldsymbol{u})^T \left(\boldsymbol{S}(\boldsymbol{u}) - \sum\nolimits_{i=1}^{d} \boldsymbol{B}_i(\boldsymbol{u}) \partial_{x_i} \boldsymbol{u} \right) \leq 0. \tag{3.5}$$

A sufficient condition is given by

$$\nabla_{\boldsymbol{u}} U(\boldsymbol{u})^T \, \boldsymbol{S}(\boldsymbol{u}) \leq 0 \quad \text{and} \quad \nabla_{\boldsymbol{u}} U(\boldsymbol{u})^T \sum\nolimits_{i=1}^{d} \boldsymbol{B}_i(\boldsymbol{u}) \partial_{x_i} \boldsymbol{u} = 0. \tag{3.6}$$

In the following we derive two pairs of entropy-entropy flux pairs for isothermal fluids starting from a thermodynamical point of view and mechanical point of view, see Sects. 3.1 and 3.2. Note that for general barotropic fluids and, in particular, isentropic fluids only the latter will provide an entropy-entropy flux pair.

3.1. **Entropy-entropy flux pair: thermodynamical approach**

Here we confine ourselves on isothermal fluids and extend the analysis for the isothermal Euler equations, cf. [54], to our model (1.8) where we have to account for heat exchange in the energy equation. We therefore have to start from the full Baer-Nunziato system (1.1) to derive an appropriate entropy-entropy flux pair. From (1.1) we derive equations for the density and internal energy

$$\partial_t \rho_k + \frac{\rho_k}{\alpha_k} (\boldsymbol{v}_k - \boldsymbol{V}_I) \cdot \nabla \alpha_k + \boldsymbol{v}_k \cdot \nabla \rho_k + \rho_k \nabla \cdot \boldsymbol{v}_k = S_{\rho,k}, \tag{3.7}$$

$$\partial_t e_k + \boldsymbol{v}_k \cdot (\nabla e_k) - \frac{1}{\alpha_k \rho_k} \sum\nolimits_{l=1, \neq k}^{K} P_{k,l} \, (\boldsymbol{v}_k - \boldsymbol{V}_I) \cdot \nabla \alpha_l + \frac{p_k}{\rho_k} \nabla \cdot \boldsymbol{v}_k = -\frac{1}{\alpha_k \rho_k} \nabla \cdot (\alpha_k \boldsymbol{q}_k) + S_{e,k} \tag{3.8}$$

with expected relaxation terms

$$S_{\rho,k} := \frac{1}{\alpha_k} \left(S_{\alpha\rho,k} - \rho_k S_{\alpha,k} \right), \tag{3.9}$$

$$S_{e,k} := \frac{1}{\alpha_k \rho_k} \left(S_{\alpha\rho E,k} - \boldsymbol{v}_k \cdot \boldsymbol{S}_{\alpha\rho\boldsymbol{v},k} + S_{\alpha\rho,k}(u_k - e_k) \right). \tag{3.10}$$

In order to derive the evolution equation for the entropy $s_k = s_k(\rho_k)$ we rewrite (1.18) as

$$T_{ref} ds_k = de_k - \frac{p_k}{\rho_k^2} d\rho_k = \left(e_k' - \frac{p_k}{\rho_k^2} \right) d\rho_k, \tag{3.11}$$

where due to (1.5) it holds $T_k = T_{ref} = const$, $k = 1, \dots, K$. By means of the evolution equations (3.7) and (3.8) for the density and the internal energy we then deduce the entropy law

$$\partial_t s_k + \boldsymbol{v}_k \cdot \nabla s_k = \frac{1}{\alpha_k \rho_k} \left(\Pi_k - \frac{1}{T_{ref}} \nabla \cdot (\alpha_k \boldsymbol{q}_k) \right) + S_{s,k} \tag{3.12}$$

with relaxation term

$$S_{s,k} := \frac{1}{\alpha_k \rho_k T_{ref}} \left(\alpha_k \rho_k S_{e,k} - \frac{p_k}{\rho_k} S_{\alpha\rho,k} + p_k S_{\alpha,k} \right) \tag{3.13}$$

and production term

$$\Pi_k := \frac{1}{T_{ref}} \left(\sum\nolimits_{l=1,\neq k}^{K} P_{k,l} \left(\boldsymbol{v}_k - \boldsymbol{V}_I \right) \cdot \nabla \alpha_l + p_k (\boldsymbol{v}_k - \boldsymbol{V}_I) \cdot \nabla \alpha_k \right). \tag{3.14}$$

Finally, we obtain together with (1.1b) for the volume specific entropy

$$\partial_t (\alpha_k \rho_k s_k) + \nabla \cdot (\alpha_k \rho_k s_k \boldsymbol{v}_k) + \frac{1}{T_{ref}} \nabla \cdot (\alpha_k \boldsymbol{q}_k) = \Pi_k S_{\alpha\rho s,k} \tag{3.15}$$

with relaxation term

$$S_{\alpha\rho s,k} := \alpha_k \rho_k S_{s,k} + s_k S_{\alpha\rho,k}. \tag{3.16}$$

Next we determine from the energy equation (1.1d) the heat flux

$$\nabla \cdot (\alpha_k \, \boldsymbol{q}_k) = S_{\alpha\rho E,k} - \partial_t (\alpha_k \, \rho_k \, E_k) - \nabla \cdot (\alpha_k \, \rho_k \, \boldsymbol{v}_k \, (E_k + p_k/\rho_k)) - \sum\nolimits_{l=1,\neq k}^{K} P_{k,l} \, \boldsymbol{V}_I \cdot \nabla \alpha_l. \tag{3.17}$$

Plugging the heat flux into the entropy equation (3.15) we obtain

$$\begin{aligned} \partial_t (\alpha_k \, \rho_k \, (T_{ref} s_k - E_k)) + \nabla \cdot (\alpha_k \, \rho_k \, \boldsymbol{v}_k \, (T_{ref} s_k - E_k - p_k/\rho_k)) = \\ T_{ref} \Pi_k + \sum\nolimits_{l=1,\neq k}^{K} P_{k,l} \, \boldsymbol{V}_I \cdot \nabla \alpha_l + T_{ref} S_{\alpha\rho E,k}. \end{aligned} \tag{3.18}$$

Summing these evolution equations over all components and using the conservation constraint (1.11) we obtain

$$\begin{aligned} \partial_t \left(\sum\nolimits_{k=1}^{K} \alpha_k \, \rho_k \, (E_k - T_{ref} s_k) \right) + \nabla \cdot \left(\sum\nolimits_{k=1}^{K} (\alpha_k \, \rho_k \, \boldsymbol{v}_k \, (E_k - T_{ref} s_k + p_k/\rho_k)) \right) = \\ - T_{ref} \sum\nolimits_{k=1}^{K} S_{\alpha\rho s,k} - T_{ref} \sum\nolimits_{k=1}^{K} \Pi_k. \end{aligned} \tag{3.19}$$

Comparing (3.19) with (3.4) we may choose for the entropy-entropy flux pair

$$U(\boldsymbol{u}) := \sum\nolimits_{k=1}^{K} \alpha_k \, \rho_k \, (E_k - T_{ref} \, s_k) = \sum\nolimits_{k=1}^{K} \alpha_k \, \rho_k \, (f_k + 1/2 \boldsymbol{v}_k^2), \tag{3.20a}$$

$$F_i(\boldsymbol{u}) := \sum\nolimits_{k=1}^{K} (\alpha_k \, \rho_k \, \boldsymbol{v}_k \, (E_k - T_{ref} s_k + p_k/\rho_k)) = \sum\nolimits_{k=1}^{K} (\alpha_k \, \rho_k \, \boldsymbol{v}_k \, (g_k + 1/2 \boldsymbol{v}_k^2)) \tag{3.20b}$$

with the specific free energy f_k and Gibbs free energy g_k

$$f_k = e_k - T_{ref} s_k, \quad g_k = f_k + p_k/\rho_k. \tag{3.21}$$

The compatibility conditions (3.3) hold by derivation of (3.59), see also [38] in case of the full Baer-Nunziato model.

Theorem 3.1. *(Convexity of entropy function) Let e_k be a convex function of τ_k, $k = 1, \ldots, K$. Then the Hessian of U given in* (3.20a) *is positive semi-definite, i.e., the entropy U is a convex function of $\boldsymbol{u}$ but not necessarily strictly convex.*

Proof. To verify that

$$U(\boldsymbol{u}) := \sum\nolimits_{k=1}^{K} \alpha_k \rho_k (e_k - T_{ref} s_k + \boldsymbol{v}_k^2/2) = \sum\nolimits_{k=1}^{K} \alpha_k U_k(\boldsymbol{u}_k) \tag{3.22}$$

is a convex function of the quantities $\boldsymbol{u} := (\boldsymbol{\alpha}^t, \alpha_1 \boldsymbol{u}_1^t, \ldots, \alpha_K \boldsymbol{u}_K^t)^t$ with $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_{K-1})^t$ and $\boldsymbol{u}_k := (\rho_k, \rho_k \boldsymbol{v}_k^t)^t$ we proceed in analogy to [38] for the full multi-component Baer-Nunziato model but using an isothermal fluid.

For this purpose, we need to prove that the Hessian is positive semi-definite. First of all, we note that by (1.2)

$$\frac{\partial \alpha_k}{\partial \alpha_l} = \delta_{k,l} - \delta_{k,K}, \quad \frac{\partial \boldsymbol{u}_k}{\partial \alpha_l} = -\frac{1}{\alpha_k} \boldsymbol{u}_k (\delta_{k,l} - \delta_{k,K}), \ \frac{\partial \boldsymbol{u}_k}{\partial \alpha_l \boldsymbol{u}_l} = \frac{1}{\alpha_k} \delta_{k,l} \boldsymbol{I}_{d+1}$$

holds for $k = 1, \ldots, K, \ l = 1, \ldots, K-1$. Then it follows for the gradient of U

$$\frac{\partial U}{\partial \alpha_l}(\boldsymbol{u}) = U_l(\boldsymbol{u}_l) - U_K(\boldsymbol{u}_K) - \frac{\partial U_l}{\partial \boldsymbol{u}_l}(\boldsymbol{u}_l) \cdot \boldsymbol{u}_l + \frac{\partial U_K}{\partial \boldsymbol{u}_K}(\boldsymbol{u}_K) \cdot \boldsymbol{u}_K, \ \frac{\partial U}{\partial \alpha_l \boldsymbol{u}_l}(\boldsymbol{u}) = \frac{\partial U_l}{\partial \boldsymbol{u}_l}(\boldsymbol{u}_l). \tag{3.23}$$

The Hessian of U is determined by the second order derivatives

$$\begin{aligned}
\frac{\partial^2 U}{\partial \alpha_k \partial \alpha_l}(\boldsymbol{u}) &= \delta_{k,l} \frac{1}{\alpha_l} \boldsymbol{u}_l^t \frac{\partial^2 U_l}{\partial^2 \boldsymbol{u}_l}(\boldsymbol{u}_l) \boldsymbol{u}_l + \frac{1}{\alpha_K} \boldsymbol{u}_K^t \frac{\partial^2 U_K}{\partial^2 \boldsymbol{u}_K}(\boldsymbol{u}_K) \boldsymbol{u}_K, \ k,l = 1, \ldots, K, \\
\frac{\partial^2 U}{\partial \alpha_k \partial \alpha_l \boldsymbol{u}_l}(\boldsymbol{u}) &= -\frac{1}{\alpha_l} (\delta_{k,l} - \delta_{K,l}) \frac{\partial^2 U_l}{\partial^2 \boldsymbol{u}_l}(\boldsymbol{u}_l) \boldsymbol{u}_l, l = 1, \ldots, K, \ k = 1, \ldots, K-1, \\
\frac{\partial^2 U}{\partial \alpha_k \boldsymbol{u}_k \partial \alpha_l \boldsymbol{u}_l}(\boldsymbol{u}) &= \frac{1}{\alpha_l} \delta_{k,l} \frac{\partial^2 U_l}{\partial^2 \boldsymbol{u}_l}(\boldsymbol{u}_l), \ k,l = 1, \ldots, K.
\end{aligned}$$

For a compact representation of the Hessian we introduce the notation

$$\begin{aligned}
\boldsymbol{U}_{\boldsymbol{\alpha},\boldsymbol{\alpha}} &:= \left(\frac{\partial^2 U}{\partial \alpha_k \partial \alpha_l}(\boldsymbol{u}) \right)_{l,k=1,\ldots,K-1} \in \mathbb{R}^{(K-1)\times(K-1)}, \\
\boldsymbol{U}_{\alpha_k \boldsymbol{u_k}, \alpha_k \boldsymbol{u_k}} &:= \frac{\partial^2 U}{\partial \alpha_k \boldsymbol{u}_k \partial \alpha_k \boldsymbol{u}_k}(\boldsymbol{u}) \in \mathbb{R}^{(d+1)\times(d+1)}, \\
\boldsymbol{U}_{\alpha_k \boldsymbol{u_k}, \boldsymbol{\alpha}} &:= \left(\frac{\partial^2 U}{\partial \alpha_k \boldsymbol{u}_k \partial \alpha_1}(\boldsymbol{u}), \ldots, \frac{\partial^2 U}{\partial \alpha_k \boldsymbol{u}_k \partial \alpha_{K-1}}(\boldsymbol{u}) \right) \in \mathbb{R}^{(d+1)\times(K-1)}.
\end{aligned}$$

According to the above second order derivatives these are determined by

$$\boldsymbol{U}_{\boldsymbol{\alpha},\boldsymbol{\alpha}} = \frac{1}{\alpha_K} \boldsymbol{u}_K^t \boldsymbol{U}_K'' \boldsymbol{u}_K \mathbf{1}_{K-1} + \operatorname{diag} \left(\left(\frac{1}{\alpha_k} \boldsymbol{u}_k^t \boldsymbol{U}_k'' \boldsymbol{u}_k \right)_{k=1,\ldots,K-1} \right), \tag{3.24}$$

$$\boldsymbol{U}_{\alpha_k \boldsymbol{u_k}, \alpha_k \boldsymbol{u_k}} = \frac{1}{\alpha_k} \boldsymbol{U}_k'', \tag{3.25}$$

$$\boldsymbol{U}_{\alpha_k \boldsymbol{u_k}, \boldsymbol{\alpha}} = \left(-\frac{1}{\alpha_k} (\delta_{k,l} - \delta_{K,k}) \boldsymbol{U}_k'' \boldsymbol{u}_k \right)_{l=1,\ldots,K-1} = \boldsymbol{U}_{\boldsymbol{\alpha}, \alpha_k \boldsymbol{u}_k}^t, \tag{3.26}$$

where $\boldsymbol{U}_k''$ denotes the Hessian of the entropy $U_k = U_k(\boldsymbol{u}_k)$ of component k. Then the Hessian can be represented as block-matrix

$$\boldsymbol{U}''(\boldsymbol{u}) = \begin{pmatrix} \boldsymbol{U}_{\boldsymbol{\alpha},\boldsymbol{\alpha}} & \boldsymbol{U}_{\boldsymbol{\alpha},\alpha_1 \boldsymbol{u_1}} & \cdots & \boldsymbol{U}_{\boldsymbol{\alpha},\alpha_1 \boldsymbol{u_K}} \\ \boldsymbol{U}_{\alpha_1 \boldsymbol{u_1},\boldsymbol{\alpha}} & \boldsymbol{U}_{\alpha_1 \boldsymbol{u_1},\alpha_1 \boldsymbol{u_1}} & & \\ \vdots & & \ddots & \\ \boldsymbol{U}_{\alpha_K \boldsymbol{u_K},\boldsymbol{\alpha}} & & & \boldsymbol{U}_{\alpha_1 \boldsymbol{u_K},\alpha_K \boldsymbol{u_K}} \end{pmatrix}. \tag{3.27}$$

To verify positive semi-definiteness of the Hessian we introduce the vector $\boldsymbol{x} = (\boldsymbol{a}^t, \boldsymbol{b}_1^t, \dots, \boldsymbol{b}_K^t)^t$ with $\boldsymbol{a} \in \mathbb{R}^{K-1}$ and $\boldsymbol{b}_k \in \mathbb{R}^{d+1}$, $k = 1, \dots, K$. Then we obtain by the block-structure (3.27) of the Hessian

$$\boldsymbol{x}^t \boldsymbol{U}'' \boldsymbol{x} = \boldsymbol{a}^t \boldsymbol{U}_{\boldsymbol{\alpha},\boldsymbol{\alpha}} \boldsymbol{a} + \sum\nolimits_{k=1}^{K} \boldsymbol{a}^t \boldsymbol{U}_{\boldsymbol{\alpha},\alpha_k \boldsymbol{u_k}} \boldsymbol{b}_k + \sum\nolimits_{k=1}^{K} \boldsymbol{b}_k^t \left(\boldsymbol{U}_{\alpha_k \boldsymbol{u_k},\boldsymbol{\alpha}} \boldsymbol{a} + \boldsymbol{U}_{\alpha_k \boldsymbol{u_k},\alpha_k \boldsymbol{u_k}} \boldsymbol{b}_k \right). \tag{3.28}$$

By means of (3.24), (3.25) and (3.26) we determine

$$\begin{aligned}
&\boldsymbol{a}^t \boldsymbol{U}_{\boldsymbol{\alpha},\boldsymbol{\alpha}} \boldsymbol{a} = \frac{1}{\alpha_K} (a \boldsymbol{u}_K)^t \boldsymbol{U}_K'' (a \boldsymbol{u}_K) + \sum\nolimits_{k=1}^{K-1} \frac{1}{\alpha_k} (a_k \boldsymbol{u}_k)^t \boldsymbol{U}_k'' (a_k \boldsymbol{u}_k), \;\; a := \sum\nolimits_{l=1}^{K-1} a_l, \\
&\boldsymbol{a}^t \boldsymbol{U}_{\boldsymbol{\alpha},\alpha_k \boldsymbol{u_k}} \boldsymbol{b}_k = - \sum\nolimits_{l=1}^{K-1} \frac{1}{\alpha_k} (\delta_{k,l} - \delta_{k,K}) \boldsymbol{b}_k^t \boldsymbol{U}_k'' (a_l \boldsymbol{u}_k), \\
&\boldsymbol{b}_k^t \boldsymbol{U}_{\boldsymbol{\alpha},\alpha_k \boldsymbol{u_k}} \boldsymbol{a} = \frac{1}{\alpha_K} \delta_{k,K} \boldsymbol{b}_K^t \boldsymbol{U}_K'' (a \boldsymbol{u}_K) - \frac{1}{\alpha_k} (1 - \delta_{k,K}) \boldsymbol{b}_k^t \boldsymbol{U}_k'' (a_k \boldsymbol{u}_k), \\
&\boldsymbol{b}_k^t \boldsymbol{U}_{\alpha_k \boldsymbol{u_k},\alpha_k \boldsymbol{u_k}} \boldsymbol{b}_k = \frac{1}{\alpha_k} \boldsymbol{b}_k^t \boldsymbol{U}_k'' \boldsymbol{b}_k.
\end{aligned}$$

Incorporating this into (3.28) we finally conclude with

$$\boldsymbol{x}^t \boldsymbol{U}'' \boldsymbol{x} = \sum\nolimits_{k=1}^{K-1} \frac{1}{\alpha_k} (\boldsymbol{b}_k - a_k \boldsymbol{u}_k)^t \boldsymbol{U}_k'' (\boldsymbol{b}_k - a_k \boldsymbol{u}_k) + \frac{1}{\alpha_K} (\boldsymbol{b}_K - a \boldsymbol{u}_K)^t \boldsymbol{U}_K'' (\boldsymbol{b}_K - a \boldsymbol{u}_K).$$

Finally, we check that the Hessians $\boldsymbol{U}_k''$ are positive semi-definite where $\boldsymbol{U}_k$ is introduced in (3.22). For this purpose we consider the energy and the entropy to be given in terms of volume and temperature, i.e. $\tilde{e}_k(\tau_k, T) = e_k(\tau_k, s_k(\tau_k, T))$. The first derivatives of the energy and the pressure with respect to the volume are given by

$$\frac{\partial \tilde{e}_k}{\partial \tau_k} = \frac{\partial e_k}{\partial \tau_k} + \frac{\partial e_k}{\partial s_k} \frac{\partial s_k}{\partial \tau_k} \overset{(1.16)}{=} -p_k + T_{ref} \frac{\partial s_k}{\partial \tau_k} \quad \text{and} \quad \frac{\partial p_k}{\partial \tau_k} \overset{(1.29)}{=} -\frac{\tilde{c}_k^2}{\tau_k^2}. \tag{3.29}$$

From this we conclude

$$\frac{\partial}{\partial \rho_k} (\rho_k (e_k - T_{ref} s_k)) = e_k - T_{ref} s_k + \frac{p_k}{\rho_k}, \quad \frac{\partial^2}{\partial \rho_k^2} (\rho_k (e_k - T_{ref} s_k)) = \frac{\tilde{c}_k^2}{\rho_k} > 0. \tag{3.30}$$

For the Hessian we then determine

$$\boldsymbol{U}_k'' = \frac{1}{\rho_k} \begin{pmatrix} \tilde{c}_k^2 + \boldsymbol{v}_k^2 & -\boldsymbol{v}_k^t \\ -\boldsymbol{v}_k & \boldsymbol{I}_d \end{pmatrix} \tag{3.31}$$

that can be easily verified to be positive definite. This implies $\boldsymbol{x}^t \boldsymbol{U}'' \boldsymbol{x} \geq 0$. Note that for $\boldsymbol{x} \neq \boldsymbol{0}$ we cannot ensure $\boldsymbol{x}^t \boldsymbol{U}'' \boldsymbol{x}$ to be positive even if $\boldsymbol{U}_k''$ is strictly convex because all the terms $\boldsymbol{b}_k - a_k \boldsymbol{u}_k$, $k = 1, \dots, K-1$, and $\boldsymbol{b}_K - a \boldsymbol{u}_K$ may vanish at the same time.

□

Furthermore, to ensure that the entropy inequality (3.2) holds we have to verify that the right-hand side of (3.19) is not positive, see also [38] in case of the full Baer-Nunziato model.

3.1.1. *Entropy production due to interfacial velocity and pressure*

Since we cannot control the sign of Π_k, we determine the interfacial pressures $P_{k,l}$ and the interfacial velocity $\boldsymbol{V}_I$ such that the sum $\Pi := \sum_{k=1}^{K} \Pi_k$ vanishes with Π_k given in Eqn. (3.14) and the conservation constraints (1.11) or, equivalently, (1.12) hold. Assuming that the interfacial velocity can be written as a linear combination of the velocities of the components according to (2.38) there exists a unique closure for the interfacial pressures.

Theorem 3.2. *(Entropy production due to interfacial states) For the interfacial velocity $\boldsymbol{V}_I$ of the form* (2.38) *there uniquely exist interfacial pressures*

$$P_{k,l} = \beta_k p_l + p_k(1-\beta_k) \tag{3.32}$$

such that the production term $\Pi = T_{ref}\sum_{k=1}^K \Pi_k$ vanishes and the conservation constraints (1.11) or, equivalently, (1.12) hold. In particular, the interfacial pressures are all positive and

$$P_I = \sum\nolimits_{k=1}^K p_k(1-\beta_k). \tag{3.33}$$

Proof. The proof is in complete analogy to the proof of Theorem 6 in [38] for the full multi-component Baer-Nunziato model assuming local thermal equilibrium where we have to replace the temperatures T_k by T_{ref}. This is due to the fact that the terms Π_k as well as the conservation constraints coincide. Thus, the resulting interfacial pressures (3.32) are a special case of those presented in [38]. We emphasize that the resulting interfacial states are the same for both isentropic and isothermal fluids, see Theorem 3.6.

□

We would like to mention that the above result is motivated by Gallouët et al. [10] and Hérard [24] for a two-phase and a three-phase model, respectively, in case of the full Baer-Nunziato model. In [38] it was proven for a multi-component fluid with an arbitrary number of components. More recently, Bussac [3] verified the result in case of a non-equilibrium multi-component model with miscible conditions instead of immiscible conditions as in the aforementioned references.

Using for the convex combination (2.38) the coefficients (2.39), then the interfacial pressure and the interfacial velocity are given by

$$P_I = \sum\nolimits_{k=1}^K p_k(1-\overline{\gamma}_k\alpha_k\rho_k/\hat{\rho}), \quad \boldsymbol{V}_I = \sum\nolimits_{i=1}^K \overline{\gamma}_i\alpha_i\rho_i\boldsymbol{v}_i \Big/ \sum\nolimits_{k=1}^K \overline{\gamma}_k\alpha_k\rho_k. \tag{3.34}$$

This closure of the interfacial states ensures consistency with an entropy inequality and the fields corresponding to the material waves are linearly degenerated.

For special choices of $\overline{\boldsymbol{\gamma}} \in [0,1]^K$ the interfacial values coincide with those in the literature in case of two and three components. For instance, choosing $\overline{\boldsymbol{\gamma}} = \boldsymbol{e}_i$ for some $i \in \{1,\dots,K\}$ we obtain

$$P_I = \sum\nolimits_{k=1,\neq i}^K p_k, \quad \boldsymbol{V}_I = \boldsymbol{v}_i. \tag{3.35}$$

For $i=1$ these coincide with those given in [10] and [24] for $K=2$ and $K=3$, respectively. In case of uniform coefficients $\overline{\gamma}_k = 1/K$, $k=1,\dots,K$, the interfacial pressure and velocity are given by

$$P_I = \sum\nolimits_{k=1}^K p_k(1-\alpha_k\rho_k/\rho) = \sum\nolimits_{k=1}^K p_k \sum\nolimits_{j=1,\neq k}^K \alpha_j\rho_j/\rho, \quad \boldsymbol{V}_I = \sum\nolimits_{i=1}^K \alpha_i\rho_i\boldsymbol{v}_i \Big/ \sum\nolimits_{k=1}^K \alpha_k\rho_k = \boldsymbol{v}. \tag{3.36}$$

where ρ and $\boldsymbol{v}$ are the density and the velocity of the mixture, see (1.13), respectively.

Obviously, in case of two components, i.e., $K=2$, the Abgrall-Saurel closure [51]

$$P_I = \alpha_1 p_1 + \alpha_2 p_2, \quad \boldsymbol{V}_I = \frac{\alpha_1\rho_1}{\rho}\boldsymbol{v}_1 + \frac{\alpha_2\rho_2}{\rho}\boldsymbol{v}_2 \tag{3.37}$$

may violate the 2nd law of thermodynamics because the sum of the entropy production terms due to the interfacial states does not vanish. Furthermore, the choice for the interfacial velocity in [51] is not motivated by the linearly degeneration structure of the associated characteristic field, see Prop. 2.2.

Remark 3.1. *Finally, we would like to point out that for barotropic fluids another entropy-entropy flux pair (EEP) has been derived in [2, 50] based on the "energies" $\overline{e}_k$, see Remark 1.1. As these "energies" coincide with the energies e_k for isentropic fluids, we present this entropy law in Sect. 3.2. Since isothermal fluids are a special case of a barotropic fluid, this EEP is also an EEP in the isothermal case. It is worthwhile mentioning that the other EEP results in the same interfacial pressures as given in Thm. 3.2.*

3.1.2. *Entropy production due to relaxation*

To be admissible with the entropy inequality (3.6) the production term

$$S_U := -T_{ref} \sum\nolimits_{k=1}^{K} S_{\alpha\rho s,k} \tag{3.38}$$

has to be non-positive. Here we use (3.16), (3.13) and (3.10) to determine the phasic terms

$$S_{\alpha\rho s,k} = \frac{1}{T_{ref}} \left(S_{\alpha\rho E,k} + \left(\frac{1}{2} \boldsymbol{v}_k^2 - g_k \right) S_{\alpha\rho,k} + p_k S_{\alpha,k} - \boldsymbol{v}_k \cdot \boldsymbol{S}_{\alpha\rho\boldsymbol{v},k} \right). \tag{3.39}$$

Note that in abuse of physical notation we call this term entropy production rather than energy production with respect to the mathematical concept of entropy-entropy flux pairs because energy is known to be conserved. In the following we consider one-by-one the contributions of mechanical relaxation and chemical relaxation. The analysis is similar to [38] in case of the physical entropy production. Note that all the relaxation processes are in agreement with the constraints due to conservation (1.10).

Theorem 3.3. *(Entropy production due to mechanical relaxation) Let the relaxation parameters be non-negative, i.e., θ_p, $\theta_v \geq 0$. Then the entropy production due to mechanical relaxation is non-positive, i.e.,*

$$S_U^{p,v} = T_{ref} \sum\nolimits_{k=1}^{K} S_{\alpha\rho s,k}^{p,v} \leq 0. \tag{3.40}$$

Proof. According to (3.39) and (1.46), (1.47) the phasic entropy production terms for velocity and pressure relaxation are determined by

$$S_{\alpha\rho s,k}^{v} = -\theta_v \frac{\alpha_k \rho_k}{T_{ref}} (\boldsymbol{v} - \boldsymbol{v}_k)^2, \quad S_{\alpha\rho s,k}^{p} = -\theta_p \frac{\alpha_k}{T_{ref}} (p - p_k)^2. \tag{3.41}$$

This immediately implies the assertion. □

For the relaxation of chemical potentials we discuss different scenarios for two-phase and three-phase models.

In case of the two-component mixture, the entropy production term due to relaxation of the chemical potentials is determined by (3.39) and (1.49) as

$$S_{\alpha\rho s,k}^{\mu} = (-1)^k \theta_\mu \, \dot{m} \, \frac{1}{T_{ref}} \left(\epsilon - g_k + \frac{p_k}{\varrho} - \frac{1}{2} (\boldsymbol{v}_k - \hat{\boldsymbol{v}})^2 \right), \quad k = 1, 2. \tag{3.42}$$

However choosing the parameters ϵ, ϱ and $\hat{\boldsymbol{v}}$ appropriately the sum of the entropy production terms is non-negative.

Theorem 3.4. *(Entropy production due to relaxation of Gibbs free energies) Let the relaxation parameter θ_μ be positive. The mass flux is chosen as the kinetic relation*

$$\dot{m} = a(g_2 - g_1), \ a \geq 0. \tag{3.43}$$

Choosing in (1.49) the parameters

$$\beta_1^v = \beta_2^v = \frac{1}{2}, \quad \varrho = \frac{1}{c}\frac{p_2 - p_1}{g_2 - g_1},\ c \leq 1, \tag{3.44}$$

then the entropy production is non-positive, i.e.,

$$S_U^\mu = -T_{ref} \sum\nolimits_{k=1}^{2} S_{\alpha\rho s,k}^\mu \leq 0. \tag{3.45}$$

Proof. First of all, we note that by (3.44) the velocity $\hat{\boldsymbol{v}}$ is the average of the phasic velocities and, thus, the entropy production does not depend on the velocity because

$$\frac{1}{2}\left((\hat{\boldsymbol{v}} - \boldsymbol{v}_1)^2 - (\hat{\boldsymbol{v}} - \boldsymbol{v}_2)^2\right) = \frac{1}{2}(\boldsymbol{v}_2 - \boldsymbol{v}_1) \cdot (2\hat{\boldsymbol{v}} - \boldsymbol{v}_1 - \boldsymbol{v}_2) = \boldsymbol{0}. \tag{3.46}$$

Then the entropy production reduces to

$$S_U^\mu = -\theta_\mu\, \dot{m} \left(g_2 - g_1 + \frac{p_1 - p_2}{\varrho}\right) = -\theta_\mu\, a\, (g_2 - g_1)^2 (1 - c) \leq 0.$$

Thus, the energy production S_U^μ is non-positive.

□

Note that the kinetic relation (3.43) is in agreement with the kinetic relation in [8]. Furthermore, choosing $\hat{\boldsymbol{v}}$ as the average of the phasic velocities is reasonable because the entropy production (3.45) should be a product of the relaxed mass flux $\theta_\mu\, \dot{m}$ and an interfacial entropy should not depend on any velocity.

Note that (3.34) and the mixture velocities (1.50) with coefficients (3.44) differ. In [26] it was recommended to use different interfacial velocities for the convective system and the relaxation terms. This is admissible because the conservation constraint (1.10) is satisfied for any convex combination (2.38). Moreover, the source term cannot be derived from the ensemble averaging procedure, see [7], Chapter 11, but the averaged model has to be closed by modeling these terms appropriately. Therefore, we are free to choose another velocity in the chemical relaxation model. Note that in the Drew-Passman model different interfacial velocities have been introduced in the evolution equations for volume fraction, momentum and energy, see [7], Eqns. (11.8), (11.39) and (11.41).

In case of the three-component mixture, the entropy production terms due to the relaxation of chemical potentials are determined by (3.39) and (1.51) as

$$S_{\alpha\rho s,1}^\mu = \theta_\mu\, \dot{m}\, \frac{1}{T_{ref}} \left(\epsilon_1 - g_1 + \frac{p_1}{\varrho_1} + \frac{1}{2}(\hat{\boldsymbol{v}} - \boldsymbol{v}_1)^2\right), \tag{3.47a}$$

$$S_{\alpha\rho s,2}^\mu = -\theta_\mu\, \dot{m}\, \frac{1}{T_{ref}} \left(\epsilon_2 - g_2 - \frac{p_2}{\varrho_2} + \frac{1}{2}(\hat{\boldsymbol{v}} - \boldsymbol{v}_2)^2\right), \tag{3.47b}$$

$$S_{\alpha\rho s,3}^\mu = \theta_\mu\, \dot{m}\, \frac{1}{T_{ref}} \left(\epsilon_2 - \epsilon_1 - p_3 \left(\frac{1}{\varrho_1} + \frac{1}{\varrho_2}\right)\right). \tag{3.47c}$$

The phasic relaxation terms may become negative also when θ_μ and $\dot{m}$ are positive. However, choosing the parameters ϱ_1, ϱ_2 and $\hat{\boldsymbol{v}}$ appropriately the total production term is non-negative.

In particular, we are interested in the three-component model, i.e., $K = 3$, with water vapor ($k = 1$), liquid water ($k = 2$) and inert gas ($k = 3$). The source terms describing the mass transfer depend on the chemical potentials of the vapor and the liquid. In the model under consideration the gas is a mixture of water vapor and some other constituent, where both components are modeled as an ideal gas. Accordingly the chemical

potential of the vapor is given by

$$\mu_1 = g_1 + (\gamma_1 - 1)c_{v,1}T_{ref} \ln\left(\frac{\alpha_1}{\alpha_1 + \alpha_3}\right). \tag{3.48}$$

In the special case of vanishing third component, i.e., $\alpha_3 = 0$, the chemical potential of the vapor reduces to the vapor Gibbs free energy. The chemical potential μ_2 of the liquid equals its Gibbs free energy g_2, i.e.,

$$\mu_2 = g_2\,.$$

In chemical equilibrium the chemical potentials of the vapor and the liquid equal each other. For details see the book of Müller and Müller [36], Section 8.2.4.

In the following we consider a homogeneous mixture where velocity, pressure and temperature are in equilibrium. In the total entropy we therefore additionally have to account for the mixture entropy given by

$$S_M = -\sum\nolimits_{k=1,3} \alpha_k \rho_k (\gamma_k - 1) c_{v,k} \ln\left(\frac{\alpha_k}{\alpha_1 + \alpha_3}\right), \tag{3.49}$$

see [35], p. 54, 298, 320. Again, the chemical potential of the vapor phase is given by (3.48) and the chemical potential of the liquid phase equals its Gibbs free energy, i.e., $\mu_2 = g_2$. In chemical equilibrium the chemical potentials of the vapor and the liquid phase equal each other. Accordingly, the mass flux is now a function of $\mu_2 - \mu_1$, i.e.,

$$\dot{m} = a(\mu_2 - \mu_1) \tag{3.50}$$

with $a \geq 0$ that again is an agreement with the kinetic relation in [8].

Theorem 3.5. *(Entropy production due to relaxation of chemical potentials) For a homogeneous mixture, i.e., $\boldsymbol{v}_k = \boldsymbol{v} = const$, $p_k = p = const$ and $T_k = T = const$, the total entropy production is non-positive, i.e.,*

$$S_U^\mu - T_{ref} S_{S_M}^\mu \leq 0, \tag{3.51}$$

if the relaxation parameter θ_μ as well as the equilibrium temperature T are positive.

Proof. For a homogeneous mixture the entropy production (3.38) reduces to

$$S_U^\mu = -\dot{m}\,\theta_\mu\,(g_2 - g_1) = -\dot{m}\,\theta_\mu\,\left(\mu_2 - \mu_1 + (\gamma_1 - 1)c_{v,1}T_{ref}\ln\left(\frac{\alpha_1}{\alpha_1 + \alpha_3}\right)\right).$$

In analogy to the derivation of the entropy production, we determine the entropy production of the mixture entropy S_M

$$S_{S_M}^\mu = -\dot{m}\,\theta_\mu\,(\gamma_1 - 1)c_{v,1}\ln\left(\frac{\alpha_1}{\alpha_1 + \alpha_3}\right). \tag{3.52}$$

Then the total entropy production is given by

$$S_U^\mu - T_{ref} S_{S_M}^\mu = -\dot{m}\,\theta_\mu\,(\mu_2 - \mu_1)\,. \tag{3.53}$$

Thus, by (3.50) we conclude (3.51). □

3.2. **Entropy-entropy flux pair: mechanical approach**

The entropy-entropy flux pair derived in case of isothermal fluids in the previous section does not apply for general barotropic fluids, but for isothermal fluids only. Instead of the mixture of total free energies typically

the mixture of total energies is used for a mathematical entropy in the barotropic case. For the derivation of an entropy-entropy flux pair we introduce the "energies" $\overline{e}_k$ according to Remark 1.1

$$\overline{e}_k'(\rho_k) = \frac{p_k(\rho_k)}{\rho_k^2} \quad \text{or, equiv.,} \quad \overline{e}_k'(\tau_k) = -p_k(\tau_k) \tag{3.54}$$

Because of (1.15) this energy coincides with the thermodynamical energy e_k in case of isentropic fluids. For the mathematical entropy function the total specific energy $\overline{E}_k = \overline{e}_k(\rho_k) + \boldsymbol{v}_k^2/2$ is used. For smooth solutions of the model (1.8) we obtain the evolution equations for the fractional total energies

$$\partial_t(\alpha_k\,\rho_k\,\overline{E}_k) + \nabla\cdot(\alpha_k\,\rho_k\,\boldsymbol{v}_k\,(\overline{E}_k + p_k/\rho_k)) = \overline{S}_{\alpha\rho\overline{E}_k,k} + \overline{\Pi}_k, \tag{3.55}$$

with relaxation term $\overline{S}_{\alpha\rho\overline{E}_k,k}$ and production term $\overline{\Pi}_k$

$$\overline{S}_{\alpha\rho\overline{E},k} = (\overline{e}_k + p_k/\rho_k - \boldsymbol{v}_k^2/2)S_{\alpha\rho,k} - p_k S_{\alpha,k} + \boldsymbol{v}_k\cdot\boldsymbol{S}_{\alpha\rho\boldsymbol{v},k}, \tag{3.56}$$

$$\overline{\Pi}_k = p_k(\boldsymbol{V}_I - \boldsymbol{v}_k)\cdot\nabla\alpha_k - \sum\nolimits_{l=1,\neq k}^{K} P_{k,l}\boldsymbol{v}_k\cdot\nabla\alpha_l = -\sum\nolimits_{l=1,\neq k}^{K}(p_k(\boldsymbol{V}_I - \boldsymbol{v}_k) + P_{k,l}\boldsymbol{v}_k\,)\cdot\nabla\alpha_l. \tag{3.57}$$

We emphasize that the relaxation term $\overline{S}_{\alpha\rho\overline{E},k}$ is different from the corresponding term determined by the relaxation models (1.46), (1.47), (1.49) and (1.51). It is interesting to note that in contrast to the single-phase case the energy equation is not redundant in the multi-component case due to the non-conservative products and the relaxation terms. Similar to the isothermal case, see Eqn. (3.17), we determine the heat flux as

$$\nabla\cdot(\alpha_k\,\boldsymbol{q}_k) = S_{\alpha\rho E,k} - \overline{S}_{\alpha\rho\overline{E},k} - \sum\nolimits_{l=1,\neq k}^{K} P_{k,l}\,\boldsymbol{V}_I\cdot\nabla\alpha_l - \overline{\Pi}_k \tag{3.58}$$

comparing the difference of (1.1d) and (3.55).

Summing these evolution equations over all components we obtain the evolution equation of the total energy

$$\partial_t\left(\sum\nolimits_{k=1}^{K}\alpha_k\,\rho_k\,\overline{E}_k\right) + \nabla\cdot\left(\sum\nolimits_{k=1}^{K}(\alpha_k\,\rho_k\,\boldsymbol{v}_k\,(\overline{E}_k + p_k/\rho_k))\right) = \sum\nolimits_{k=1}^{K}\overline{S}_{\alpha\rho\overline{E},k} + \sum\nolimits_{k=1}^{K}\overline{\Pi}_k, \tag{3.59}$$

Comparing (3.59) with (3.4) we may choose for the entropy-entropy flux pair

$$U(\boldsymbol{u}) := \sum\nolimits_{k=1}^{K}\alpha_k\rho_k\overline{E}_k, \tag{3.60a}$$

$$F_i(\boldsymbol{u}) := \sum\nolimits_{k=1}^{K}(\alpha_k\,\rho_k\,v_{k,i}\,(E_k + p_k/\rho_k)),\ i = 1,\ldots,d. \tag{3.60b}$$

The compatibility conditions (3.3) hold by derivation of (3.59). In the barotropic case non-strict convexity of U has been proven in [50]. Alternatively, the proof of Theorem 3.1 for the isothermal case also applies in case of isentropic fluids where

$$U(\boldsymbol{u}) = \sum\nolimits_{k=1}^{K}\alpha_k\rho_k(e_k + \boldsymbol{v}_k^2/2) = \sum\nolimits_{k=1}^{K}\alpha_k U_k(\boldsymbol{u}_k) \tag{3.61}$$

with $e_k = e_k(\tau_k, s_k)$ and $s_k = const.$ For the Hessian $\boldsymbol{U}_k''$ we again obtain (3.31). Then convexity of e_k with respect to τ_k according to (1.17) ensures that the Hessian is positive semi-definite.

We emphasize that in case of an isothermal fluid the mixture total energy $\rho E = \sum_{k=1}^{K}\alpha_k\rho_k E_k$ is *not* an admissible entropy function because (3.55) does not account for heat exchange. To ensure the entropy inequality (3.6) we have to verify that the right-hand side of (3.59) is not positive. This will provide us with admissibility criteria for the interfacial pressures as well as the relaxation terms.

3.2.1. *Entropy production due to interfacial velocity and pressure*

Obviously, we cannot control the sign of Π_k. Therefore we determine the interfacial pressures $P_{k,l}$ and the interfacial velocity $\boldsymbol{V}_I$ such that the sum $\overline{\Pi} := \sum_{k=1}^K \overline{\Pi}_k$ vanishes and the conservation constraints (1.11) or, equivalently, (1.12) hold. Assuming that the interfacial velocity can be written as a linear combination of the velocities of the components according to (2.38) there exists a unique closure for the interfacial pressures.

Theorem 3.6. *(Entropy production due to interfacial states) For any convex combination (2.38) for the interfacial velocity $\boldsymbol{V}_I$ there uniquely exist interfacial pressures*

$$P_{k,l} = \beta_k p_l + p_k(1-\beta_k) \tag{3.62}$$

such that the production term $\overline{\Pi} = \sum_{k=1}^K \overline{\Pi}_k$ vanishes and the conservation constraints (1.11) or, equivalently, (1.12) hold. In particular, the interfacial pressures are all positive and

$$P_I = \sum_{k=1}^K p_k(1-\beta_k). \tag{3.63}$$

Proof. First of all, we note that by the conservation constraint (1.11) and the definition (3.57) of $\overline{\Pi}_k$ it holds

$$\overline{\Pi} = \sum_{k=1}^K \overline{\Pi}_k = \sum_{k=1}^K \sum_{l=1,\neq k}^K (P_{k,l} - p_k)(\boldsymbol{V}_I - \boldsymbol{v}_k)\cdot\nabla\alpha_l. \tag{3.64}$$

Obviously, this term coincides with $T_{ref}\Pi = T_{ref}\sum_{k=1}^K \Pi_k$ with Π_k determined by (3.14) in the isothermal case. Since by the saturation constraint (1.2) we may substitute $\nabla\alpha_K$ by the linearly independent gradients $\nabla\alpha_k$, $k = 1,\dots,K-1$, we may rewrite $\overline{\Pi}$ in terms of the linearly independent gradients. Then $\overline{\Pi}$ vanishes if and only if the cofactors vanish, i.e.,

$$\sum_{k=1,k\neq l}^{K-1} \left((P_{k,l}-p_k) - (P_{k,K}-p_k)\right)(\boldsymbol{v}_k - \boldsymbol{V}_I) + (P_{K,l}-p_K)(\boldsymbol{v}_K - \boldsymbol{V}_I) = \boldsymbol{0}, \quad l = 1,\dots,K-1. \tag{3.65}$$

Next we use that the interfacial velocity $\boldsymbol{V}_I$ is a convex combination of the single component velocities $\boldsymbol{v}_k$, i.e., (2.38) holds. Then we may rewrite the velocity differences $\boldsymbol{v}_k - \boldsymbol{V}_I$ in terms of the differences $\boldsymbol{v}_i - \boldsymbol{v}_{i+1}$, $i = 1,\dots,K-1$. Rearranging (3.65) in terms of these yields $(K-1)^2$ equations for the $(K-1)K$ unknowns $P_{k,l}$, $k,l = 1,\dots,K$, $k \neq l$. Additional $K-1$ equations are provided by the conservation constraints (1.12). Altogether we obtain a linear system of equations for the interfacial pressures. This system is identical to the linear system derived in [38], Eqns. (6.20) and (6.22), where the temperatures are formally replaced by 1. This system has a unique solution. The proof is identical to the one of the analogue theorem in [38] in case of the full Baer-Nunziato model.

□

We would like to mention that in [25] the interfacial pressures in case of barotropic fluids are given for $K = 2,3,4$. The proof is shortly sketched without investigating the regularity of the resulting linear system. Furthermore, as in the isothermal case the interfacial pressures (3.62) coincide with those presented in [38] for the full multi-component Baer-Nunziato model at local thermal equilibrium.

Remark 3.2 (Interfacial pressures for isothermal fluids)**.** *We emphasize that in the isothermal case using for the convex combination* (2.38) *with the coefficients* (2.39)*, the interfacial pressures* (3.62) *coincide with the interfacial pressures* (3.32) *although the production terms due to the interfacial states* (3.14) *and* (3.57) *differ. Thus, for isothermal fluids the two admissible entropy-entropy flux pairs result in the same interfacial pressures.*

3.2.2. *Entropy production due to relaxation*

To be admissible with the entropy inequality (3.6) the production $\sum_{k=1}^K \overline{S}_{\alpha\rho E,k}$ has to be non-positive. Note that in abuse of physical notation we call this term entropy production rather than energy production with

respect to the mathematical concept of entropy-entropy flux pairs because energy is known to be conserved. In the following we consider one-by-one the contributions of pressure and velocity relaxation. The analysis is similar to [38] in case of the physical entropy production. Note that all the relaxation processes are in agreement with the constraints due to conservation (1.10).

Theorem 3.7. *(Entropy production due to pressure relaxation) Let the relaxation parameter be non-negative, i.e., $\theta_p \geq 0$. Furthermore, let the coefficients $d_{k,l}$ be positive and symmetric, i.e.*

$$0 \leq d_{k,l} = d_{l,k}, \quad l,k = 1,\dots,K,\ l \neq k. \tag{3.66}$$

Then the entropy production due to pressure relaxation (1.48a) is non-positive, i.e.,

$$S_U^p = \sum\nolimits_{k=1}^{K} S_{U,k}^p \leq 0 \tag{3.67}$$

with entropy production of component $k = 1,\dots,K$

$$S_{U,k}^p = \overline{S}_{\alpha\rho\overline{E},k}^p = -\theta_p p_k \sum\nolimits_{l=1}^{K} d_{k,l}(p_k - p_l). \tag{3.68}$$

In particular, this holds true for the model (1.46).

Proof. First of all, we note that by symmetry of arbitrary coefficients $a_{k,l} = a_{l,k}$ it holds

$$\sum\nolimits_{k=1}^{K-1}\sum\nolimits_{l=k+1}^{K} a_{k,l} = \sum\nolimits_{l=2}^{K}\sum\nolimits_{k=1}^{l-1} a_{k,l} = \sum\nolimits_{k=2}^{K}\sum\nolimits_{l=1}^{k-1} a_{l,k}$$

From this we conclude with $a_{k,l} = p_k^2 d_{k,l}$ and $a_{k,l} = p_k d_{k,l} p_l$

$$\sum\nolimits_{l,k=1,l\neq k}^{K} p_k^2 d_{k,l} = \sum\nolimits_{k=2}^{K}\sum\nolimits_{l=1}^{k-1} p_k^2 d_{k,l} + \sum\nolimits_{k=1}^{K-1}\sum\nolimits_{l=k+1}^{K} p_k^2 d_{k,l} = \sum\nolimits_{k=2}^{K}\sum\nolimits_{l=1}^{k-1}(p_k^2 + p_l^2) d_{k,l},$$
$$\sum\nolimits_{l,k=1,l\neq k}^{K} p_k p_l d_{k,l} = \sum\nolimits_{k=2}^{K}\sum\nolimits_{l=1}^{k-1} p_k p_l d_{k,l} + \sum\nolimits_{k=1}^{K-1}\sum\nolimits_{l=k+1}^{K} p_k p_l d_{k,l} = 2\sum\nolimits_{k=2}^{K}\sum\nolimits_{l=1}^{k-1} p_k p_l d_{k,l}.$$

Combining these results we obtain

$$\sum\nolimits_{l,k=1,l\neq k}^{K} p_k d_{k,l}(p_k - p_l) = \sum\nolimits_{k=2}^{K}\sum\nolimits_{l=1}^{k-1} d_{k,l}(p_k^2 - 2p_k p_l + p_l^2) = \sum\nolimits_{k=2}^{K}\sum\nolimits_{l=1}^{k-1} d_{k,l}(p_k - p_l)^2.$$

Obviously, this term is non-negative and vanishes only if all pressures coincide. Thus, by assumption the energy production $\overline{S}_{\alpha\rho\overline{E},k}^p$ is non-positive. In particular, this holds true for the special model (1.46) because $d_{k,l} = \alpha_k\alpha_l$ satisfies the constraint (3.66)

□

Since the proof holds for an arbitrary number of components K the theorem generalizes the result in [25] for $K = 4$. For $K = 2$ the result is well-known.

For the velocity relaxation we obtain a similar result.

Theorem 3.8. *(Entropy production due to velocity relaxation) Let the relaxation parameter be non-negative, i.e., $\theta_v \geq 0$. Furthermore, let the coefficients $e_{k,l}$ be non-negative and symmetric, i.e.*

$$0 \leq e_{k,l} = e_{l,k}, \quad l,k = 1,\dots,K,\ l \neq k. \tag{3.69}$$

Then the entropy production due to velocity relaxation (1.48b) is non-positive, i.e.,

$$S_U^v = \sum\nolimits_{k=1}^{K} S_{U,k}^v \leq 0 \tag{3.70}$$

with entropy production of component $k = 1, \dots, K$

$$S^v_{U,k} = \overline{S}^v_{\alpha\rho\overline{E},k} = -\theta_v \sum\nolimits_{l=1}^{K} e_{k,l}\, \boldsymbol{v}_k \cdot (\boldsymbol{v}_k - \boldsymbol{v}_l). \tag{3.71}$$

In particular, this holds true for the model (1.47).

Proof. The proof is in full analogy to the pressure relaxation where we replace $d_{k,l}$, p_k and p_l by $e_{k,l}$, $v_{k,i}$ and $v_{l,i}$, respectively. Then we obtain

$$\sum\nolimits_{l,k=1,l\neq k}^{K} e_{k,l}\boldsymbol{v}_k \cdot (\boldsymbol{v}_k - \boldsymbol{v}_l) = \sum\nolimits_{i=1}^{d} \sum\nolimits_{k=2}^{K} \sum\nolimits_{l=1}^{k-1} e_{k,l}(v_{k,i} - v_{l,i})^2 = \sum\nolimits_{k=2}^{K} \sum\nolimits_{l=1}^{k-1} e_{k,l}(\boldsymbol{v}_k - \boldsymbol{v}_l)^2.$$

Obviously, this term is non-negative and vanishes only if all velocities coincide. Thus, by assumption the energy production $\overline{S}^v_{\alpha\rho\overline{E},k}$ is non-positive. In particular, this holds true for the special model (1.47) because $e_{k,l} = \alpha_k\rho_k\alpha_l\rho_l / \sum_{j=1}^{K} \alpha_j\rho_j$ satisfies the constraint (3.69).

□

Since there is no temperature relaxation and no relaxation of chemical potentials, there is no corresponding energy production.

4. Conclusion

In this work we have investigated Baer-Nunziato-type models for barotropic fluids with an arbitrary number of components. In the literature frequently barotropic fluids are confined to isentropic fluids, cf. [2, 25, 50] whereas here we also consider isothermal fluids.

First of all, we have determined eigenvalues and eigenvectors for the quasi-1D model in case of general barotropic fluids, i.e., the phasic pressures only depend on the phasic densities. For non-resonance states a full set of linearly independent eigenvectors has been derived. The different fields have been characterized as either genuinely nonlinear or linearly degenerated assuming that the phasic fundamental derivatives of thermodynamics are positive and choosing a particular convex combination for the interfacial velocity. For all fields we have given a full set of Riemann invariants whose gradients are linearly independent. Furthermore we have verified that the quasi-1D equations is symmetrizable away from non-resonance states constructing a symmetrizer.

Two entropy-entropy flux pairs have been derived. While both of them are admissible for isothermal fluids there is only one of the two that is admissible for general barotropic fluids including isentropic fluids. For these entropy-entropy flux pairs we have verified that (i) the (mathematical) entropy functions are convex but not strictly convex even if the phasic entropy functions are strictly convex and (ii) the (mathematical) entropy inequality holds, i.e., the sum of the entropy production due to the non-conservative products and the relaxation processes is non-increasing. It is worthwhile mentioning that for isothermal fluids and isentropic fluids we have used different entropy functions, namely, the mixture of the phasic free total energies and the mixture of phasic total energies, respectively. In both cases we have to account for the heat flux in the balance laws for the phasic total energies to render these equations to be redundant in case of barotropic fluids. From a physical point of view this is mandatory. In the isothermal case heat production due to the work performed in the system has to be removed from the system to keep the temperatures constant. On the other hand, for isentropic fluids there is an energy production due to the relaxation processes. For both cases corresponding heat fluxes have been identified. Furthermore, since we cannot control the sign of the energy production due to the non-conservative products we have determined the interfacial velocity and the interfacial pressures such that this production term vanishes. Starting from an interfacial velocity determined by a convex combination of phasic velocities a unique set of interfacial pressures has been identified that coincides for both isothermal fluids and isentropic fluids.

Appendix A

Let denote the vector of conservative variables by

$$\boldsymbol{u} = (\alpha_1, \dots, \alpha_{K-1}, \boldsymbol{u}_1^t, \dots, \boldsymbol{u}_K^t)^t,\ \boldsymbol{u}_k = (\alpha_k \rho_k, \alpha_k \rho_k \boldsymbol{v}_k^t)^t. \tag{4.1}$$

Then the projected system in conservative variables can be written in quasi-conservative form as

$$\partial_t \boldsymbol{w} + \overline{\boldsymbol{A}}_n(\boldsymbol{u})\, \partial_\xi \boldsymbol{u} = \boldsymbol{0}., \quad \overline{\boldsymbol{A}}_n(\boldsymbol{u}) = \boldsymbol{J}(\boldsymbol{w}(\boldsymbol{u}))\, \boldsymbol{B}_n(\boldsymbol{w}(\boldsymbol{u}))\, \boldsymbol{J}(\boldsymbol{w}(\boldsymbol{u}))^{-1}, \tag{4.2}$$

where the Jacobian $\boldsymbol{J}(\boldsymbol{w}) = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{w}}(\boldsymbol{w})$ of the variable transformation and its inverse $\boldsymbol{J}(\boldsymbol{w})^{-1} = \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{u}}(\boldsymbol{u})$ are determined by

$$\boldsymbol{J}(\boldsymbol{w}) = \begin{pmatrix} \boldsymbol{I}_{K-1} & & & \\ \frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{\alpha}} & \frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{w}_1} & & \\ \vdots & & \ddots & \\ \frac{\partial \boldsymbol{u}_K}{\partial \boldsymbol{\alpha}} & & & \frac{\partial \boldsymbol{u}_K}{\partial \boldsymbol{w}_K} \end{pmatrix}, \quad \boldsymbol{J}(\boldsymbol{w})^{-1} = \begin{pmatrix} \boldsymbol{I}_{K-1} & & & \\ \frac{\partial \boldsymbol{w}_1}{\partial \boldsymbol{\alpha}} & \frac{\partial \boldsymbol{w}_1}{\partial \boldsymbol{u}_1} & & \\ \vdots & & \ddots & \\ \frac{\partial \boldsymbol{w}_K}{\partial \boldsymbol{\alpha}} & & & \frac{\partial \boldsymbol{w}_K}{\partial \boldsymbol{u}_K} \end{pmatrix} \tag{4.3}$$

with

$$\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{\alpha}} = \begin{pmatrix} \rho_k(p_k) \\ \rho_k(p_k)\boldsymbol{v}_k \end{pmatrix} \boldsymbol{e}_{K-1,k}^t,\ k = 1, \dots, K-1,\ \frac{\partial \boldsymbol{u}_K}{\partial \boldsymbol{\alpha}} = -\begin{pmatrix} \rho_K(p_K) \\ \rho_K(p_K)\boldsymbol{v}_K \end{pmatrix} \boldsymbol{1}_{K-1}^t, \tag{4.4}$$

$$\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{w}_k} = \alpha_k \begin{pmatrix} \boldsymbol{0}_d^T & \rho_k'(p_k) \\ \rho_k(p_k)\boldsymbol{I}_d & \rho_k'(p_k)\boldsymbol{v}_k \end{pmatrix},\ k = 1, \dots, K, \tag{4.5}$$

$$\frac{\partial \boldsymbol{w}_k}{\partial \boldsymbol{\alpha}} = \begin{pmatrix} \boldsymbol{0}_d \\ -\bar{c}_k^2(\rho_k)\rho_k/\alpha_k \end{pmatrix} \boldsymbol{e}_{K-1,k}^t,\ k = 1, \dots, K-1,\ \frac{\partial \boldsymbol{w}_K}{\partial \boldsymbol{\alpha}} = \begin{pmatrix} \boldsymbol{0}_d \\ \bar{c}_K^2(\rho_K)\rho_K/\alpha_K \end{pmatrix} \boldsymbol{1}_{K-1}^t \tag{4.6}$$

$$\frac{\partial \boldsymbol{w}_k}{\partial \boldsymbol{u}_k} = \begin{pmatrix} -(\alpha_k\rho_k)^{-1}\boldsymbol{v}_k & (\alpha_k\rho_k)^{-1}\boldsymbol{I}_d \\ \bar{c}_k^2(\rho_k)/\alpha_k & \boldsymbol{0}_d^T \end{pmatrix},\ k = 1, \dots, K. \tag{4.7}$$

Then the eigensystem for the matrices $\boldsymbol{B}_n(\boldsymbol{w})$ and $\overline{\boldsymbol{A}}_n(\boldsymbol{u})$ are related by

$$\overline{\boldsymbol{R}}_n(\boldsymbol{u}) = \boldsymbol{J}(\boldsymbol{w}(\boldsymbol{u}))\, \boldsymbol{R}_n(\boldsymbol{w}(\boldsymbol{u}))\, \boldsymbol{D}(\boldsymbol{u})^{-1},\ \overline{\boldsymbol{L}}_n(\boldsymbol{u}) = \boldsymbol{D}(\boldsymbol{u})\, \boldsymbol{L}_n(\boldsymbol{w}(\boldsymbol{u}))\, \boldsymbol{J}(\boldsymbol{w}(\boldsymbol{u}))^{-1},\ \overline{\boldsymbol{\Lambda}}_n(\boldsymbol{u}) = \boldsymbol{\Lambda}_n(\boldsymbol{w}(\boldsymbol{u})), \tag{4.8}$$

$$\overline{\boldsymbol{L}}_n(\boldsymbol{u})\, \overline{\boldsymbol{A}}_n(\boldsymbol{u})\, \overline{\boldsymbol{R}}_n(\boldsymbol{u}) = \overline{\boldsymbol{\Lambda}}_n(\boldsymbol{u}),\ \overline{\boldsymbol{L}}_n(\boldsymbol{u})\, \overline{\boldsymbol{R}}_n(\boldsymbol{u}) = \boldsymbol{I} \tag{4.9}$$

with scaling matrices given by the block diagonal matrix

$$\boldsymbol{D}(\boldsymbol{u}) = \mathrm{diag}(\boldsymbol{I}_{K-1}, \boldsymbol{D}_1(\boldsymbol{u}_1), \dots, \boldsymbol{D}_K(\boldsymbol{u}_K)),\ \boldsymbol{D}_k(\boldsymbol{u}_k) = \alpha_k \mathrm{diag}(1, \rho_k \boldsymbol{1}_{d-1}, 1),\ k = 1, \dots, K. \tag{4.10}$$

Finally, we obtain

$$\overline{\boldsymbol{R}}_n(\boldsymbol{u}) = \begin{pmatrix} \boldsymbol{I}_{K-1} & & & \\ \overline{\boldsymbol{R}}_{I,n}^1 & \overline{\boldsymbol{R}}_{1,n} & & \\ \vdots & & \ddots & \\ \overline{\boldsymbol{R}}_{I,n}^K & & & \overline{\boldsymbol{R}}_{K,n} \end{pmatrix}, \quad \overline{\boldsymbol{L}}_n(\boldsymbol{u}) = \begin{pmatrix} \boldsymbol{I}_{K-1} & & & \\ \overline{\boldsymbol{L}}_{I,n}^1 & \overline{\boldsymbol{L}}_{1,n} & & \\ \vdots & & \ddots & \\ \overline{\boldsymbol{L}}_{I,n}^K & & & \overline{\boldsymbol{L}}_{K,n} \end{pmatrix} \tag{4.11}$$

with blocks

$$\overline{\boldsymbol{R}}_{k,n} = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ \boldsymbol{v}_k - \overline{c}_k \boldsymbol{n} & \boldsymbol{t}_1 & \dots & \boldsymbol{t}_{d-1} & \boldsymbol{v}_k + \overline{c}_k \boldsymbol{n} \end{pmatrix}, \ k = 1, \dots, K, \tag{4.12}$$

$$\overline{\boldsymbol{R}}^k_{I,n} = \rho_k \left(\begin{pmatrix} 1 \\ \boldsymbol{v}_k \end{pmatrix} \boldsymbol{e}^t_{K-1,k} + \sigma_k^{-1} \begin{pmatrix} \boldsymbol{\beta}^t_k - \overline{c}_k^{-2} (\delta^n_k)^2 \boldsymbol{\gamma}^t_k \\ \boldsymbol{n}(\boldsymbol{\gamma}^t_k - \boldsymbol{\beta}^t_k)\delta^n_k + \boldsymbol{v}_k(\boldsymbol{\beta}^t_k - \overline{c}_k^{-2}(\delta^n_k)^2 \boldsymbol{\gamma}^t_k) \end{pmatrix} \right), \ k = 1, \dots, K-1, \tag{4.13}$$

$$\overline{\boldsymbol{R}}^K_{I,n} = \rho_K \left(- \begin{pmatrix} 1 \\ \boldsymbol{v}_K \end{pmatrix} \boldsymbol{1}^t_{K-1} + \sigma_K^{-1} \begin{pmatrix} \boldsymbol{\beta}^t_K - \overline{c}_K^{-2} (\delta^n_K)^2 \boldsymbol{\gamma}^t_K \\ \boldsymbol{n}(\boldsymbol{\gamma}^t_K - \boldsymbol{\beta}^t_K)\delta^n_K + \boldsymbol{v}_K(\boldsymbol{\beta}^t_K - \overline{c}_K^{-2}(\delta^n_K)^2 \boldsymbol{\gamma}^t_K) \end{pmatrix} \right) \tag{4.14}$$

$$\overline{\boldsymbol{L}}_{k,n} = \frac{1}{2\overline{c}_k} \begin{pmatrix} \overline{c}_k + v_{k,n} & -2\overline{c}_k \boldsymbol{v}_k \cdot \boldsymbol{t}_1 & \dots & -2\overline{c}_k \boldsymbol{v}_k \cdot \boldsymbol{t}_{d-1} & \overline{c}_k - v_{k,n} \\ -\boldsymbol{n} & 2\overline{c}_k \boldsymbol{t}_1 & \dots & 2\overline{c}_k \boldsymbol{t}_{d-1} & \boldsymbol{n} \end{pmatrix}^t, \ k = 1, \dots, K, \tag{4.15}$$

$$\overline{\boldsymbol{L}}^k_{I,n} = -\frac{\rho_k}{2\overline{c}_k^2} \left(\begin{pmatrix} \overline{c}_k^2 \\ \boldsymbol{0}_{d-1} \\ \overline{c}_k^2 \end{pmatrix} \boldsymbol{e}^t_{K-1,k} + \sigma_k^{-1} \begin{pmatrix} \overline{c}_k^2 \boldsymbol{\beta}^t_k - (\delta^n_k)^2 \boldsymbol{\gamma}^t_k + \overline{c}_k(\boldsymbol{\beta}^t_k - \boldsymbol{\gamma}^t_k)\delta^n_k \\ \boldsymbol{0}_{d-1} \\ \overline{c}_k^2 \boldsymbol{\beta}^t_k - (\delta^n_k)^2 \boldsymbol{\gamma}^t_k - \overline{c}_k(\boldsymbol{\beta}^t_k - \boldsymbol{\gamma}^t_k)\delta^n_k \end{pmatrix} \right), \ k = 1, \dots, K-1, \tag{4.16}$$

$$\overline{\boldsymbol{L}}^K_{I,n} = -\frac{\rho_K}{2\overline{c}_K^2} \left(- \begin{pmatrix} \overline{c}_K^2 \\ \boldsymbol{0}_{d-1} \\ \overline{c}_K^2 \end{pmatrix} \boldsymbol{1}^t_{K-1} + \sigma_K^{-1} \begin{pmatrix} \overline{c}_K^2 \boldsymbol{\beta}^t_K - (\delta^n_K)^2 \boldsymbol{\gamma}^t_K + \overline{c}_K(\boldsymbol{\beta}^t_K - \boldsymbol{\gamma}^t_K)\delta^n_K \\ \boldsymbol{0}_{d-1} \\ \overline{c}_K^2 \boldsymbol{\beta}^t_K - (\delta^n_K)^2 \boldsymbol{\gamma}^t_K - \overline{c}_K(\boldsymbol{\beta}^t_K - \boldsymbol{\gamma}^t_K)\delta^n_K \end{pmatrix} \right). \tag{4.17}$$

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