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RELATIVE ENTROPY FOR THE NUMERICAL DIFFUSIVE LIMIT OF THE LINEAR JIN-XIN SYSTEM

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Abstract. This paper deals with the diffusive limit of the Jin and Xin model and its approximation by an asymptotic preserving finite volume scheme. At the continuous level, we determine a convergence rate to the diffusive limit by means of a relative entropy method. Considering a semi-discrete approximation (discrete in space and continuous in time), we adapt the method to this semi-discrete framework and establish that the approximated solutions converge towards the discrete convection-diffusion limit with the same convergence rate.

1. Introduction

Jin and Xin introduced in [JX95] a relaxation technique in order to build robust numerical schemes for the Cauchy problem associated with the nonlinear scalar equation

$$\partial_t u + \partial_x f(u) = 0, \tag{1}$$

where f is typically a Lipschitz-continuous and nonlinear function. Relaxation consists in augmenting the equation into a system which reads

$$\partial_t u^\nu + \partial_x v^\nu = 0, \tag{2}$$

$$\partial_t v^\nu + \lambda^2 \partial_x u^\nu = \frac{1}{\nu}(f(u^\nu) - v^\nu), \tag{3}$$

where u^ν, $v^\nu : Q_T \to \mathbb{R}$ are the unknowns, with $Q_T = [0, T) \times \mathbb{R}$ and $T > 0$, $\lambda > 0$ is a given constant and ν is a relaxation parameter. The hyperbolic part of (2)-(3) is linear, of wave velocities $\pm\lambda$. The nonlinearity of f is shifted to the right hand side of the second equation. Formally, as ν tends to zero, one observes that the second equation gives $v^\nu = f(u^\nu)$ and then solutions to (2)-(3) converge formally to the solution of (1). The question of the convergence of solutions of u^ν towards a weak entropy solution u of (1) has been addressed, for example, in [CLL94, Nat96, Ser00]. The latter reference includes the proof of the existence of invariant domains for the relaxed system (2)-(3) as long as they exist for the scalar equation (1). Moreover any convex entropy $\eta \in C^2(\mathbb{R})$ of the scalar equation (1) extends to an entropy $E \in C^2(\mathbb{R}^2)$ for the relaxed system (2)-(3). These two ingredients make it possible to prove, by a method of compensated compactness, convergence in long time to the scalar equation for arbitrary initial data.

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In the present paper we focus on a slightly different scaling of the original Jin and Xin model. The system we consider reads

$$\partial_t u + \partial_x v = 0, \tag{4}$$

$$\varepsilon^2 \partial_t v + \lambda^2 \partial_x u = au - v, \tag{5}$$

where ε is the relaxation parameter. Here we consider the linear case $f(u) = au$, where a is a given coefficient. For the sake of readability, the ε dependency is not explicitly noted for u and v. This system is endowed with an entropy-flux pair $(E^\varepsilon, F^\varepsilon)$ which complies with the entropy inequality

$$\partial_t E^\varepsilon(u,v) + \partial_x F^\varepsilon(u,v) \leq -(au-v)^2. \tag{6}$$

In the specific case of a linear relaxation, the entropy function reads

$$E^\varepsilon(u,v) = \frac{\lambda^2}{2}u^2 + \frac{\varepsilon^2}{2}v^2 - \varepsilon^2 auv, \tag{7}$$

and the entropy flux F^ε is defined by

$$F^\varepsilon(u,v) = -\frac{\lambda^2 a}{2}u^2 - \frac{\varepsilon^2 a}{2}v^2 + \lambda^2 uv. \tag{8}$$

Assuming the subcharacteristic condition [Whi74]

$$\lambda > \varepsilon\,|a|, \tag{9}$$

the entropy functional $E^\varepsilon(u,v)$ is strictly convex in the sense that there exists $\beta_1 \geq \beta_0 > 0$ such that

$$\mathrm{spec}(\nabla^2 E^\varepsilon) \subset [\beta_0, \beta_1]. \tag{10}$$

In this article, we focus on the diffusion limit of the solutions $w = (u,v)$ of (4)-(5). Indeed in the limit $\varepsilon \to 0$, the solutions $w = (u,v)$ of (4)-(5) converge, in a sense to be prescribed, to the solutions $\bar{w} = (\bar{u},\bar{v})$ of the following convection-diffusion equation

$$\partial_t \bar{u} + a\partial_x \bar{u} = \lambda^2 \partial_{xx}\bar{u}, \tag{11}$$

$$\bar{v} = a\bar{u} - \lambda^2 \partial_x \bar{u}. \tag{12}$$

In [JL98], the authors exhibited a convergence result as the relaxation parameter ε tends to zero, using a priori estimates in appropriate Sobolev spaces, for initial data close to ones producing a travelling-wave solution. Very recently Crin-Barat and Shou [CBS23] have studied the diffusive relaxation limit of the system (4)-(5) toward viscous conservation laws (11)-(12) in the multi-dimensional setting. They prove global well-posedness of strong solutions for initial data close to constant state in suitable Besov spaces.

Adapting the technique in [Bou99], Bianchini considered in [Bia18] the diffusive relaxation process of the Jin-Xin model in terms of BGK type approximations. She established the convergence towards a nonlinear heat equation in the relaxation limit and managed to derive a convergence rate in $O(\sqrt{\varepsilon})$ in the L^2 norm. However, by applying a numerical scheme preserving the hyperbolic-parabolic asymptotics proposed in [JPT98] to the model (4)-(5), a rate of convergence in ε^2 is observed. Figure 1 presents the numerical evidence, the details of the numerical scheme and the test case being given in Section 3.3.

Our present work is motivated by the observation of this different convergence rate in numerical simulations. It turns out that we already observed similar numerical rates of convergence when studying the convergence of the discrete solutions of the p-system with damping towards the discrete solutions of the porous media

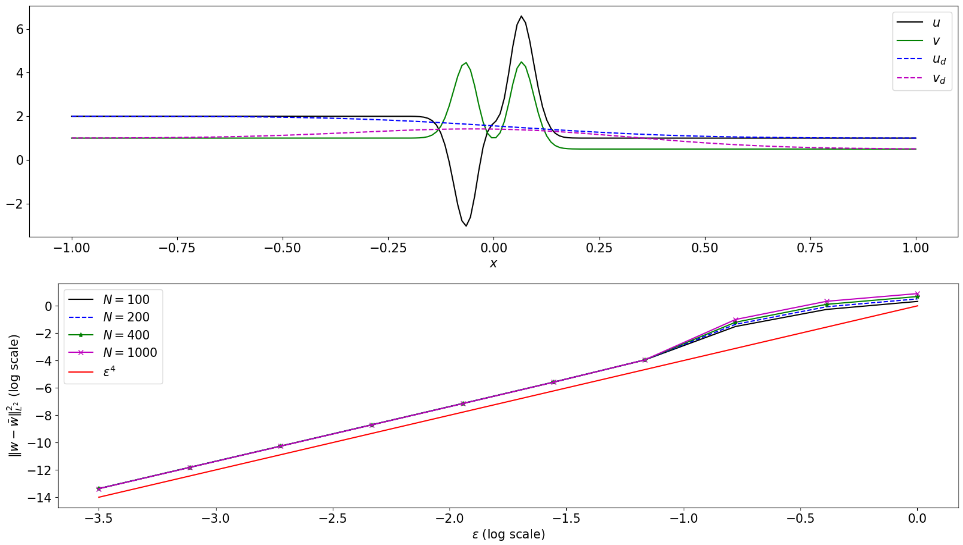

FIGURE 1. Linear test case. Top: profile of (u, v) in space, compared to $(\bar{u}, \bar{v})$. Bottom: L^2 norm of the error $\|(u,v) - (\bar{u},\bar{v})\|^2_{L^2(Q_T)}$ with respect to ε in log scale.

equation [BBCM16, BBBC19]. The numerical analysis, performed in the latter reference, is an adaptation at the discrete level of the relative entropy method used by Lattanzio and Tzavaras in [LT13] to prove the convergence in the continuous setting. The key tool is the relative entropy functional, which behaves as a squared L^2 norm of the difference between a solution of the p-system and a solution of the porous media equation. Actually the relative entropy method has been applied successfully to a large number of problems concerning hyperbolic systems, for instance: adaptation of the weak-strong uniqueness results of Dafermos [Daf79] and Di Perna [DiP79] to solutions of hyperbolic systems of conservation laws with weaker regularity [GJK21, Vas23], uniqueness of measure-valued solutions to hyperbolic-parabolic systems [CT18, Chr21], asymptotic stability of stationary solutions to hyperbolic systems with singular geometry terms and nonconservative products [Seg18]. Besides in [Bia21] Bianchini also makes use of the relative entropy method as well and applies it to an intermediate BGK-type model.

Our purpose is to make use of the relative entropy technique to establish the convergence of the solutions of the system (4)–(5) towards the solutions of (11)–(12), both at the continuous and the discrete level. The relative entropy $E^\varepsilon(w|\bar{w})$ of the system (4)–(5) is defined as the first order Taylor expansion of E around a smooth solution $\bar{w}$ of (11)–(12):

$$\begin{aligned} E^\varepsilon(w|\bar{w}) &= E^\varepsilon(w) - E^\varepsilon(\bar{w}) - \nabla E^\varepsilon(\bar{w}) \cdot (w - \bar{w}) \\ &= \frac{\lambda^2}{2}(u-\bar{u})^2 + \frac{\varepsilon^2}{2}(v-\bar{v})^2 - \varepsilon^2 a(u-\bar{u})(v-\bar{v}) \end{aligned} \tag{13}$$

for w a classical solution of (4)–(5). Thanks to the strict convexity (10) of E^ε, the relative entropy behaves like a squared L^2 norm of the difference between the solution (u, v) of the hyperbolic relaxation system (4)–(5) and the solution $(\bar{u}, \bar{v})$ of the convection-diffusion limit problem (11)–(12), namely

$$\frac{\beta_0}{2}\left(|u-\bar{u}|^2 + |v-\bar{v}|^2\right) \le E^\varepsilon(w|\bar{w}) \le \frac{\beta_1}{2}\left(|u-\bar{u}|^2 + |v-\bar{v}|^2\right). \tag{14}$$

The purpose of this article is to present similar results as in [BBCM16] for convergence of solutions of (4)–(5) towards solutions to (11)–(12). Section 2 is devoted to the continuous result. We first establish a relative entropy identity. Then under some regularity assumptions on the parabolic solutions, we establish a convergence result in relative entropy with the expected convergence rate of ε^2. The result is then extended to the semi-discrete level, by introducing a discrete in space and continuous in time numerical scheme in Section 3. In Section 3.2 we construct a discrete relative entropy identity with numerical residuals which we manage to control, leading to the convergence result with the expected convergence rate. The section 3.3 concludes with the details of the numerical result presented in Figure 1. In conclusion, we provide some perspectives for general nonlinear relaxation terms.

2. The continuous setting

In this section, we first study the continuous case, adapting the diffusive relative entropy method developed in [LT13] to the case of the linear Jin-Xin relaxation system (4)–(5).

We first establish an evolution law satisfied by the relative entropy (13).

Lemma 2.1. *Let $w = (u, v)$ be a strong entropy solution of (4)–(5) and $\bar{w} = (\bar{u}, \bar{v})$ be a smooth solution of the limit problem (11)–(12). Then the relative entropy $E^\varepsilon(w|\bar{w})$, defined by (13), satisfies the following evolution law:*

$$\partial_t E^\varepsilon(w|\bar{w}) + \partial_x F^\varepsilon(w|\bar{w}) = -\left(a(u-\bar{u}) - (v-\bar{v})\right)^2 + \left((v-\bar{v}) - a(u-\bar{u})\right)\varepsilon^2 \partial_t \bar{v}, \tag{15}$$

where the relative entropy flux is given by

$$F^\varepsilon(w|\bar{w}) = -\frac{\lambda^2 a}{2}(u-\bar{u})^2 - \frac{\varepsilon^2 a}{2}(v-\bar{v})^2 + \lambda^2(u-\bar{u})(v-\bar{v}). \tag{16}$$

Proof. Using the definition (13), the time derivative of the relative entropy satisfies

$$\begin{aligned}\partial_t E^\varepsilon(w|\bar{w}) = &\left[\lambda^2(u-\bar{u}) - \varepsilon^2 a(v-\bar{v})\right]\partial_t(u-\bar{u}) \\ &+ \left[(v-\bar{v}) - a(u-\bar{u})\right]\varepsilon^2\partial_t(v-\bar{v}).\end{aligned}$$

Remarking that the limit problem (11)–(12) can be written such that we get the same left hand side than for the Jin-Xin system (4)–(5)

$$\begin{aligned}&\partial_t \bar{u} + \partial_x \bar{v} = 0,\\ &\varepsilon^2\partial_t\bar{v} + \lambda^2\partial_x\bar{u} = a\bar{u} - \bar{v} + \varepsilon^2\partial_t\bar{u},\end{aligned}$$

it yields

$$\begin{aligned}\partial_t E^\varepsilon(w|\bar{w}) = &-\left[\lambda^2(u-\bar{u}) - \varepsilon^2 a(v-\bar{v})\right]\partial_x(v-\bar{v}) \\ &- \left[(v-\bar{v}) - a(u-\bar{u})\right]\lambda^2\partial_x(u-\bar{u}) \\ &- \left(a(u-\bar{u}) - (v-\bar{v})\right)^2 + \left((v-\bar{v}) - a(u-\bar{u})\right)\varepsilon^2\partial_t\bar{v},\end{aligned}$$

which concludes the proof using the definition of the relative entropy flux (16). □

In addition, we now suppose that the systems (4)–(5) and (11)–(12) are endowed with initial conditions such that the following limits hold:

$$\lim_{x\to\pm\infty} w(t,x) = \lim_{x\to\pm\infty} \bar{w}(t,x) = w_\pm, \tag{17}$$

where $w_\pm$ are constant states.

Now, to compare w solution of (4)–(5) and $\bar{w}$ solution of (11)–(12), let us introduce the positive error estimate given by

$$\phi^\varepsilon(t) = \int_{\mathbb{R}} E^\varepsilon(w|\bar{w})\,\mathrm{d}x. \tag{18}$$

The following convergence result, with an explicit rate, is established.

Theorem 2.2. *Consider initial data w_0 for (4)–(5) and $\bar{w}_0$ for (11)–(12) such that $\phi^\varepsilon(0) < +\infty$. Endowed with these initial data, let $\bar{w}$ be the smooth solution of (11)–(12) defined on $Q_T = [0,T) \times \mathbb{R}$, and w be a strong entropy solution of (4)–(5). Let us assume that there exists $K > 0$ such that $\|\partial_t \bar{v}\|_{L^2(Q_T)} \leq K$. Then the following stability estimate holds*

$$\phi^\varepsilon(t) \leq \phi^\varepsilon(0) + \frac{K}{2}\varepsilon^4, \quad t \in [0,T). \tag{19}$$

Moreover, if $\phi^\varepsilon(0) \to 0$ as $\varepsilon \to 0$, then

$$\sup_{t\in[0,T)} \phi^\varepsilon(t) \to 0 \text{ as } \varepsilon \to 0. \tag{20}$$

Proof. Using the limit assumptions (17), we first remark that $F^\varepsilon(w|\bar{w}) \to 0$ in the limit $x \to \pm\infty$. Then, integrating (15) on $[0,t) \times \mathbb{R}$, $t < T$, yields

$$\begin{aligned}\phi^\varepsilon(t) - \phi^\varepsilon(0) \leq &- \int_0^t \int_{\mathbb{R}} \left| a(u-\bar{u}) - (v-\bar{v}) \right|^2 (s,x) \,\mathrm{d}x\,\mathrm{d}s \\ &+ \varepsilon^2 \int_0^t \int_{\mathbb{R}} \left| (v-\bar{v}) - a(u-\bar{u}) \right| (s,x) \left| \partial_t \bar{v} \right| (s,x) \,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{21}$$

Concerning the last integral in this estimate, applying Cauchy-Schwarz and Young inequalities together with the assumption $\|\partial_t \bar{v}\|_{L^2(Q_T)} \leq K$, we obtain

$$\begin{aligned}\varepsilon^2 \int_0^t \int_{\mathbb{R}} \left| (v-\bar{v}) - a(u-\bar{u}) \right| \left| \partial_t \bar{v} \right| \,\mathrm{d}x\,\mathrm{d}s &\leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} \left| a(u-\bar{u}) - (v-\bar{v}) \right|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \frac{\varepsilon^4}{2} \int_0^t \int_{\mathbb{R}} \left| \partial_t \bar{v} \right|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} \left| a(u-\bar{u}) - (v-\bar{v}) \right|^2 \,\mathrm{d}x\,\mathrm{d}s + \frac{K}{2}\varepsilon^4.\end{aligned}$$

Then, inequality (21) becomes

$$\phi^\varepsilon(t) - \phi^\varepsilon(0) \leq -\frac{1}{2} \int_0^t \int_{\mathbb{R}} \left| a(u-\bar{u}) - (v-\bar{v}) \right|^2 (s,x) \,\mathrm{d}x\,\mathrm{d}s + \frac{K}{2}\varepsilon^4,$$

which concludes the proof. □

3. Semi-discrete finite volume scheme and numerical convergence rate

From a numerical point of view, the key ingredient is to consider a numerical scheme for (4)-(5) which provides the required discretization of (11)-(12) in the limit of ε to zero.

Such schemes refer to *Asymptotic Preserving Schemes*, notion introduced by Jin in [Jin99]. Such schemes 1) have to provide a consistent discretization of the hyperbolic solutions of (4)-(5) and of the parabolic solutions of (11)-(12) at the limit $\varepsilon \to 0$, 2) admit a CFL condition which does not degenerate as $\varepsilon \to 0$.

In the following, we focus on a scheme which is continuous in time and discrete in space. Therefore the point 2) is not restrictive here. However the numerical results given in the Introduction, see Figure 1, have been obtained by an asymptotic preserving scheme, satisfying both 1) and 2), introduced in [JPT98] and defined in Section 3.3. Note that the relative entropy method has been applied to a full discrete scheme by Bulteau *et al* [BBBC19], for the p-system asymptotic towards the porous media equation.

3.1. Definition of the semi-discrete scheme

Let us consider a uniform mesh made of cells $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})_{i\in\mathbb{Z}}$ with uniform size step Δx. Here, the centers of cells are denoted $x_i = i\Delta x$ for all $i \in \mathbb{Z}$. On each cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, the solutions of (4)-(5) are approximated by time dependent piecewise constant functions $w_i(t) = {}^t(u_i(t), v_i(t))$. The space discretization scheme is based on the standard HLL numerical fluxes (see [HLvL83]). Hence the continuous in time and discrete in space numerical scheme reads

$$\begin{cases} \dfrac{d}{dt}u_i = -\dfrac{1}{2\Delta x}\left(v_{i+1} - v_{i-1}\right) + \dfrac{\lambda}{2\Delta x}(u_{i+1} - 2u_i + u_{i-1}), \\ \dfrac{d}{dt}v_i = -\dfrac{\lambda^2}{2\varepsilon^2\Delta x}(u_{i+1} - u_{i-1}) + \dfrac{\lambda}{2\Delta x}(v_{i+1} - 2v_i + v_{i+1}) + \dfrac{1}{\varepsilon^2}(au_i - v_i). \end{cases} \tag{22}$$

As soon as ε goes to zero, this finite volume scheme provides a consistent approximation of the parabolic limit (11)-(12): the pair $\bar{w}_i(t) = {}^t(\bar{u}_i(t), \bar{v}_i(t))$ evolves in time as follows

$$\begin{cases} \dfrac{d}{dt}\bar{u}_i = -\dfrac{1}{2\Delta x}(\bar{v}_{i+1} - \bar{v}_{i-1}) + \dfrac{\lambda}{2\,\Delta x}(\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}), \\ \dfrac{\lambda^2}{2\Delta x}(\bar{u}_{i+1} - \bar{u}_{i-1}) = a\bar{u}_i - \bar{v}_i. \end{cases} \tag{23}$$

The numerical scheme is endowed with convenient limit conditions, in agreement with (17) to be imposed to the approximate solution as follows:

$$\begin{aligned} \lim_{i\to\pm\infty} u_i &= \lim_{i\to\pm\infty} \bar{u}_i = u_\pm, \\ \lim_{i\to\pm\infty} v_i &= \lim_{i\to\pm\infty} \bar{v}_i = v_\pm. \end{aligned} \tag{24}$$

Finally, to simplify the forthcoming computations, we introduce some notations. Let $w(t) = (w_i(t))_{i\in\mathbb{Z}}$ be a function of time $t \in [0, T)$, piecewise constant on cells $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$. Then we define for $i \in \mathbb{Z}$

$$(D_x w)_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{\Delta x}, \quad (D_{xx} w)_i = \frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta x^2}. \tag{25}$$

We also introduce the following norm:

$$\|w\|_{L^2(Q_t)} = \left(\int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, |w_i(s)|^2\, \mathrm{d}s \right)^{1/2}.$$

3.2. Convergence rate

We adapt the discrete relative entropy method of [BBCM16], inspired of the continuous approach introduced by Lattanzio and Tzavaras [LT13], to the semi-discrete scheme (22).

First, according to the definition of the relative entropy given by (13), we define the discrete relative entropy function by

$$\begin{aligned} E_i^\varepsilon(t) &= E^\varepsilon(u_i, v_i|\bar{u}_i, \bar{v}_i)(t) \\ &= \frac{\lambda^2}{2}(u_i(t) - \bar{u}_i(t))^2 + \frac{\varepsilon^2}{2}(v_i(t) - \bar{v}_i(t))^2 - \varepsilon^2 a(u_i(t) - \bar{u}_i(t))(v_i(t) - \bar{v}_i(t)). \end{aligned} \tag{26}$$

Mimicking the continuous framework, we introduce $\phi^\varepsilon(t)$ to denote the discrete space integral of $E_i^\varepsilon(t)$ as follows:

$$\phi^\varepsilon(t) = \sum_{i\in\mathbb{Z}} \Delta x\; E_i^\varepsilon(t). \tag{27}$$

Without ambiguity and for the sake of clarity, the time dependence is omitted in the sequel.

Now, we state the discrete counterpart of Theorem 2.2.

Theorem 3.1. *Let $\bar{w}_i(t) = (\bar{u}_i(t), \bar{v}_i(t))_{i\in\mathbb{Z}}$ be a smooth solution of (11)-(12) defined on $Q_T = [0, T) \times \mathbb{R}$. We assume the existence of a positive constant $K < +\infty$ such that the following estimates are satisfied:*

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t}\bar{v} \right\|_{L^2(Q_T)} \leq K, \qquad \|D_{xx}\bar{v}\|_{L^2(Q_T)} \leq K, \tag{28}$$

where the discrete operator D_{xx} is defined in (25). Let $w_i(t) = (u_i(t), v_i(t))_{i\in\mathbb{Z}}$ be a solution of (22) such that $\phi^\varepsilon(0) < +\infty$. We assume that the subcharacteristic condition (9) is fulfilled, as well as the assumptions on the limit conditions (24). Then we have

$$\phi^\varepsilon(t) \leq \phi^\varepsilon(0) + B\,\varepsilon^4, \quad t \in [0, T), \tag{29}$$

where B is a positive constant which depends only on λ and K. Moreover if $\phi^\varepsilon(0) \to 0$ as $\varepsilon \to 0$ then $\sup_{t\in[0,T)} \phi^\varepsilon(t) \to 0$ when $\varepsilon \to 0$.

Because the relative entropy behaves like a squared L^2 norm of $w - \bar{w}$, see (14), it follows from (29) that the convergence rate of $\|w - \bar{w}\|^2_{L^2(Q_T)}$ behaves like $O(\varepsilon^4)$.

To establish this convergence result, the first step is to exhibit the evolution equation satisfied by the relative entropy E_i^ε, namely the discrete counterpart of Lemma 2.1.

Lemma 3.2. *Let $(\bar{u}_i, \bar{v}_i)_{i\in\mathbb{Z}}$ be a smooth solution of (23) and let $(\tau_i, u_i)_{i\in\mathbb{Z}}$ be a solution of (22). The relative entropy E_i^ε, defined by (26), verifies the following evolution law:*

$$\begin{aligned} \frac{\mathrm{d}E_i^\varepsilon}{\mathrm{d}t} + \frac{1}{\Delta x}(F^\varepsilon_{i+1/2} - F^\varepsilon_{i-1/2}) = &- \left[a(u_i - \bar{u}_i) - (v_i - \bar{v}_i)\right]^2 + \varepsilon^2 \left[a(u_i - \bar{u}_i) - (v_i - \bar{v}_i)\right] \frac{\mathrm{d}}{\mathrm{d}t}\bar{v}_i \\ &+ R_i^1 + R_i^2 + R_i^3 + R_i^4, \end{aligned} \tag{30}$$

where $F^\varepsilon_{i+1/2}$ corresponds to an approximation of the relative entropy flux $F^\varepsilon(w|\bar{w})$ at the interface $x_{i+1/2}$ given by

$$\begin{aligned} F^\varepsilon_{i+1/2} = &-\frac{\varepsilon^2}{2}a(v_i - \bar{v}_i)(v_{i+1} - \bar{v}_{i+1}) - \frac{\lambda^2}{2}a(u_i - \bar{u}_i)(u_{i+1} - \bar{u}_{i+1}) \\ &+ \frac{\lambda^2}{2}\left[(u_i - \bar{u}_i)(v_{i+1} - \bar{v}_{i+1}) + (u_{i+1} - \bar{u}_{i+1})(v_i - \bar{v}_i)\right], \end{aligned} \tag{31}$$

and the quantities R_i^j, $j = 1, \dots, 4$ denote numerical residuals given by

$$\begin{aligned} R_i^1 &= \frac{\lambda^3}{2}\Delta x(u_i - \bar{u}_i)\left(D_{xx}(u - \bar{u})\right)_i, \\ R_i^2 &= \frac{\varepsilon^2\lambda}{2}\,\Delta x\,(v_i - \bar{v}_i)\left(D_{xx}(v - \bar{v})\right)_i, \\ R_i^3 &= \frac{\varepsilon^2\lambda}{2}\,\Delta x\,\left[(v_i - \bar{v}_i) - a(u_i - \bar{u}_i)\right](D_{xx}\bar{v})_i, \\ R_i^4 &= -\varepsilon^2\,a\frac{\lambda}{2}\,\Delta x\,\left[(v_i - \bar{v}_i)\left(D_{xx}(u - \bar{u})\right)_i + (u_i - \bar{u}_i)\left(D_{xx}(v - \bar{v})\right)_i\right]. \end{aligned} \tag{32}$$

Observe that this evolution equation turns out to be a discrete form of (15) supplemented by numerical viscosity terms.

Proof. According to the definition (26), the time derivative of the semi-discrete relative entropy E_i reads

$$\begin{aligned}\frac{\mathrm{d}E_i^\varepsilon}{\mathrm{d}t} =&\lambda^2(u_i-\bar{u}_i)\frac{\mathrm{d}}{\mathrm{d}t}(u_i-\bar{u}_i)+\varepsilon^2(v_i-\bar{v}_i)\frac{\mathrm{d}}{\mathrm{d}t}(v_i-\bar{v}_i)\\ &-\varepsilon^2 a\frac{\mathrm{d}}{\mathrm{d}t}(u_i-\bar{u}_i)(v_i-\bar{v}_i)-\varepsilon^2 a(u_i-\bar{u}_i)\frac{\mathrm{d}}{\mathrm{d}t}(v_i-\bar{v}_i).\end{aligned} \tag{33}$$

Now, we rewrite the second equation of the scheme (23) as follows:

$$\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}_i=-\frac{\lambda^2}{2\Delta x}(\bar{u}_{i+1}-\bar{u}_{i-1})+(a\bar{u}_i-\bar{v}_i)+\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}_i.$$

Using (22), we obtain

$$\begin{aligned}\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}(v_i-\bar{v}_i)=&-\frac{\lambda^2}{2\Delta x}\left[(u_{i+1}-\bar{u}_{i+1})-(u_{i-1}-\bar{u}_{i-1})\right]+\varepsilon^2\frac{\lambda}{2}\Delta x\,(D_{xx}v)_i\\ &+\left[a(u_i-\bar{u}_i)-(v_i-\bar{v}_i)\right]-\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}_i.\end{aligned}$$

Plugging this equality in (33) and using the first equations of (22) and (23) lead to

$$\begin{aligned}\frac{\mathrm{d}E_i^\varepsilon}{\mathrm{d}t}=&-\lambda^2(u_i-\bar{u}_i)\frac{1}{2\Delta x}\left[(v_{i+1}-\bar{v}_{i+1})-(v_{i-1}-\bar{v}_{i-1})\right]+\lambda^2(u_i-\bar{u}_i)\frac{\lambda}{2}\Delta x\left(D_{xx}(u-\bar{u})\right)_i\\ &-(v_i-\bar{v}_i)\frac{\lambda^2}{2\Delta x}\left[(u_{i+1}-\bar{u}_{i+1})-(u_{i-1}-\bar{u}_{i-1})\right]+(v_i-\bar{v}_i)\frac{\varepsilon^2\lambda}{2}\Delta x\left(D_{xx}v\right)_i\\ &+(v_i-\bar{v}_i)\left[a(u_i-\bar{u}_i)-(v_i-\bar{v}_i)\right]-(v_i-\bar{v}_i)\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}_i\\ &+\varepsilon^2 a(v_i-\bar{v}_i)\frac{1}{2\Delta x}\left[(v_{i+1}-\bar{v}_{i+1})-(v_{i-1}-\bar{v}_{i-1})\right]-\varepsilon^2 a(v_i-\bar{v}_i)\frac{\lambda}{2}\Delta x\left(D_{xx}(u-\bar{u})\right)_i\\ &+a(u_i-\bar{u}_i)\frac{\lambda^2}{2\Delta x}\left[(u_{i+1}-\bar{u}_{i+1})-(u_{i-1}-\bar{u}_{i-1})\right]-\varepsilon^2 a(u_i-\bar{u}_i)\left(D_{xx}v\right)_i\\ &-a(u_i-\bar{u}_i)\left[a(u_i-\bar{u}_i)-(v_i-\bar{v}_i)\right]+a(u_i-\bar{u}_i)\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}_i.\end{aligned}$$

By rearranging the terms and using the definition (31) of the relative entropy flux, it yields

$$\begin{aligned}\frac{\mathrm{d}E_i^\varepsilon}{\mathrm{d}t}+\frac{1}{\Delta x}(F^\varepsilon_{i+\frac{1}{2}}-F^\varepsilon_{i-\frac{1}{2}})=&-\left[a(u_i-\bar{u}_i)-(v_i-\bar{v}_i)\right]^2+\left[a(u_i-\bar{u}_i)-(v_i-\bar{v}_i)\right]\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}_i\\ &+\lambda^2(u_i-\bar{u}_i)\frac{\lambda}{2}\Delta x\left(D_{xx}(u-\bar{u})\right)_i+(v_i-\bar{v}_i)\frac{\varepsilon^2\lambda}{2}\Delta x\left(D_{xx}v\right)_i\\ &-\varepsilon^2 a(v_i-\bar{v}_i)\frac{\lambda}{2}\Delta x\left(D_{xx}(u-\bar{u})\right)_i-\varepsilon^2 a(u_i-\bar{u}_i)\frac{\lambda}{2}\Delta x\left(D_{xx}v\right)_i.\end{aligned}$$

Writing $D_{xx}v$ as $D_{xx}(v-\bar{v})+D_{xx}\bar{v}$, we are now able to identify the remainder terms R_i^j, $j=1,\dots,4$, which concludes the proof. □

From now on, we state estimates satisfied by residuals R_i^j, $j=1,\dots,4$.

Lemma 3.3. *Under the subcharacteristic condition (9), for all $\theta > 0$, we have the following estimates*

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x R_i^1 \,\mathrm{d}s = -\frac{\lambda^3}{2}\Delta x \,\|D_x(u-\bar{u})\|^2_{L^2(Q_t)}, \tag{34}$$

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x R_i^2 \,\mathrm{d}s = -\frac{\varepsilon^2\lambda}{2}\Delta x \,\|D_x(v-\bar{v})\|^2_{L^2(Q_t)}, \tag{35}$$

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x R_i^3 \,\mathrm{d}s \le \varepsilon^4\frac{\lambda^2}{8\theta}\Delta x^2 \|D_{xx}\bar{v}\|^2_{L^2(Q_t)} + \frac{\theta}{2}\int_0^t \sum_{i\in\mathbb{Z}} \Delta x \left[(v_i-\bar{v}_i) - a(u_i-\bar{u}_i)\right]^2 \,\mathrm{d}s, \tag{36}$$

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x R_i^4 \,\mathrm{d}s \le \frac{\lambda^3}{2}\Delta x \,\|D_x(u-\bar{u})\|^2_{L^2(Q_t)} + \frac{\lambda}{2}\varepsilon^2\,\Delta x\,\|D_x(v-\bar{v})\|^2_{L^2(Q_t)}, \tag{37}$$

where the discrete operators D_x and D_{xx} are defined in (25).

Proof. Equality (34) (resp. (35)) is directly obtained by summing R_i^1 (resp. R_i^2) over $i \in \mathbb{Z}$, applying a discrete integration by parts, and integrating with respect to time.

Then, by definition of R_i^3 (32), summing over $i \in \mathbb{Z}$, integrating with respect to t and using the Cauchy-Schwarz inequality, we obtain

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x R_i^3 \,\mathrm{d}s \le \frac{\varepsilon^2\lambda}{2}\Delta x \left(\int_0^t \sum_{i\in\mathbb{Z}} \Delta x \left[(v_i-\bar{v}_i) - a(u_i-\bar{u}_i)\right]^2 \,\mathrm{d}s\right)^{1/2} \|D_{xx}\bar{v}\|_{L^2(Q_t)}.$$

Finally, we apply the Young inequality with $\theta > 0$ to get (36).

At last, we prove estimate (37). To do this, we first perform a discrete integration by parts and apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}\int_0^t \sum_{i\in\mathbb{Z}} \Delta x R_i^4 \,\mathrm{d}s &= \varepsilon^2 a\,\lambda\,\Delta x \int_0^t \sum_{i\in\mathbb{Z}} \Delta x \,(D_x(u-\bar{u}))_{i+\frac{1}{2}}\,(D_x(v-\bar{v}))_{i+\frac{1}{2}} \,\mathrm{d}s\\ &\le \varepsilon^2 |a|\lambda\,\Delta x\,\|D_x(u-\bar{u})\|_{L^2(Q_t)}\,\|D_x(v-\bar{v})\|_{L^2(Q_t)}.\end{aligned}$$

But using the subcharacteristic condition, we have $\varepsilon|a| < \lambda$, which yields

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x R_i^4 \,\mathrm{d}s \le \lambda^{3/2}\sqrt{\Delta x}\,\|D_x(u-\bar{u})\|_{L^2(Q_t)}\lambda^{1/2}\varepsilon\sqrt{\Delta x}\|D_x(v-\bar{v})\|_{L^2(Q_t)},$$

and we finally get (37) thanks to Young inequality. □

With these results, we can now establish the proof of Theorem 3.1.

Proof of Theorem 3.1. First of all, let us remark that thanks to Lemma 3.3, we have

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x (R_i^1 + R_i^2 + R_i^4) \,\mathrm{d}s \le 0.$$

Then, integrating with respect to t and summing for $i \in \mathbb{Z}$ the semi-discrete evolution law derived in Lemma 3.2, and using the estimate (36), we obtain

$$\phi^\varepsilon(t) - \phi^\varepsilon(0) \leq -\int_0^t \sum_{i\in\mathbb{Z}} \Delta x \left[a(u_i - \bar{u}_i) - (v_i - \bar{v}_i)\right]^2 \, \mathrm{d}s + \varepsilon^2 \int_0^t \sum_{i\in\mathbb{Z}} \Delta x \left[a(u_i - \bar{u}_i) - (v_i - \bar{v}_i)\right] \frac{\mathrm{d}\bar{v}_i}{\mathrm{d}t} \, \mathrm{d}s$$
$$+ \varepsilon^4 \frac{\lambda^2}{8\theta} \Delta x^2 \|D_{xx}\bar{v}\|^2_{L^2(Q_t)} + \frac{\theta}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x \left[(v_i - \bar{v}_i) - a(u_i - \bar{u}_i)\right]^2 \, \mathrm{d}s.$$

Applying Cauchy-Schwarz et Young inequalities with $\alpha > 0$ on the second term of the right hand side, we get

$$\phi^\varepsilon(t) - \phi^\varepsilon(0) \leq \left(-1 + \frac{\alpha}{2} + \frac{\theta}{2}\right) \int_0^t \sum_{i\in\mathbb{Z}} \Delta x \left[a(u_i - \bar{u}_i) - (v_i - \bar{v}_i)\right]^2 \, \mathrm{d}s$$
$$+ \varepsilon^4 \left(\left\| \frac{\mathrm{d}}{\mathrm{d}t} \bar{v} \right\|^2_{L^2(Q_t)} + \frac{\lambda^2}{8\theta} \Delta x^2 \|D_{xx}\bar{v}\|^2_{L^2(Q_t)} \right).$$

Choosing, for example, $\alpha = \frac{1}{2} = \theta$ concludes the proof thanks to assumptions (28). □

3.3. Numerical experiments

The numerical results presented in Figure 1 have been obtained by a fully discrete scheme as proposed by Jin, Pareschi and Toscani [JPT98]. This scheme is based on a reformulation of system (4)-(5)

$$\begin{cases} \partial_t u - \partial_x v = 0, \\ \partial_t v + \lambda^2 \partial_x u = -\dfrac{1}{\varepsilon^2} \left(au - v - (1-\varepsilon^2)\lambda^2 \partial_x u\right). \end{cases}$$

According to [JPT98], it is convenient to is to use an explicit scheme for to approximate the convection step and to treat the stiff source term implicitly. In the first step, the convective and non-stiff system is approximated thanks to a classical HLL scheme [HLvL83]:

$$u_i^{n+\frac{1}{2}} = u_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}^u_{i+\frac{1}{2}} - \mathcal{F}^u_{i-\frac{1}{2}}\right), \tag{38a}$$
$$v_i^{n+\frac{1}{2}} = v_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}^v_{i+\frac{1}{2}} - \mathcal{F}^v_{i-\frac{1}{2}}\right), \tag{38b}$$

where the numerical fluxes are defined by

$$\mathcal{F}^u_{i+\frac{1}{2}} = \frac{1}{2}(v_i^n + v_{i+1}^n) - \frac{\lambda}{2}(u_{i+1}^n - u_i^n),$$
$$\mathcal{F}^v_{i+\frac{1}{2}} = \frac{\lambda^2}{2}(u_i^n + u_{i+1}^n) - \frac{\lambda}{2}(v_{i+1}^n - v_i^n).$$

This scheme is stable under the CFL condition $\dfrac{\Delta t}{\Delta x}\lambda \leq \dfrac{1}{2}$ which does not depend on ε (see for instance [Bou04, section 2.3]). Next, the stiff source term is treated in an implicit way to obtain unconditional stability:

$$u_i^{n+1} = u_i^{n+\frac{1}{2}},$$
$$\frac{v_i^{n+1} - v_i^{n+\frac{1}{2}}}{\Delta t} = \frac{1}{\varepsilon^2} \left(a u_i^{n+1} - v_i^{n+1} - (1-\varepsilon^2)\lambda^2 \frac{u_{i+\frac{1}{2}}^{n+1} - u_{i-\frac{1}{2}}^{n+1}}{\Delta x} \right),$$

where we take $u_{i+\frac{1}{2}}^{n+1} = \dfrac{u_{i+1}^{n+1} + u_i^{n+1}}{2}$.

Since $u_i^{n+1} = u_i^{n+\frac{1}{2}}$, let us emphasize that v_i^{n+1} can be computed explicitly from $(u_i^n, v_i^n)_{i\in\mathbb{Z}}$. Finally, the relaxation step can be written as

$$u_i^{n+1} = u_i^{n+\frac{1}{2}}, \tag{39a}$$

$$v_i^{n+1} = \left(\frac{\varepsilon^2}{\varepsilon^2 + \Delta t}\right) v_i^{n+\frac{1}{2}} + \frac{\Delta t}{\varepsilon^2 + \Delta t}\left(a u_i^{n+\frac{1}{2}} - (1-\varepsilon^2)\frac{\lambda^2}{2\Delta x}(u_{i+1}^{n+\frac{1}{2}} - u_{i-1}^{n+\frac{1}{2}})\right). \tag{39b}$$

Considering the scheme (38)-(39) in the limit of ε to zero provides a numerical scheme approximating the solutions of the convection-diffusion problem (11)-(12). It reads:

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{2\Delta x}(\bar{v}_{i+1}^n - \bar{v}_{i-1}^n) + \frac{\lambda\Delta t}{2\Delta x}(\bar{u}_{i+1}^n - 2\bar{u}_i^n + \bar{u}_{i-1}^n), \tag{40}$$

$$\bar{v}_i^{n+1} = a\bar{u}_i^{n+1} - \frac{\lambda^2}{2\Delta x}(\bar{u}_{i+1}^{n+1} - \bar{u}_{i-1}^{n+1}). \tag{41}$$

The numerical results presented in Figure 1-top correspond to a Riemann initial condition $u(0,x) = 2\cdot\mathbf{1}_{x<0} + \mathbf{1}_{x\geq 0}$, the initial condition $v(0,x)$ is set to local equilibrium, namely $v(0,x) = au(0,x)$. The final time of computation is $T = 0.1$, the domain $[0,1]$ is discretized with $N_x = 200$ cells and the CFL parameter is set to 0.95. The wave speed is $\lambda = 0.72$, $a = 0.5$ and the relaxation parameter ε is set to 1, according to the subcharacteristic condition (9). Hence the profiles of (u,v) present the two waves of speed $\pm\lambda$ and are strongly mollified by the relaxation term. On the other hand the asymptotic limit $\bar{u}$ corresponds to the solution of the parabolic equation (11) and $\bar{v}$ to (12).

The convergence rate presented in Figure 1-bottom has been computed with different values of ε from 10^{-1} to 0.5×3.10^{-3} with the same initial condition and various space discretizations. The log-log scale figure shows the convergence rate in $O(\varepsilon^4)$, which is in agreement with theorems 2.2 and 3.1.

4. Conclusion and prospects

In this work, we establish the convergence of the solution of the linear Jin and Xin model to a solution of the convection-diffusion equation obtained in the limit $\varepsilon \to 0$ by a relative entropy method. The estimates give a rate of convergence in $O(\varepsilon^4)$, which is found numerically. Moreover, the technique adapted to the discrete setting again gives the same rate of convergence for a class of finite volume schemes that are discrete in space and continuous in time.

Actually the estimate holds for the nonlinear case, considering a relaxation term of type $f(u) - v$, with nonlinear function f. In this case, relative entropy computations are more technical than in the linear case. This is because the entropy of the Jin and Xin system with nonlinear f is not explicit, see [Ser00]. In fact, the relative entropy identity contains terms that are more difficult to handle at both continuous and discrete levels.

However the same rate of convergence in ε^4 can be observed. As a numerical evidence, we present in Figure 2 the numerical results for the Jin and Xin model for $f(u) = u^2/2$. Figure 2-top presents the profile of (u,v) for the system (4)-(5). The initial data is at local equilibrium namely $u(0,x) = 2\cdot\mathbf{1}_{x<0} + \mathbf{1}_{x\geq 0}$ and $v(0,x) = f(u(0,x))$. One sets $\varepsilon = 1$ and $\lambda = 3$ in order to satisfy the subcharacteristic condition. The others parameters are identical to the linear test case. Figure 2-bottom shows the behaviour of ϕ with respect to ε in log-log scale.

A natural perspective is to adapt the relative entropy method to exhibit the convergence rate in the nonlinear case.

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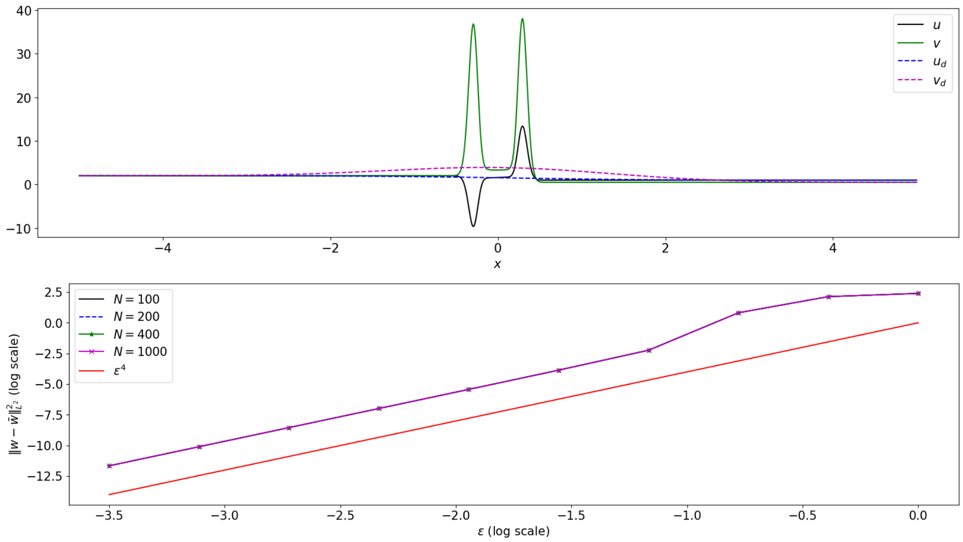

FIGURE 2. Nonlinear test case. Top: profile of (u, v) in space, compared to $(\bar{u}, \bar{v})$. Bottom: L^2 norm of the error $\|(u,v) - (\bar{u}, \bar{v})\|^2_{L^2(Q_T)}$ with respect to ε in log scale.

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