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LIMIT THEOREMS: SOME RECENT RESULTS

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Abstract. The article presents a series of recent results in the field of limit theorems which allow to appreciate the richness of this theme. These results include: the study of the asymptotic behaviour of cumulative processes with applications to Hawkes processes; maximal inequalities and functional central limit theorems for U-statistics; scaling limits for network traffic models that reveal a scaling transition phenomenon; central limit theorems for weakly dependent stationary random fields.

Résumé. Cet article présente une série de résultats récents sur le thème des théorèmes limites qui permettent d'apprécier la richesse de la thématique. Les résultats présentés regroupent : l'étude du comportement asymptotique des processus cumulatifs avec des applications aux processus de Hawkes; des inégalités maximales et des théorèmes limites centraux fonctionnels pour des U-statistiques; des limites d'échelles pour des modèles de trafic de réseau révélant un phénomène de transition d'échelle; des théorèmes limites centraux pour des champs aléatoires stationnaires faiblement dépendants.

Introduction

In probability, the term *limit theorem* refers to any convergence result for sequences of random variables or for more involved stochastic processes. The study of limit theorems for theoretical aspects, for statistical applications, or the study of limit theorems for specific probabilistic models (e.g. obtaining of scaling limits for discrete models) encompass a large part of the research activity in probability. The classical law of large number and the central limit theorem for sums of i.i.d. random variables were the first to be obtained and remain the most important to this day. They have been extended to continuous time processes (e.g. Donsker invariance principle) or to more complicated objects in various directions (e.g. heavy-tailed random variables, extreme values theory, ...). Limit theorems have also been extended from the classical independent setting to some weak dependence settings, and then also to long range dependence settings. Among many others, the monographes [14, 33, 43, 49, 58, 71] give an overview on limit theorems theory in these directions.

The aim of this article is to illustrate the variety of the current research activity in this field by showing some recent limit theorems obtained in several directions. It is the synthesis of the session of the *Journées MAS 2022*

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entitled "Limit Theorems" and organized by O. Durieu. The four sections of the article are based on or related to each of the talks of the session.

In Section 1, after introducing the notion of *cumulative process* that generalizes the one of renewal process, L. Colombani presents several results concerning their asymptotic properties. In particular, large deviation principles have been recently obtained by the author in [11] with P. Cattiaux and M. Costa. Applications of these general results to Hawkes processes [12] are also presented.

In Section 2, D. Giraudo gives a general introduction to the study of limit theorems for U-*statistics*: a general class of estimators highlighted by Hoeffding [37] in 1948. Recently, thanks to new maximal inequalities, D. Giraudo [29] has obtained functional central limit theorems in Hölder spaces.

In Section 3, V. Pilipauskaitė exposes recent advances for some *network traffic models*. After a general introduction on these models, the results of [45] obtained by the author in collaboration with R. Leipus and D. Surgailis are presented, focusing on the phenomenon of *scaling transition*.

Section 4 concerns functional central limit theorem for random fields. The notion of *quenched CLT*, known for Markov chains, is extended to random fields. L. Reding, in collaboration with N. Zhang and M. Peligrad, has obtained results [66, 79] for some weakly dependent random fields. The dependence is determined trough a projective criterion that generalizes *Hannan's criterion* for sequences.

In the following, $\mathcal{N}(m, \sigma^2)$ denotes the normal distribution with mean m and variance σ^2 and $\xrightarrow{a.s.}$, $\xrightarrow{\text{law}}$, and $\xrightarrow{\text{fdd}}$ denote respectively the almost sure convergence, the convergence in distribution (or weak convergence), and the weak convergence of the finite dimensional distributions.

1. Cumulative processes (L. Colombani)

In this section, we are interested in asymptotic properties for cumulative processes. This process, introduced by Smith [72], is a generalization of the renewal process. It can also be called *compound-renewal* process or *renewal-reward* process. A cumulative process is a continuous time process cumulating independent random variables occurring in time intervals given by a renewal process.
This process can be applied in various fields, such as finance [67], insurance [22], sport [41], queues [62], etc.

Definition 1.1. A real valued process $(Z_t)_{t\geq 0}$ is a cumulative process if

- $Z_0 = 0$,
- there exists a renewal process $(S_n)_n$ such that for any n, $(Z_{S_n+t} - Z_{S_n})_{t\geq 0}$ is independent of $S_0, \dots, S_n$ and $(Z_s)_{t\leq S_n}$,
- the distribution of $(Z_{S_n+t} - Z_{S_n})_{t\geq 0}$ is independent of n.

Such a process can be written, for all $t \geq 0$

$$Z_t = W_0(t) + W_1 + \cdots + W_{M_t} + r_t$$

where $W_0(t) = Z_{t\wedge S_0}$, $W_n = Z_{S_n} - Z_{S_{n-1}}$, $r_t = Z_t - Z_{M_t}$ and $M_t = \sup\{n \geq 0, S_n \leq t\}$. Then the $(W_k)_k$ are i.i.d.
By denoting by $(\tau_i)_i$ the times associated with the renewal process $\tau_n = S_n - S_{n-1}$ and $\tau_0 = 0$, we have $(\tau_k, W_k)_k$ i.i.d.

Remark 1.2. In practice, we can define a cumulative process by considering a sequence of couples $(\tau_k, W_k)_k$ i.i.d., valued in $(0, \infty) \times \mathbb{R}$, directly as

$$Z_t = \sum_{i=1}^{M_t} W_i,$$

where $M_t = \sup\{n \geq 0, \sum_{i=1}^{n} \tau_i \leq t\}$. Then, the leftover $r_t = 0$ a.s. and the first component $W_0(t) = 0$ a.s.

Example 1.3.

- A renewal process is a cumulative process, with $W_i = 1$ a.s.
- A compound Poisson process, $Z_t = \sum_{i=1}^{M_t} W_i$, with M_t a Poisson process and $(W_i)_i$ an i.i.d. sequence, independent of M_t, is a cumulative process.
- For $(\tau_i)_i$ an i.i.d sequence of random non-negative variables and M_t the renewal counting associated to it, for F a deterministic function, $\sum_{i=1}^{M_t} F(\tau_i)$ is a cumulative process.
- $Z_t = \int_0^t X_s ds$, where $(X_s)_s$ is a regenerative process with i.i.d. cycles, is a cumulative process.

1.1. Asymptotic properties

As for the classical renewal process, we know some properties of the asymptotic behaviour of Z_t. Firstly, we can enunciate a law of large numbers:

Proposition 1.4. *Suppose* $\mathbb{E}[\tau_1] < \infty$ *and* $\mathbb{E}[|W_1|] < \infty$. *Then*

$$\frac{1}{t} Z_t \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$$

if and only if $\mathbb{E}\left[\max_{S_0 \le t < S_1} |r_t|\right] < \infty$.

This property, which can be found in Asmussen's book [3], can be proved quite directly by decomposing the process as

$$\frac{1}{t}\sum_{i=1}^{M_t} W_i = \frac{M_t}{t}\frac{1}{M_t}\sum_{i=1}^{M_t} W_i$$

and by using the well-known Renewal Theorem for the renewal process.

This last property is also known as Renewal Reward Theorem in financial mathematics (see Chapter 3.6 from [68]) and is frequently completed by the following:

Proposition 1.5. *Suppose* $\mathbb{E}[\tau_1] < \infty$ *and* $\mathbb{E}[|W_1|] < \infty$. *Then*

$$\frac{1}{t}\mathbb{E}[Z_t] \xrightarrow[t\to\infty]{} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$$

if and only if $\mathbb{E}\left[\max_{S_0 \le t < S_1} |r_t|\right] < \infty$.

We also have a Central Limit Theorem, which can be also found in Asmussen's book. It doesn't need any condition on the law of r_t.

Proposition 1.6. *Suppose* $\mathrm{Var}[\tau_1] < \infty$ *and* $\mathrm{Var}[W_1] < \infty$. *Then*

$$\sqrt{t}\left(\frac{1}{t}Z_t - m\right) \xrightarrow[t\to\infty]{\text{law}} \mathcal{N}\left(0, \sigma^2\right),$$

where $m = \dfrac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$, $\sigma^2 = \mathrm{Var}\left[W_1 - \dfrac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}\tau_1\right]$.

Some authors have also generalized some properties, known for the sums of random values, to the random processes. Thus, Frolov [24] has proved a Law of the Iterated Logarithm, which allows us to describe the magnitude of the fluctuations of the process. Note that this property has been proved in the simpler context of Remark 1.2, under the assumption that the $(W_i)_i$ are non-negative.

Proposition 1.7. *We assume $r_t = 0$ a.s., $W_0(t) = 0$ a.s. and $(W_i)_i$ is a non-negative sequence. If $\mathbb{E}[W_1^2] < \infty$ and $\mathbb{E}[\tau_1^2] < \infty$, then*

$$\limsup_{t\to+\infty} \frac{1}{\sqrt{2(\sigma^2/\mathbb{E}[\tau_1])t\log\log t}}\,(Z_t - mt) = 1 \text{ a.s. },$$

where $m = \dfrac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$ *and* $\sigma^2 = \mathrm{Var}\left[W_1 - \dfrac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}\tau_1\right]$.

Remark 1.8. Note that if $\dfrac{r_t}{\sqrt{t\log\log t}} \to 0$ a.s., the proposition stays true.

Thereafter, a Chung law, which completes the Law of the Iterated Logarithm, has been developed by Martikainen and Frolov [48]. It has been proved in the context of Remark 1.2, as before.

Proposition 1.9. *We assume $r_t = 0$ a.s., $W_0(t) = 0$ a.s. and $(W_i)_i$ is a non-negative sequence. If $\mathbb{E}[W_1^2] < \infty$ and $\mathbb{E}[\tau_1^2] < \infty$, then*

$$\liminf_{T\to+\infty} \sqrt{\frac{\mathbb{E}[\tau_1]}{\sigma^2}}\sqrt{\frac{\log\log T}{T}} \sup_{1\le t\le T} |Z(t) - mt| = \frac{\pi}{\sqrt{8}} \text{ a.s.},$$

where $m = \dfrac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$ *and* $\sigma^2 = \mathrm{Var}\left[W_1 - \dfrac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}\tau_1\right]$.

The properties of large deviations have also been studied, in [8], [11] and [78], in order to control the deviation of the process from the mean.

Proposition 1.10. *We assume $r_t = 0$ a.s., $W_0(t) = 0$ a.s. We also assume there exists an exponential moment for τ_1 and W_1, that is to say,*

i) $\exists\theta_0 \in (0,\infty]$ such that $\mathbb{E}[\mathrm{e}^{\theta\tau_1}] < \infty$ for $\theta < \theta_0$,

ii) $\exists\beta_0 \in (0,\infty]$ such that $\mathbb{E}[\mathrm{e}^{\beta|W_1|}] < \infty$, for $\beta < \beta_0$.

Let introduce the classical Cramer transforms, for $(a,b) \in \mathbb{R}^2$,

$$\Lambda^*(a,b) = \sup_{x,y}\left\{ax + by - \log\mathbb{E}(e^{x\tau + yW})\right\},$$

and the rate function

$$J(y) = \inf_{\beta>0}\ \beta\,\Lambda^*\left(\frac{1}{\beta}, \frac{y}{\beta}\right). \tag{1}$$

We define $\bar{J}$ as

$$\bar{J}(y) = J(y) \text{ for } m \neq 0,$$
$$\bar{J}(0) = \min(J(0), \theta_0).$$

Then, we have

- *If $\beta_0 = \infty$ (in particular if W is bounded) then Z_t/t satisfies a large deviations principle with good rate function $\bar{J}$:*
  (1) *for any closed set $\mathcal{C} \in \mathbb{R}$,*
  $$\limsup_{t\to\infty} \frac{1}{t}\ \ln\mathbb{P}\left(Z_t/t \in \mathcal{C}\right) \le -\inf_{m\in\mathcal{C}} \bar{J}(m)$$
  (2) *for any open set $\mathcal{O} \in \mathbb{R}$,*
  $$\liminf_{t\to\infty} \frac{1}{t}\ \ln\mathbb{P}\left(Z_t/t \in \mathcal{O}\right) \ge -\inf_{m\in\mathcal{O}} \bar{J}(m),$$

- *If $\beta_0 \leq \infty$, denoting $m = \mathbb{E}(W)/\mathbb{E}(\tau)$ we have for all $a > 0$ and $\kappa \in (0,1)$*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{Z_t}{t} \geq m + a\right) \leq -\min\left[\inf_{z\geq m+(a/2)} \bar{J}(z)\,,\ \frac{\beta_0 a(1-\kappa)}{4}\right],$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{Z_t}{t} \leq m - a\right) \leq -\min\left[\inf_{z\leq m-(a/2)} \bar{J}(z)\,,\ \frac{\beta_0 a(1-\kappa)}{4}\right].$$

Moreover, some properties of the trajectories of the process have been developed by Borovkov, Mogulskii and Logachov, and can be found in [10], [9], [7] and [46].

1.2. Application to the Hawkes process

Hawkes processes have been defined by Hawkes [35] and are used in many applications (seismology, finance, neurology, ...). They are self-influenced jump processes: the probability that a jump happens between t and $t + dt$ depends on the jumps before the time t.
A Hawkes process $t \mapsto N^h([0,t])$ is a point process on $\mathbb{R}$, characterized by its intensity process $t \mapsto \Lambda(t)$ through the infinitesimal relation

$$\mathbb{P}(N^h((t,t+dt)) = 1|\mathcal{F}_t) = \Lambda(t)dt$$

and we can defined

$$\Lambda(t) = \left(\lambda + \int_0^t h(t-s)N^h(ds)\right)^+,$$

where $x^+ = \max(0,x)$, $\lambda > 0$ is the initial intensity and h is the reproduction function.
Note that this definition may not be totally general, and can vary among authors. In particular, here, we suppose that the initial number of jumps is null, and we allow inhibition of the process.

When h is non-negative, the process is linear and self-excited. It can be described as Poissonnian arrivals with Galton-Watson clusters associated with each arrival. Such a structure is not known when h can be negative.

When h has compact support, it has been shown in [12] that the Hawkes process is a cumulative process. Thus, the properties above can be applied to the Hawkes process.

1.2.1. *Construction of the cumulative process*

Let's denote by $U_1, U_2, \ldots$ the successive jumps of a Hawkes process N^h. We suppose that h has compact support included in $[0, L(h)]$. We define the stopping time τ_1 which is the first time after U_1 such that there has been no jump during a time $L(h)$:

$$\tau_1 = \inf\{t > U_1, N^h((t - L(h), t]) = 0\}.$$

We set $S_1 = \tau_1$. We define $W_1 = N^h([0, S_1])$, the number of jumps of N^h on this first interval.
Then, we can define recursively (τ_n, W_n): let $n \in \mathbb{N}^*$ such that $(\tau_1, W_1), \ldots, (\tau_n, W_n)$ are defined. Let $S_n = \sum_{i=1}^n \tau_i$. We define

$$\tau_{n+1} = \inf\{t > U_{W_1+\cdots+W_n+1}, N^h((t - L(h), t]) = 0\} - S_n,$$

and

$$W_{n+1} = N^h([S_n, S_n + \tau_{n+1})).$$

Lemma 1.11. *With this construction, the $(\tau_i, W_i)_i$ are i.i.d random variables, and the $(\tau_i)_i$ are a.s. finite.*

In fact, the renewal relies on the fact that if for a $t > 0$, we have $U_i + L(h) < t$ for all $U_i < t$, then the intensity is equal to λ and the self-dependency is null until the next jump.
We define $M_t = \sup\{n \geq 0, S_n \leq t\}$.

Proposition 1.12. *$t \mapsto N^h([0,t])$ can be considered as a cumulative process with:*

$$N^h([0,t]) = \sum_{i=1}^{M_t} W_i + r_t,$$

with $r_t = N^h((S_{M_t}, t])$ such that $0 \le r_t \le W_{M_t+1}$ a.s.

1.2.2. *Applications of the properties of the cumulative process*

Even if the law of (τ_1, W_1) has to be computed according to the parameters λ and h of the process, we can show, in all generality, that τ_1 and W_1 have an exponential moment, i.e.

$$\exists \theta_0 \in (0,\infty], \forall \theta < \theta_0, \mathbb{E}[\mathrm{e}^{\theta \tau_1}] < \infty$$
$$\exists \beta_0 \in (0,\infty], \forall \beta < \beta_0, \mathbb{E}[\mathrm{e}^{\beta |W_1|}] < \infty.$$

This follows from the properties of the linear Hawkes process and of the busy time of a queue.

Thus, τ_1 and W_1 admit all the moments. Moreover, the law of r_t can be pretty well controlled (see [12]) and we have the following lemma:

Lemma 1.13. *We have*

$$\frac{W_{M_t+1}}{\sqrt{t}} \xrightarrow[t\to\infty]{a.s.} 0.$$

Thus

$$\frac{r_t}{\sqrt{t}} \xrightarrow[t\to\infty]{a.s.} 0 \quad \text{and} \quad \frac{r_t}{\sqrt{t \log\log t}} \xrightarrow[t\to\infty]{a.s.} 0.$$

We can then apply the propositions 1.4, 1.5, 1.6, 1.7 and 1.10, and we have:

Proposition 1.14. *For $\lambda > 0$, h a function with a compact support, and N^h a Hawkes process. We construct τ_1 and W_1 as in the last subsection.*
Let's denote $m = \dfrac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$ and $\sigma^2 = \mathrm{Var}\left[W_1 - \dfrac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}\tau_1\right]$.
Then we have

i) *a law of large number :* $\dfrac{1}{t} N^h([0,t]) \xrightarrow[t\to\infty]{a.s.} \dfrac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$, *and :* $\dfrac{1}{t}\mathbb{E}(N^h([0,t])) \xrightarrow[t\to\infty]{} \dfrac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$,

ii) *a central limit theorem :* $\sqrt{t}\left(\dfrac{1}{t}N^h([0,t]) - m\right) \xrightarrow[t\to\infty]{\text{law}} \mathcal{N}\left(0, \sigma^2\right)$,

iii) *a law of iterated logarithm :* $\limsup_{t\to+\infty} \dfrac{1}{\sqrt{2(\sigma^2/\mathbb{E}[\tau_1]) t \log\log t}} \left(N^h([0,t]) - mt\right) = 1$ *a.s.* .

The i) and ii) can be found in [12]. The iii) follows easily with the Remark 1.8.

We also have some large deviations inequalities and a large deviation principle.

Proposition 1.15. *We refer to the notations of the Proposition 1.10: for $(a,b) \in \mathbb{R}^2$,*

$$\Lambda^*(a,b) = \sup_{x,y} \left\{ ax + by - \ln\left(\mathbb{E}\left[e^{x\tau_1 + yW_1}\right]\right)\right\},$$

and for $z \in \mathbb{R}^+$,

$$J(z) = \inf_{\beta > 0} \left(\beta\, \Lambda^* \left(\frac{1}{\beta}, \frac{z}{\beta} \right) \right),$$

and

$$\bar{J}(z) = J(z) \text{ for } m \neq 0,$$
$$\bar{J}(0) = \min(J(0), \theta_0)\,.$$

Then, we obtain the following

(i) *If $\beta_0 = +\infty$, N_t^h/t satisfies a large deviation principle with rate function $\bar{J}$.*

(ii) *If $\beta_0 \leq +\infty$, we have for all $a > 0$ and $\kappa \in (0,1)$*

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{N_t^h}{t} \geq m + a\right) \leq -\min\left[\inf_{z-m\geq\kappa a} J(z)\,,\, \frac{1-\kappa}{4}\beta_0 a\right],$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{N_t^h}{t} \leq m - a\right) \leq -\min\left[\inf_{z-m\leq\kappa a} J(z)\,,\, \frac{(1-\kappa)}{4}\beta_0 a\right].$$

2. Limit theorems for U-statistics of independent data via an exponential inequality (D. Giraudo)

2.1. Introduction to limit theorems for U-statistics

Let $(X_i)_{i\geq 1}$ be an i.i.d. sequence of random variables taking values in a measurable space $(S, \mathcal{S})$ and let $h\colon S^2 \to \mathbb{R}$ be a measurable function, S^2 is endowed with the product σ-algebra and $\mathbb{R}$ with the Borel σ-algebra. The U-statistic of kernel h and data $(X_i)_{i\geq 1}$ is defined as

$$U_{n,h} = \frac{1}{\binom{n}{2}} \sum_{1\leq i<j\leq n} h(X_i, X_j)\,, \quad n \geq 2.$$

It is known that if $\mathbb{E}\left[|h(X_1, X_2)|\right]$ is finite, then $U_{n,h} \xrightarrow{a.s.} \mathbb{E}\left[h(X_1, X_2)\right]$. Therefore, $U_{n,h}$ is an unbiased estimator of $\mathbb{E}\left[h(X_1, X_2)\right]$, and the letter "U" in U-statistic stands for unbiased.

U-statistics of higher order can be defined by

$$U_{n,h} = \frac{1}{\binom{n}{r}} \sum_{(i_\ell)_{\ell=1}^r \in I_n^r} h(X_{i_1}, \dots, X_{i_r})\,, \quad n \geq r,$$

where I_n^r denotes the set of r-uples of integers such that $1 \leq i_1 < \dots < i_r \leq n$. However, in order to ease the exposure of the paper, we will restrict ourselves to U-statistics of order two.

A natural question is that of the asymptotic normality of $U_{n,h}$. Let us look at some particular kernels. We assume that X_1 is centered and has variance one.

- Assume that $h(x,y) = x + y$. Then an elementary computation shows that

$$U_{n,h} = \frac{2}{n}\sum_{i=1}^n X_i$$

  hence we get by the usual central limit theorem that $(\sqrt{n}U_{n,h})_{n\geq 2}$ converges in distribution to a centered normal distribution with variance 4.
- Assume that $h(x,y) = xy$. Then

$$U_{n,h} = \frac{1}{n(n-1)}\left(\left(\sum_{i=1}^n X_i\right)^2 - \sum_{i=1}^n X_i^2\right)$$

hence $(nU_{n,h})_{n\geq 2}$ converges in distribution to N^2-1, where N has a standard normal distribution.

These two examples shows that the normalization required for a central limit theorem and the limiting process crucially depends on the kernel. Degeneracy of the random variable $\mathbb{E}\left[h\left(X_1,X_2\right)\mid X_1\right]$ plays a decisive role. Let us define this concept.

Definition 2.1. We say that the kernel $h\colon S^2\to\mathbb{R}$ is degenerated with respect to $(X_i)_{i\geq 1}$ if

$$\mathbb{E}\left[h\left(X_1,X_2\right)\mid\sigma\left(X_1\right)\right]=\mathbb{E}\left[h\left(X_1,X_2\right)\mid\sigma\left(X_2\right)\right]=0. \tag{2}$$

If one of the random variables involved in (2) is not degenerated, then we say that h is not degenerated with respect to $(X_i)_{i\geq 1}$.

For example, if $S=\mathbb{R}$ and X_1 is centered, then $h(x,y)=xy$ is degenerated but $h(x,y)=x+y$ is not, unless X_1 is identically 0.

Hoeffding's decompositions shows that a U-statistic of order two can be decomposed as a sum of a so-called linear term and degenerated U-statistic. More precisely, let

$$\theta:=\mathbb{E}\left[h\left(X_1,X_2\right)\right],\quad h_1(x)=\mathbb{E}\left[h\left(x,X_1\right)\right]-\theta,\quad h_2(y)=\mathbb{E}\left[h\left(X_1,y\right)\right]-\theta$$

and

$$h_3(x,y)=h(x,y)-h_1(x)-h_2(y)-\theta.$$

Then the following equality takes place:

$$U_{n,h}=\theta+\frac{2}{n(n-1)}\sum_{i=1}^{n}(n-i)\,h_1\left(X_i\right)+\frac{2}{n(n-1)}\sum_{j=1}^{n}(j-1)\,h_2\left(X_j\right)+U_{n,h_3} \tag{3}$$

and h_3 is degenerated with respect to $(X_i)_{i\geq 1}$. When h is symmetric, that is $h(x,y)=h(y,x)$ for each $x,y\in S$, we have $h_1=h_2$ hence (3) simplifies as

$$U_{n,h}=\theta+\frac{2}{n}\sum_{i=1}^{n}h_1\left(X_i\right)+U_{n,h_3}.$$

As we have seen before, degeneracy plays an important role in the central limit theorem since it determines the normalization and the limiting process. It is also the case for other limit theorems.

(1) Law of large numbers: let $1\leq p<2$; if $\mathbb{E}\left[\left|h\left(X_1,X_2\right)\right|\right]<\infty$, then the following convergence holds (see [28])

$$n^{1-1/p}\left(U_{n,h}-\mathbb{E}\left[U_{n,h}\right]\right)\xrightarrow[n\to\infty]{a.s.}0. \tag{4}$$

Moreover, if h is degenerated with respect to $(X_i)_{i\geq 1}$, then (see [28])

$$n^{2-2/p}U_{n,h}\xrightarrow[n\to\infty]{a.s.}0. \tag{5}$$

(2) Bounded law of the iterated logarithms: suppose that $\mathbb{E}\left[h\left(X_1,X_2\right)^2\right]<\infty$. Then the random variable

$$\sup_{n\geq 1}\frac{\sqrt{n}}{\sqrt{\mathrm{LL}(n)}}\left|U_{n,h}-\mathbb{E}\left[U_{n,h}\right]\right| \tag{6}$$

is almost surely finite, where $L\colon \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $L(x) := \max\{\ln x, 1\}$ and $\mathrm{LL}(x) := L \circ L(x)$. Moreover, if h is degenerated with respect to $(X_i)_{i\geq 1}$, then

$$\sup_{n\geq 1} \frac{n}{\mathrm{LL}(n)} |U_{n,h}| \tag{7}$$

is almost surely finite (see [2]).

(3) Central limit theorem: suppose that $\mathbb{E}\left[h(X_1,X_2)^2\right] < \infty$. Then it was shown in [37] that

$$\sqrt{n}\left(U_{n,h} - \mathbb{E}[U_{n,h}]\right) \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0,\sigma^2), \tag{8}$$

where $\sigma^2 = 4\operatorname{Var}\left(\mathbb{E}[h(X_1,X_2) \mid X_1]\right)$.

If h is degenerated with respect to $(X_i)_{i\geq 1}$, σ is zero and we would get a degenerated random variable. In order to get a non-degenerated limit, we have to consider an other normalization. When h is symmetric, there exist a sequence $(a_k)_{k\geq 1}$ of real numbers such that $\sum_{k\geq 1} a_k^2 < \infty$ and an i.i.d. sequence $(N_k)_{k\geq 1}$ of standard normal random variables for which

$$nU_{n,h} \xrightarrow[n\to\infty]{\text{law}} \sum_{k\geq 1} a_k\left(N_k^2 - 1\right).$$

Such theorems have then been extended to the case of dependent data. For example, the question of the law of large numbers was treated in [1,18] for absolutely regular processes, [32] for Bernoulli shifts, that of the central limit theorem in [17] for α and β-mixing sequences.

2.2. Exponential inequalities for U-statistics

Let us state the exponential inequality obtained in [29].

Theorem 2.2. *Let $(S,\mathcal{S})$ be a measurable space, $h\colon S^2 \to \mathbb{R}$ be a measurable function (with S^2 induced with the product σ-algebra) and let $(X_i)_{i\geq 1}$ be an i.i.d. sequence of S-valued random variables. Suppose that $\mathbb{E}[h(X_1,X_2)] = 0$. Then for all positive x and y,*

$$\begin{aligned}\mathbb{P}\left\{\max_{2\leq n\leq N}\binom{n}{2}|U_{n,h}| > Nx\right\} &\leq A\exp\left(-\frac{x}{y}\right) + B\int_1^\infty \mathbb{P}\left\{\left|\mathbb{E}[h(X_1,X_2)\mid X_1]\right| > yN^{-1/2}\sqrt{u}C\right\}\mathrm{d}u \\ &+ B\int_1^\infty \mathbb{P}\left\{\left|\mathbb{E}[h(X_1,X_2)\mid X_2]\right| > yN^{-1/2}\sqrt{u}C\right\}\mathrm{d}u + B\int_1^\infty \mathbb{P}\left\{|h(X_1,X_2)| > y\sqrt{u}C\right\}(1+\ln(u))^2\,\mathrm{d}u, \end{aligned} \tag{9}$$

where the constants A, B and C are numerical constants.

Let us make some comments on Theorem 2.2. When h is degenerated with respect to $(X_i)_{i\geq 1}$, the second and third terms of the right hand side of (9) vanish hence in this case, the right hand side of (9) does not depend on N. When h is not degenerated, we can replace x by $x\sqrt{N}$ and y by $y\sqrt{N}$. In this way, the first three terms of (9) are independent of N and the fourth has a contribution that becomes negligible as N goes to infinity.

The proof of Theorem 2.2 relies on the following martingale inequality, which is a combination of a general deviation inequality given in Theorem 2.1 in [23] with a result on convex ordering given in Theorem 6 of [69].

Proposition 2.3. *Let $(D_i)_{i\geq 1}$ be a martingale differences sequence with respect to the filtration $(\mathcal{F}_i)_{i\geq 0}$. Suppose that $\mathbb{E}\left[D_i^2\right]$ is finite for all $i \geq 1$. Suppose that there exists a random variable Y such that for all $1 \leq i \leq n$, and each convex increasing function $\varphi\colon \mathbb{R} \to \mathbb{R}$, we have $\mathbb{E}\left[\varphi\left(D_i^2\right)\right] \leq \mathbb{E}\left[\varphi\left(Y^2\right)\right]$. Then for all $x, y > 0$ and* *each $n \geq 1$, the following inequality holds:*

$$\mathbb{P}\left\{\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} D_i\right| > xn^{1/2}\right\} \leq 2\exp\left(-\frac{1}{2}\left(\frac{x}{y}\right)^2\right) + 2\int_1^{\infty}\mathbb{P}\left\{Y^2 > y^2u/4\right\}\mathrm{d}u. \tag{10}$$

Let us give a sketch of proof of Theorem 2.2. We start from the decomposition (3). The terms associated with h_1 and h_2 can be treated directly with the use of Proposition 2.3. For the terms associated to the degenerated part, we use Proposition 2.3 with $D_j = N^{-1/2}\sum_{i=1}^{j-1} h_3(X_i, X_j)$ and $Y = N^{-1/2}\max_{1\leq j\leq N}\sum_{i=1}^{j-1} h_3(X_i, X_0)$ then with $D'_i = h_3(X_{-1}, X_0)$.

In the application to functional limit theorems, it will be important to be able to control increments of a U-statistic. For simplicity, we will state only the degenerated case, the general one can be deduced from decomposition (3).

Proposition 2.4. *Let $(S, \mathcal{S})$ be a measurable space, $h\colon S^2 \to \mathbb{R}$ be a measurable function (with S^2 induced with the product σ-algebra) and let $(X_i)_{i\geq 1}$ be an i.i.d. sequence of S-valued random variables. Suppose that h is degenerated with respect to $(X_i)_{i\geq 1}$. Then the following inequality holds for all positive x and y, and all $n_2 > n_1 \geq 2$,*

$$\mathbb{P}\left\{\frac{1}{\sqrt{n_2 - n_1}\sqrt{n_2}}\left|\binom{n_2}{2}U_{n_2,h} - \binom{n_1}{2}U_{n_1,h}\right| > x\right\} \leq A\exp\left(-\frac{x}{y}\right) + B\int_1^{\infty}\mathbb{P}\left\{\left|h(X_1, X_2)\right| > y\sqrt{u}C\right\}(1+\ln(u))^2\,\mathrm{d}u, \tag{11}$$

where A, B and C are numerical constants.

2.3. An application to the law of large numbers

A direct application of Theorem 2.2 gives convergence rates in the law of large numbers.

Theorem 2.5. *Let $(S, \mathcal{S})$ be a measurable space, $h\colon S^2 \to \mathbb{R}$ be a symmetric function and let $(X_i)_{i\geq 1}$ be an i.i.d. sequence of random variables with values in S. If there exists a positive $\gamma > 0$ such that $M :=$ $\sup_{t>0}\exp(t^\gamma)\mathbb{P}\{|h(X_1, X_2)| > t\}$ is finite, then for all $x > 0$,*

$$\mathbb{P}\left\{\max_{2\leq n\leq N}|U_{n,h}| > x\right\} \leq K_1\exp\left(-K_2 N^{\frac{\gamma}{2+\gamma}} x^{\frac{2\gamma}{2+\gamma}}\right),$$

where K_1 and K_2 depend on γ and M. Moreover, if h is degenerated with respect to $(X_i)_{i\geq 1}$, then

$$\mathbb{P}\left\{\max_{2\leq n\leq N}|U_{n,h}| > x\right\} \leq K_1\exp\left(-K_2 N^{\frac{\gamma}{1+\gamma}} x^{\frac{\gamma}{1+\gamma}}\right).$$

2.4. An application to the functional central limit theorem in some Hölder spaces

A partial sum process can be associated to U-statistics in the following way:

$$\sigma_n(t) := \left(U_{[nt],h} - (nt - [nt])\left(U_{[nt]+1,h} - U_{[nt],h}\right)\right), t \in [0,1], n \geq r, \tag{12}$$

where for $x \in \mathbb{R}$, $[x]$ is the unique integer satisfying $[x] \leq x < [x] + 1$. In other words, $\sigma_n(k/n) = U_{k,h}$ and the random function $t \mapsto \sigma_n(t)$ is affine on the intervals $[k/n, (k+1)/n]$. In [47], the convergence in distribution in the Skorokhod space $D[0,1]$ of the process $\left(U_{[n\cdot]}/a_n\right)_{n\geq 2}$ is studied, where $(a_n)_{n\geq 1}$ is a normalizing sequence. In Corollary 1, it is shown that if h is degenerated with respect to $(X_i)_{i\geq 1}$, then $\left(nU_{[n\cdot]}\right)_{n\geq 2}$ converges in distribution to $\left(\sum_{k\geq 1}\lambda_k\left(B_{k,t}^2-t\right)\right)_{t\in[0,1]}$, where $(\lambda_k)_{k\geq 1}$ is a sequence of real numbers such that $\sum_{k\geq 1}\lambda_k^2<\infty$ and $(B_{k,\cdot})_{k\geq 1}$ is a sequence of independent standard Brownian motions. If h is not degenerated, then for some constant $\sigma>0$, $\left(\sqrt{n}U_{[n\cdot]}\right)_{n\geq 2}$ converges in distribution to $(\sigma B_t)_{t\in[0,1]}$, where $(B_t)_{t\in[0,1]}$ is a standard Brownian motion. It can be shown that in the previous convergences, $U_{[n\cdot]}$ may be replaced by $\sigma_n(\cdot)$.

Define

$$\rho_{\beta,c}\colon t\mapsto t^{1/2}\left(\ln(c/t)\right)^{\beta},\quad t\in(0,1]\quad\text{ and } \rho_{\beta,c}(0)=0, \tag{13}$$

where $\beta>0$ and $c>0$ are such that the function $\rho_{\beta,c}$ is increasing on $(0,1]$, for instance, $\ln c>2\beta$. We denote by $\mathcal{H}_{\rho_{\beta,c}}$ the Hölder space associated to the modulus of regularity $\rho_{\beta,c}$, that is, the set of function $x\colon[0,1]\to\mathbb{R}$ such that

$$\|x\|_{\rho_{\beta,c}}:=\sup_{0\leq s<t\leq 1}|x(t)-x(s)|/\rho_{\beta,c}(t-s)+|x(0)|$$

is finite. Since $\rho_{\beta,c}$ vanishes only at 0, such a norm is well defined. Instead of dealing with the convergence in $\mathcal{H}_{\rho_{\beta,c}}$, we will work with a subspace which is more adapted to the study of convergence in distribution. Let

$$\mathcal{H}^o_{\rho_{\beta,c}}:=\left\{x\colon[0,1]\to\mathbb{R}\mid\lim_{\delta\to 0}\sup_{\substack{s,t\in[0,1]\\0<t-s<\delta}}\frac{|x(t)-x(s)|}{\rho_{\beta,c}(t-s)}=0\right\}.$$

The convergence of partial sum processes of the form

$$W_n(t):=\frac{1}{a_n}\left(\sum_{i=1}^{[nt]}Y_i+(nt-[nt])Y_{[nt]+1}\right),\quad t\in[0,1], \tag{14}$$

when $(Y_i)_{i\geq 1}$ is i.i.d. has been studied in [64,65]. The convergence of $(W_n(\cdot))_{n\geq 1}$ in $\mathcal{H}^o_{\rho_{\beta,c}}$ for $\beta>1/2$ holds if and only if

$$\forall A>0,\lim_{t\to\infty}t\mathbb{P}\left\{|Y_1|>At^{1/2}\rho(1/t)\right\}=0.$$

Generally, a strategy to prove such results is to establish the convergence of the finite dimensional distributions and prove tightness, which is usually the most difficult part. In Equation (1.3) in [31], a tightness criterion for partial sum processes of the form (14) with $(Y_j)_{j\geq 1}$ stationary and ρ of the form $t\mapsto t^{\alpha}$, $0<\alpha<1/2$ was established. Its verification is done by using deviation inequalities, see for example [30] or Section 3.3 in [15].

For the purpose of the study of the convergence of $(\sigma_n(\cdot))_{n\geq 1}$ (defined by (12)), we need to extend this criterion in two directions: to partial sum processes like in (14) for which the sequence $(Y_j)_{j\geq 1}$ is not necessarily stationary and where the map $t\mapsto t^{\alpha}$ is replaced by $\rho_{\beta,c}$.

Proposition 2.6. *Let $(Y_j)_{j\geq 1}$ be a sequence of random variables. Let W_n be the partial sum process built on $(Y_j)_{j\geq 1}$ as in* (14) *with $(a_n)_{n\geq 1}$ an increasing sequence diverging to infinity and such that $\sup_{n\geq 1}a_{2n}/a_n$ is finite. Let $\rho_{\beta,c}$ be defined as in* (13)*. Suppose that for all positive ε, the following convergences hold:*

$$\lim_{J\to\infty}\limsup_{n\to\infty}\sum_{j=J}^{[\log_2 n]}\sum_{k=0}^{2^j-1}\mathbb{P}\left\{\left|S_{[n(k+1)2^{-j}]}-S_{[nk2^{-j}]}\right|>a_n\varepsilon\rho_{\beta,c}\left(2^{-j}\right)\right\}=0; \tag{15}$$

where $S_N:=\sum_{i=1}^N Y_i$. Then the partial sum process $(W_n(t))_{n\geq 1,t\in[0,1]}$ is tight in $\mathcal{H}^o_{\rho_{\beta,c}}$.

Checking this tightness criterion with the help of Proposition 2.4, we derive the following result.

Theorem 2.7. *Let $(S,\mathcal{S})$ be a measurable space and $h\colon S^2\to\mathbb{R}$ be a symmetric function. Let $(X_i)_{i\geq 1}$ be an i.i.d. sequence and let*

$$\sigma_n(t) := \left(U_{[nt],h} - (nt-[nt])\left(U_{[nt]+1,h} - U_{[nt],h}\right)\right), t\in[0,1], n\geq 2,$$

Let $\rho_{\beta,c}(t) = t^{1/2}\left(\log(c/t)\right)^{\beta}$, where β and c are such that $\rho_{\beta,c}$ is increasing on $(0,1]$.

- *Suppose that $\beta > 1/2$ and*

$$\forall A>0,\quad \mathbb{E}\left[\exp\left(A\left|h(X_1,X_2)\right|^{\frac{1}{\beta-1/2}}\right)\right]<\infty \tag{16}$$

Then there exists $\sigma\geq 0$ such that

$$\left(\sqrt{n}(\sigma_n(t)-\mathbb{E}\left[\sigma_n(t)\right])\right)_{n\geq 1,t\in[0,1]} \xrightarrow[n\to\infty]{\text{law}} (\sigma B_t)_{t\in[0,1]} \text{ in } \mathcal{H}_{\rho_{\beta,c}},$$

where $(B_t)_{t\in[0,1]}$ is a standard Brownian motion.

- *Suppose that h is degenerated with respect to $(X_i)_{i\geq 1}$, $\beta>1$ and*

$$\forall A>0,\quad \mathbb{E}\left[\exp\left(A\left|h(X_1,X_2)\right|^{\frac{1}{\beta-1}}\right)\right]<\infty$$

Then there exists a sequence $(\lambda_k)_{k\geq 1}$ of real numbers and a sequence $(B_{k,\cdot})_{k\geq 1}$ of independent standard Brownian motions such that

$$(n\sigma_n(t))_{n\geq 1,t\in[0,1]} \xrightarrow[n\to\infty]{\text{law}} \left(\sum_{k\geq 1}\lambda_k\left(B_{k,t}^2-t\right)\right)_{t\in[0,1]} \text{ in } \mathcal{H}_{\rho_{\beta,c}}.$$

3. Scaling limits for network traffic models (V. Pilipauskaitė)

This section presents some limit theorems for network traffic models, with a focus on recent results in [45]. These theorems can help explain the presence of self-similarity, long-range dependence and heavy tails in measurements of broadband network traffic, see [50, 52, 77] and related references. The network traffic model uses a stationary stochastic process $X := \{X(t),\, t\in\mathbb{R}\}$, such as an ON/OFF process or an M/G/∞ queue, in which the length of data transmission sessions follows infinite-variance Pareto-like distribution, implying that X has long-range dependence. A stationary process X with $\mathbb{E}[|X(0)|^2]<\infty$ is said to have long-range dependence if its covariance function satisfies $\int_{\mathbb{R}}|\operatorname{Cov}(X(0),X(t))|dt=\infty$. Let $X_1, X_2, \ldots$ be independent copies of such an X, representing (streams of) inputs from independent sources in the network. The aggregated input from N sources at time scale T is defined as

$$A(Tt,N) := \int_0^{Tt}\sum_{i=1}^N X_i(s)\,\mathrm{d}s,\quad t\in\mathbb{R}_+ := (0,\infty). \tag{17}$$

The limiting behaviour of its deviation from the mean is of interest when both N, T tend to infinity.

Assuming that the relationship between the time scale and the number of sources takes a power-law form, we define

$$A_{\lambda,\gamma}(t_1,t_2) := A(\lambda t_1, \lfloor\lambda^\gamma t_2\rfloor),\quad (t_1,t_2)\in\mathbb{R}_+^2, \tag{18}$$

where $\gamma>0$ is arbitrary. We then study the scaling limit:

$$\{b_{\lambda,\gamma}^{-1}(A_{\lambda,\gamma}(t_1,t_2)-\mathbb{E}[A_{\lambda,\gamma}(t_1,t_2)]),\,(t_1,t_2)\in\mathbb{R}_+^2\} \xrightarrow[\lambda\to\infty]{\text{fdd}} V_\gamma, \tag{19}$$

where $b_{\lambda,\gamma} \to \infty$ is a normalization, $V_\gamma := \{V_\gamma(t_1, t_2), (t_1, t_2) \in \mathbb{R}^2_+\}$ is a non-degenerate limiting random field. Note that γ determines the anisotropy of the scaling procedure in (19).

Anisotropic scaling limits in (19) have been studied for some classes of random fields on $\mathbb{R}^2$ or $\mathbb{Z}^2$ in a series of recent papers [5, 56, 57, 60, 61]. In these studies, $A_{\lambda,\gamma}(t_1, t_2)$ corresponds to an integral or sum of values of a stationary random field over rectangular region $(0, \lambda t_1] \times (0, \lambda^\gamma t_2]$ for an arbitrary $\gamma > 0$, which determines the relationship between the growth rates of the region in the vertical and horizontal directions. As $\lambda \to \infty$, a large class of random fields with long-range dependence exhibits a scaling transition, where different V_γ arise depending on γ. In particular, there exists a critical $\gamma_0 > 0$ such that all V_γ, $\gamma < \gamma_0$, are equal in distribution, while all V_γ, $\gamma > \gamma_0$, are also equal in distribution but differ from the case where $\gamma < \gamma_0$, with all equalities in distribution being true up to a multiplicative constant. The random fields V_γ exhibit characteristic properties, such as stationary rectangular increments and a specific form of operator-scaling property, see Remark 3.6 below. In that regard, the random field setting of (19) seems to be more natural and suited for the study of the scaling limits with aggregation, as in (17).

3.1. The ON/OFF inputs

In this section we survey fundamental results about aggregation of ON/OFF inputs. An ON/OFF source continuously transmits data at a fixed rate of 1 if it is ON and remains silent if it is OFF. Let $\{Z_{\text{on},1}, Z_{\text{on},2}, \dots\}$ and $\{Z_{\text{off},1}, Z_{\text{off},2}, \dots\}$ be independent sequences of i.i.d. positive random variables which represent the lengths of ON-periods and OFF-periods. Assume that

$$\mathbb{P}(Z_{\text{on},1} > z) \sim c_{\text{on}} z^{-\alpha_{\text{on}}}, \quad \mathbb{P}(Z_{\text{off},1} > z) \sim c_{\text{off}} z^{-\alpha_{\text{off}}}, \quad z \to \infty, \tag{20}$$

for some $1 < \alpha_{\text{on}} < \alpha_{\text{off}} < 2$ and $c_{\text{on}}, c_{\text{off}} > 0$. Note that $Z_{\text{on},1}$, $Z_{\text{off},1}$ have finite means μ_{on}, μ_{off} but their variances are infinite. Set $T_j := \sum_{i=0}^{j} (Z_{\text{on},i} + Z_{\text{off},i})$, $j = 0, 1, \dots$, where $Z_{\text{on},0}$, $Z_{\text{off},0}$ are independent random variables, independent of the sequences $\{Z_{\text{on},1}, Z_{\text{on},2}, \dots\}$, $\{Z_{\text{off},1}, Z_{\text{off},2}, \dots\}$. Choose the distribution of $Z_{\text{on},0}$, $Z_{\text{off},0}$ as in [36] so that X defined as follows is a stationary process:

$$X(t) = \mathbf{1}(0 \le t < Z_{\text{on},0}) + \sum_{j=0}^{\infty} \mathbf{1}(T_j \le t < T_j + Z_{\text{on},j+1}), \quad t \in \mathbb{R}_+. \tag{21}$$

In other words, $X(t) = 1$ if time t is in the ON-period, $X(t) = 0$ if time t is in the OFF-period. Since its covariance function $\text{Cov}(X(0), X(t)) \sim \text{const}\, t^{-(\alpha_{\text{on}}-1)}$, $t \to \infty$, see [36], is not absolutely integrable, X has long-range dependence.

Let $X_1, X_2 \dots$ be independent copies of the stationary ON/OFF process X in (21). Let $\{A(Tt, N), t \in \mathbb{R}_+\}$ be the integrated sum of $X_1, \dots, X_N$ at time scale T as in (17). Taqqu et al. [74] study its iterated limits when $N \to \infty$, then $T \to \infty$ and vice versa. Mikosch et al. [50] discuss its limits when $N, T \to \infty$ simultaneously and identify two scaling regimes of fast growth and slow growth (cases (i) and (ii) of Theorem 3.1 below, respectively). [21] complements the results of [50] by showing that the third limit process arises under intermediate scaling (case (iii) of Theorem 3.1).

Theorem 3.1 (Mikosch et al. [50], Dombry, Kaj [21]). *Let X be as in* (21) *and $\alpha := \alpha_{\text{on}}$, $\mu := \mu_{\text{on}} + \mu_{\text{off}}$ in* (20). *Let $N, T \to \infty$ so that $N/T^{\alpha-1} \to (\mu/c_{\text{on}})c^{\alpha-1} \in [0, \infty]$. Then*

$$\{b_{N,T}^{-1}(A(Tt, N) - \mathbb{E}[A(Tt, N)]), t \in \mathbb{R}_+\} \xrightarrow{\text{fdd}} \{V(t), t \in \mathbb{R}_+\}$$

holds in cases: (i) $c = \infty$, (ii) $c = 0$, (iii) $c \in (0, \infty)$, where, respectively,

(i) *$b_{N,T} := N^{1/2} T^{H_1}$, V is a fractional Brownian motion with Hurst index $H_1 := (3-\alpha)/2$ and $\mathbb{E}[|V(1)|^2] := 2c_{\text{on}}\mu_{\text{off}}^2/(\mu^3(3-\alpha)(2-\alpha)(\alpha-1))$,*

(ii) $b_{N,T} := (NT)^{1/\alpha}$, V *is an* α*-stable Lévy motion with*

$$\mathbb{E}[\exp\{i\theta V(1)\}] := \exp\{-\sigma_0^\alpha|\theta|^\alpha(1 - i\,\mathrm{sgn}(\theta)\tan(\pi\alpha/2))\}, \quad \theta \in \mathbb{R},$$

and $\sigma_0^\alpha := c_{\mathrm{on}}\mu_{\mathrm{off}}^\alpha\Gamma(2-\alpha)\cos(\pi\alpha/2)/(\mu^{\alpha+1}(1-\alpha))$,

(iii) $b_{N,T} := T$, $V := \{(\mu_{\mathrm{off}}/\mu)cJ(t/c),\, t \in \mathbb{R}_+\}$ *with* $J := \{J_Z(t,1),\, t \in \mathbb{R}_+\}$ *given below by* (23) *for* $\alpha = \alpha_{\mathrm{on}}$, $c_\alpha = c_{\mathrm{on}}$.

The stochastic process J is called the intermediate Telecom process or the fractional Poisson motion. As shown in [25, 26], J has stationary increments, infinitely divisible finite-dimensional distributions, mean zero and its covariances coincide with those of a standard fractional Brownian motion with Hurst index H_1 up to multiplicative constant. Moreover, the process J is asymptotically locally and globally self-similar with fractional Brownian motion with index H_1 and α-stable Lévy motion respectively as its tangent limits. So J can be viewed as a bridge between the limiting processes in cases (i) and (ii).

Remark 3.2. Actually, [21, 50] make a more general assumption than that in (20). Specifically, asymptotic equivalence of $z \mapsto \mathbb{P}(Z_{\mathrm{on},1} > z)$ and $z \mapsto \mathbb{P}(Z_{\mathrm{off},1} > z)$ to tails of the Pareto distribution is generalized to include a slowly varying function factor. Then the relative growth condition on $N, T \to \infty$ and normalization in cases (i), (ii) also include this factor while the limiting processes remain unchanged.

Remark 3.3. Furthermore, [21,50] show that in cases (i) and (iii) of Theorem 3.1, stochastic processes converge in distribution in $C[0,\infty)$, endowed with the topology of uniform convergence on compact sets. In case (ii) of Theorem 3.1, the convergence cannot be extended to the convergence in distribution in $D[0,\infty)$, endowed with the Skorokhod J_1-topology, see Remark 2 in [50].

Remark 3.4. Similar limit theorems hold for other network traffic models such as M/G/∞ queue (also called infinite Poisson source model), continuous flow rate model, renewal-reward process, see [39, 50, 51, 59]. Gaigalas and Kaj [26] are the first to obtain the intermediate Telecom process J for the sum of independent scaled renewal processes. Furthermore, in the random field setting, M/G/∞ queue is known as a random grain model whose scaling limits with and without aggregation have been derived in [6, 38, 56, 73].

Remark 3.5. In [54] limit theorems of this type are shown for an autoregressive process of order 1 (AR(1)) with random coefficient. If the distribution of the AR coefficient a concentrates in the neighbourhood of the unit root $a = 1$, then the process has long-range dependence. Since random-coefficient AR(1) process and network traffic models differ in their dependence structures, their slow growth limits also differ. On the other hand, a fractional Brownian motion is robust as a fast growth limit up to a choice of parameters. For aggregation of dependent random-coefficient AR(1) processes or products of their independent copies, see [44, 55].

3.2. General classes of inputs

Leipus et al. [45] investigate two classes of X that include classical network traffic models as well as new ones. For both classes, they derive simple general conditions that ensure the same scaling limits with aggregation as in cases (i) and (ii) of Theorem 3.1. The limits are discussed in the setting of random fields on $\mathbb{R}_+^2$, as in (19). It is straightforward to determine limit random fields V_γ indexed by $(t_1, t_2) \in \mathbb{R}_+^2$ from the limit processes V in Theorem 3.1, since the summands in (17) are independent. Theorem 3.1 implies that scaling transition in (19) occurs at $\gamma_0 = \alpha - 1$.

3.2.1. *Limit random fields*

Recall that a standard fractional Brownian sheet $B_{H_1,H_2} = \{B_{H_1,H_2}(t_1,t_2),\, (t_1,t_2) \in \mathbb{R}_+^2\}$ with Hurst index $(H_1, H_2) \in (0,1]^2$ is a centred Gaussian process with covariances

$$\mathbb{E}[B_{H_1,H_2}(t_1,t_2)B_{H_1,H_2}(s_1,s_2)] = \prod_{i=1}^{2}\frac{1}{2}(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i}), \quad (t_1,t_2),(s_1,s_2) \in \mathbb{R}_+^2.$$

For $\alpha \in (0,2)$, $\sigma > 0$, $\beta \in [-1,1]$, define an α-stable Lévy sheet $L_{\alpha,\sigma,\beta}$ as

$$L_{\alpha,\sigma,\beta}(t_1,t_2) := M_{\alpha,\sigma,\beta}((0,t_1] \times (0,t_2]), \quad (t_1,t_2) \in \mathbb{R}^2_+,$$

where $M_{\alpha,\sigma,\beta}$ is an α-stable random measure on $\mathbb{R}^2_+$ with control measure $\sigma^\alpha ds_1 ds_2$ and constant skewness intensity β, see [70].

Leipus et al. [45] build a general class of 'intermediate' random fields from σ-finite measures ν on $\mathbb{W} := \{w \in L^1(\mathbb{R}) : w(t) = 0,\, t < 0\}$ equipped with Borel σ-algebra $\mathcal{B}(\mathbb{W})$. Assume that there exist $\gamma \in \mathbb{R}$, $H \in \mathbb{R}$ such that $\lambda^{1+\gamma}\nu(\phi^H_\lambda(B)) = \nu(B)$, $B \in \mathcal{B}(\mathbb{W})$, $\lambda > 0$, where $\phi^H_\lambda w(t) := \lambda^{H-1}w(\lambda^{-1}t)$, $t \in \mathbb{R}$, $w \in \mathbb{W}$, moreover, $\int_{\mathbb{R}\times\mathbb{W}} \min\{|\int_0^t w(s-u)ds|, |\int_0^t w(s-u)ds|^2\} du\nu(dw) < \infty$, $t > 0$. With such a ν, [45] associates a centred (also called compensated) Poisson random measure $\tilde{M}_\nu$ on $\mathbb{R}\times\mathbb{R}\times\mathbb{W}$ with intensity measure $du_1du_2\nu(dw)$ and defines an 'intermediate' random field J_ν as

$$J_\nu(t_1,t_2) := \int_{\mathbb{R}\times(0,t_2]\times\mathbb{W}} \Big\{\int_0^{t_1} w(s-u_1)ds\Big\} \tilde{M}_\nu(du_1,du_2,dw), \quad (t_1,t_2) \in \mathbb{R}^2_+. \tag{22}$$

For properties of stochastic integrals with respect to Poisson random measure, see [63]. The most important example of (22) is the intermediate Telecom random field

$$J_Z(t_1,t_2) := \int_{\mathbb{R}\times(0,t_2]\times\mathbb{R}_+} \Big\{\int_0^{t_1} \mathbf{1}(u_1 \le s < u_1 + z)ds\Big\} \tilde{M}_Z(du_1,du_2,dz), \quad (t_1,t_2) \in \mathbb{R}^2_+, \tag{23}$$

where $\tilde{M}_Z$ is a centred Poisson random measure on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$ with intensity measure $du_1du_2\nu_Z(dz)$ and $\nu_Z(dz) := \alpha c_\alpha z^{-\alpha-1}dz$ for some $1 < \alpha < 2$, $c_\alpha > 0$. Remark the equality in distribution of J_Z and J_ν, where ν is obtained by transferring ν_Z from $\mathbb{R}_+$ to $\mathbb{W}$ via the mapping $z \mapsto \mathbf{1}(0 \le t < z)$, $t \in \mathbb{R}$.

Remark 3.6. The 'intermediate' random field J_ν in (23) is shown in [45] to satisfy a special form of operator-scaling property introduced in [4]: $\{J_\nu(\lambda t_1, \lambda^\gamma t_2),\, (t_1,t_2) \in \mathbb{R}^2_+\}$ and $\lambda^H J_\nu$ are equal in distribution. The latter property also holds for a (H_1,H_2)-multi-self-similar random field with $H := H_1 + \gamma H_2$ and all $\gamma > 0$. A random field $V = \{V(t_1,t_2),\, (t_1,t_2) \in \mathbb{R}^2_+\}$ is said to be (H_1,H_2)-multi-self-similar with index $(H_1,H_2) \in \mathbb{R}^2$ if $\{V(\lambda_1 t_1, \lambda_2 t_2),\, (t_1,t_2) \in \mathbb{R}^2_+\}$ and $\lambda_1^{H_1}\lambda_2^{H_2}V$ are equal in distribution for all $\lambda_i > 0$, $i = 1,2$, see [27]. Both B_{H_1,H_2} and $L_{\alpha,\sigma,\beta}$ are multi-self-similar, however, J_Z is not (H_1,H_2)-multi-self-similar for any H_i, $i = 1,2$.

Remark 3.7. [45] establishes that under some additional conditions on ν, the covariances of J_ν agree with those of a multiple of $B_{H_1,1/2}$ with $H_1 := H - (\gamma/2)$; moreover, the authors also obtain the asymptotic local and global self-similarity of the stochastic process $\{J_\nu(t,1),\, t \in \mathbb{R}_+\}$.

3.2.2. *Poisson shot-noise inputs*

The first class of inputs has a form

$$X(t) := \sum_{j\in\mathbb{Z}} W_j(t - T_j), \qquad t \in \mathbb{R}, \tag{24}$$

where $\{T_j,\, j \in \mathbb{Z}\}$ is a homogeneous Poisson point process on $\mathbb{R}$ with unit rate, independent of $\{W_j,\, j \in \mathbb{Z}\}$, which is a sequence of independent copies of a 'pulse' process $W = \{W(t),\, t \in \mathbb{R}\}$ with distribution $\mathbb{P}_W$ on $\mathbb{W}$ such that $\int_{\mathbb{W}}\int_0^\infty (|w(t)| + |w(t)|^2)dt\mathbb{P}_W(dw) < \infty$.

Consider $A_{\lambda,\gamma}(t_1,t_2)$ in (18) for arbitrary $\gamma > 0$. Note that summing over $\lfloor \lambda^\gamma t_2 \rfloor$ copies of independent copies of X yields the same distribution as multiplying the intensity of the underlying Poisson point process by $\lfloor \lambda^\gamma t_2 \rfloor$. Next, the centred $A_{\lambda,\gamma}(t_1,t_2)$ in (18) can be represented as

$$A_{\lambda,\gamma}(t_1,t_2) - \mathbb{E}[A_{\lambda,\gamma}(t_1,t_2)] = \int_{\mathbb{R}\times(0,\lfloor\lambda^\gamma t_2\rfloor]\times\mathbb{W}} \Big\{\int_0^{\lambda t_1} w(s-u_1)ds\Big\} \tilde{M}(du_1,du_2,dw), \quad (t_1,t_2) \in \mathbb{R}^2_+, \tag{25}$$

where $\tilde{M}$ is the centred Poisson random measure on $\mathbb{R} \times \mathbb{R} \times \mathbb{W}$ with intensity measure $du_1 du_2 \mathbb{P}_W(dw)$. This representation provides an explicit expression for the characteristic functions. In particular, [45] makes use of it to show that the long-range dependence $\mathrm{Cov}(X(0), X(t)) \sim \mathrm{const}\, t^{-2(1-H_1)}$, $t \to \infty$, for some $H_1 \in (1/2, 1)$, together with the Lyapunov condition, guarantees the convergence in (19) towards a fractional Brownian sheet with Hurst index $(H_1, 1/2)$. To obtain an α_0-stable Lévy sheet limiting distribution in (19) it seems crucial that $\int_0^\infty W(t)dt$ belongs to the normal domain of attraction of an α_0-stable law and that $\mathbb{E}[|W(t)|]^\alpha$ decays fast enough as $t \to \infty$ for some suitable $\alpha < \alpha_0$. However, the existence of 'intermediate' limit at $\gamma = \gamma_0$ in (19) is more demanding and requires an asymptotic scaling form of the distribution of the 'pulse' process W.

The most important example of (24) is M/G/∞ queue with $W(t) := \mathbf{1}(0 \le t < Z)$, $t \in \mathbb{R}$, and $Z := Z_{\mathrm{on},1}$ as in (20). Another example of (24) is an exponentially damped transmission rate model with

$$W(t) := \exp\{-Rt\}\mathbf{1}(0 \le t < Z), \quad t \in \mathbb{R}, \tag{26}$$

where $R, Z > 0$ are independent random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{P}(R \le r) \sim c_\rho r^\rho, \quad r \to 0, \qquad \mathbb{P}(Z > z) \sim c_\alpha z^{-\alpha}, \quad z \to \infty, \tag{27}$$

for some positive exponents $\rho, \alpha > 0$ and asymptotic constants $c_\rho, c_\alpha > 0$. Consider $\mathbb{R}_+^2$ with the distribution $\mathbb{P}_R \times \mathbb{P}_Z$ of (R, Z). Then the mapping from $\mathbb{R}_+^2$ to $\mathbb{W}$ given by $(r, z) \mapsto \exp\{-rt\}\mathbf{1}(0 \le t < z)$, $t \in \mathbb{R}$, induces the distribution $\mathbb{P}_W$.

Proposition 3.8 (Leipus et al. [45]). *Let X be a Poisson shot-noise process in* (24) *with W as in* (26), (27), *where $1 < \rho + \alpha < 2$. Let $\gamma_0 := \rho + \alpha - 1$. In addition, assume that if $\gamma = \gamma_0$, then R, Z have densities f_R, f_Z satisfying*

$$f_R(r) \sim \rho c_\rho r^{\rho-1}, \quad r \to 0, \qquad f_Z(z) \sim \alpha c_\alpha z^{-\alpha-1}, \quad z \to \infty,$$

and $f_R(r) \le c r^{\rho-1}$, $r > 0$, $f_Z(z) \le c z^{-\alpha-1}$, $z > 0$, for some $c > 0$. Then for all $\gamma > 0$, the convergence in (19) *holds with*

$$b_{\lambda,\gamma} := \begin{cases} \lambda^{H_1+(\gamma/2)}, & \gamma > \gamma_0, \\ \lambda^{(1+\gamma)/\alpha_0}, & \gamma < \gamma_0, \\ \lambda, & \gamma = \gamma_0, \end{cases} \qquad V_\gamma := \begin{cases} \sigma_1 B_{H_1,1/2}, & \gamma > \gamma_0, \\ L_{\alpha_0,\sigma_0,1}, & \gamma < \gamma_0, \\ J_\nu, & \gamma = \gamma_0, \end{cases}$$

where the Hurst index $H_1 := (3 - \rho - \alpha)/2$, $\sigma_1^2 := \Gamma(\rho+1)c_\rho c_\alpha c_{\rho,\alpha}/(H_1(2H_1 - 1))$, $c_{\rho,\alpha} := \int_0^\infty (1+x)^{-\alpha}(1+2x)^{-\rho}dx$ the stability index $\alpha_0 := \rho + \alpha$, $\sigma_0 := (c_0\Gamma(2-\alpha_0)\cos(\pi\alpha_0/2)/(1-\alpha_0))^{1/\alpha_0}$, $c_0 := \rho c_\rho c_\alpha \int_0^1 (1-x)^{\alpha_0-1}(\log(1/x))^{-\alpha}dx$, the measure ν is induced by the mapping $(r, z) \mapsto \exp\{-rt\}\mathbf{1}(0 \le t < z)$, $t \in \mathbb{R}$, to $\mathbb{W}$ from $\mathbb{R}_+^2$ with $\nu_R \times \nu_Z$, where $\nu_R(dr) := \rho c_\rho r^{\rho-1}dr$, $\nu_Z(dz) := \alpha c_\alpha z^{-\alpha-1}dz$.

3.2.3. *Regenerative inputs*

Let us introduce the second class of inputs X. Let $Z > 0$ be a random variable as in (27) and W be a 'pulse' process, bounded uniformly on $[0, Z)$ by a non-random constant. Let

$$(Z_j, \{W_j(t),\, t \in [0, Z_j)\}), \quad j = 1, 2, \ldots, \tag{28}$$

be i.i.d. copies of $(Z, \{W(t),\, t \in [0, Z)\})$, independent of $(Z_0, \{W_0(t),\, t \in [0, Z_0)\})$. Set $T_j := \sum_{i=0}^j Z_i$, $j = 0, 1, \ldots$, and distribution of $(Z_0, \{W_0(t),\, t \in [0, Z_0)\})$ as in [45] so that

$$X(t) := W_0(t)\mathbf{1}(0 \le t < T_0) + \sum_{j=1}^\infty W_j(t - T_{j-1})\mathbf{1}(T_{j-1} \le t < T_j), \quad t \in \mathbb{R}_+, \tag{29}$$

is a stationary process. Note that in general the 'pulses' W_j in (29) can be dependent on the regeneration intervals $Z_j = T_j - T_{j-1}$. The regeneration is a consequence of independence of the elements in (28).

The scaling limits for regenerative inputs are more difficult to study since they depend on both the length Z of the regeneration interval and the 'pulse' W through it. Leipus et al. [45] provide sufficient conditions for the scaling transition at $\gamma_0 := \alpha - 1$ in (19). Specifically, the long-range dependence of X guarantees the convergence in (19) towards a fractional Brownian sheet with Hurst index $(H_1, 1/2)$, $H_1 := (3-\alpha)/2$, for all $\gamma > \gamma_0$. On the other hand, if $\int_0^Z W(t)dt - (Z/\mu)\mu_W$ with $\mu := \mathbb{E}[Z]$, $\mu_W := \mathbb{E}[\int_0^Z W(t)dt]$, belongs to the normal domain of attraction of an α-stable law then an α-stable Lévy sheet appears as the limiting distribution V_γ for all $\gamma < \gamma_0$. Finally, for $\gamma = \gamma_0$, the convergence in (19) holds towards the intermediate Telecom random field $-(\mu_W/\mu)J_Z$ if $\int_0^Z W(t)dt$ has a lighter tail than Z, moreover, $\mathbb{E}[|W(t)\mathbf{1}(0 \le t < Z)|]$ decays fast enough as $t \to \infty$.

The class of regenerative processes is very large. It also includes the ON/OFF process with $W(t) := \mathbf{1}(0 \le t < Z_{\text{on},1})$, $t \in \mathbb{R}$, and $Z := Z_{\text{on},1} + Z_{\text{off},1}$. As noted in [21, 50], the scaling limits for the ON/OFF and M/G/∞ models are essentially the same. The following proposition demonstrates that the regenerative version of the exponentially damped transmission rate model differs from the Poisson shot-noise version in terms of scaling limits in (19) despite the same distributional assumption about (R, Z).

Proposition 3.9 (Leipus et al. [45])**.** *Let X be a stationary regenerative process in* (29) *with Z, W as in* (26), (27)*, where $1 < \alpha < \alpha + \rho < 2$. Let $\gamma_0 := \alpha - 1$. In addition, assume that if $\gamma > \gamma_0$ then for some $n \in \mathbb{N}$ the n-th convolution power of $z \mapsto \mathbb{P}(Z \le z)$ is non-singular. Then for all $\gamma > 0$, the convergence in* (19) *holds with*

$$b_{\lambda,\gamma} := \begin{cases} \lambda^{H_1 + (\gamma/2)}, & \gamma > \gamma_0, \\ \lambda^{(1+\gamma)/\alpha}, & \gamma < \gamma_0, \\ \lambda, & \gamma = \gamma_0, \end{cases} \qquad V_\gamma := \begin{cases} \sigma_1 B_{H_1,1/2}, & \gamma > \gamma_0, \\ L_{\alpha,\sigma_0,-1}, & \gamma < \gamma_0, \\ -(\mu_W/\mu)J_Z, & \gamma = \gamma_0, \end{cases}$$

where the Hurst index $H_1 := (3-\alpha)/2$, $\sigma_1^2 := c_\alpha \mu_W^2/(2\mu^3 H_1(2H_1 - 1)(1 - H_1))$, $\sigma_0^\alpha := c_\alpha(\mu_W/\mu)^\alpha \Gamma(2 - \alpha)\cos(\pi\alpha/2)/(1-\alpha)$, $\mu := \mathbb{E}[Z]$, $\mu_W := \mathbb{E}[(1 - \exp\{-RZ\})/R]$.

4. Quenched Functional Central Limit Theorems for Stationary Random Fields under a Projective Condition (L. Reding)

The following is based upon some recent joint works with Na Zhang and Magda Peligrad [66, 79].

The question of the central limit theorem for Markov chains has been the subject of many mathematical studies for the past decades now. Numerous results have been obtained in which sufficient conditions on the initial distribution and on the transition operator for the central limit theorem were derived. For example, the seminal article of Derriennic and Lin [19] established some sufficient conditions for a central limit theorem for Markov chains started not from the equilibrium but from a fixed point. Whenever this convergence in distribution holds for almost all starting point, we say that the central limit theorem is quenched. Since then, an extensive literature on these theorems has been established both in the case of sequences of random variables as well as random fields. Recently Peligrad and Volný [53] established this type of theorems in the particular case of ortho-martingales. This improvement opened up a lot of possibilities through the use of approximation methods. Using this approach, we are interested in deriving some quenched functional central limit theorems for stationary random fields under a projective criterion.

Throughout this article, we shall focus on the particular case of stationary random fields of the form

$$X_{\boldsymbol{i}} = X_{\mathbf{0}} \circ T_1^{i_1} \circ \cdots \circ T_d^{i_d}, \tag{30}$$

where $d > 1$ is an integer, $X_{\mathbf{0}}$ is a square integrable centered regular[1] random variable and $T_1, \ldots, T_d$ are invertible measure-preserving commuting transforms on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We fix, as

[1] That is for any $i \in \{1, \ldots, d\}$, we have $\mathbb{E}[X_{\mathbf{0}} | \mathcal{F}_{-\infty \boldsymbol{e}_i}] = 0$ where $\mathcal{F}_{-\infty \boldsymbol{e}_i} = \bigcap_{n=0}^{\infty} \mathcal{F}_{-n\boldsymbol{e}_i}$, $\mathcal{F}_{-n\boldsymbol{e}_i}$ is given by (31), and $\boldsymbol{e}_i$ is the multi-index whose components are all zero except for the i-th which equals to 1.

well, a sigma-field $\mathcal{F}_{\mathbf{0}} \subset \mathcal{F}$ such that $X_{\mathbf{0}}$ is $\mathcal{F}_{\mathbf{0}}$-measurable and for every $\boldsymbol{n} \in \mathbb{Z}^d$, we let

$$\mathcal{F}_{\boldsymbol{n}} = T^{-\boldsymbol{n}} \mathcal{F}_{\mathbf{0}}. \tag{31}$$

In the following, bold characters will represent multi-indexes and in particular we shall write $\mathbf{0} := (0, \ldots, 0) \in \mathbb{Z}^d$ as well as $\mathbf{1} := (1, \ldots, 1) \in \mathbb{Z}^d$. For any $\boldsymbol{n} \in \mathbb{Z}^d$, we denote $\boldsymbol{n} := (n_1, \ldots, n_d)$ and $|\boldsymbol{n}| := \prod_{i=1}^d n_i$. We will write $\boldsymbol{u} \le \boldsymbol{v}$ whenever $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}^d$ are multi-indexes such that for all $k \in \{1, \ldots, d\}, u_k \le v_k$.

4.1. Hannan's Condition for Random Fields

4.1.1. *Commuting Filtrations and the Projection Operator*

Random fields naturally appear as a generalization of sequences of random variables, however extending the one-dimensional results to greater dimension is much harder than one would think. The first problem we are faced with is to correctly define the notion of past trajectory. In order to solve this issue, we shall make use of the notion of *commuting filtrations*. While this is not the only way to define the notion of past, this property is fairly tractable and is satisfied by a large class of filtrations such as the one generated by fields of independent random variables or even by fields with independent columns (or equivalently independent rows). As a lot of processes can be expressed as a functional of i.i.d. random variables, these type of filtrations are quite common and merit interest.

Before attempting to define what a *commuting filtration* is, we first take a closer look at the case of one-dimensional filtrations. One can notice that a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ satisfy the following property: if Z is an integrable random variable, then

$$\forall n \in \mathbb{N}^*, \quad \mathbb{E}\left[\mathbb{E}\left[Z|\, \mathcal{F}_n\right]|\, \mathcal{F}_{n-1}\right] = \mathbb{E}\left[Z|\, \mathcal{F}_{n-1}\right].$$

We would like to extend that property to d-dimensional filtrations with the aim of constructing a generalization of martingales with satisfying properties. We first look at the simpler case $d = 2$. Then, we shall require the following property:

$$\forall (i,j) \in \mathbb{Z}^2, \quad \mathbb{E}\left[\mathbb{E}\left[Z|\, \mathcal{F}_{i-1,j}\right]|\, \mathcal{F}_{i,j-1}\right] = \mathbb{E}\left[Z|\, \mathcal{F}_{i-1,j-1}\right].$$

We note that this property is satisfied whenever we have

$$\forall (a,b), (u,v) \in \mathbb{Z}^2, \quad \mathbb{E}\left[\mathbb{E}\left[Z|\, \mathcal{F}_{a,b}\right]|\, \mathcal{F}_{u,v}\right] = \mathbb{E}\left[Z|\, \mathcal{F}_{a\wedge u, b\wedge v}\right],$$

where $a \wedge u$ is the minimum between a and u. In that case, we say that the filtration $(\mathcal{F}_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^2}$ is commuting and we can extend this condition to the case $d > 2$ with the following definition.

Definition 4.1 (Commuting filtration). *A filtration $(\mathcal{F}_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ is commuting if for all integrable random variable Z and for all $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^d$,*

$$\mathbb{E}\left[\mathbb{E}\left[Z|\, \mathcal{F}_{\boldsymbol{i}}\right]|\, \mathcal{F}_{\boldsymbol{j}}\right] = \mathbb{E}\left[Z|\, \mathcal{F}_{\boldsymbol{i}\wedge\boldsymbol{j}}\right],$$

where $\boldsymbol{i} \wedge \boldsymbol{j}$ is the coordinate-wise minimum between $\boldsymbol{i}$ and $\boldsymbol{j}$.

Now that we have defined the notion of commuting filtrations, we can introduce the projection operator. We consider a commuting filtration $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$, then the projection operators are defined by

$$\mathcal{P}_{\boldsymbol{i}}(f) = \left[\prod_{j=1}^{d} \left(\mathbb{E}\left[\cdot\, |\mathcal{F}_{\boldsymbol{i}}\right] - \mathbb{E}\left[\cdot\, \middle|\mathcal{F}_{\boldsymbol{i}-\boldsymbol{e}_j}\right]\right)\right](f), \quad \boldsymbol{i} \in \mathbb{Z}^d,$$

where f is an integrable function and $\boldsymbol{e}_j$ is the multi-index whose components are all zero except for the j-th which equals to 1.

4.1.2. *A Projective Criterion*

In the rest of this work, we shall require some control over the dependency of the random variables involved and, more specifically, we will make use of Hannan's condition [34]. We start by making the necessary assumption that the filtration $(\mathcal{F}_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ given by (31) is commuting. Then we say that the field X given by (30) satisfy *Hannan's condition* if

$$\sum_{\boldsymbol{u}\geq\boldsymbol{0}} \|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{u}})\|_2 < \infty. \tag{32}$$

Processes satisfying Hannan's condition have been quite extensively studied and numerous central limit theorems as well as their functional counterparts have been obtained (see for example [13, 34, 40, 75, 79]). However, some open questions about these processes still exist and deserve further investigation.

4.2. **Quenched Theorems**

An interesting problem, with many practical applications, is to study limit theorems for processes conditioned to start from a fixed past trajectory. This problem is difficult, since the stationary processes started from a fixed past trajectory, or from a point, are no longer stationary. Furthermore, the validity of a limit theorem is not enough to assure that the convergence still holds when the process is not started from its equilibrium. This type of convergence is also known under the name of almost sure conditional limit theorem or quenched limit theorem.

With the notations previously defined in mind, we consider a version $\mathbb{P}^\omega$ of the regular conditional probability $\mathbb{P}\left(\cdot|\mathcal{F}_{\boldsymbol{0}}\right)(\omega)$. We say that the random field X satisfies a *quenched functional central limit theorem* as soon as for $\mathbb{P}$-almost all $\omega\in\Omega$, it holds that, for $S_{\boldsymbol{n}} := \sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{n}} X_{\boldsymbol{i}}$,

$$\left(\frac{S_{[\boldsymbol{nt}]}}{\sqrt{|\boldsymbol{n}|}}\right)_{\boldsymbol{t}\in[0,1]^d} \xrightarrow[\boldsymbol{n}\to\infty]{\text{law}} (W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d} \qquad \text{under } \mathbb{P}^\omega.$$

where $[\boldsymbol{nt}] := ([n_1t_1],\cdots,[n_dt_d])$, with $[x]$ being the integer part of x, $(W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d}$ is a Brownian sheet and $\boldsymbol{n}\to\infty$ is to be interpreted as the convergence of $\min\{n_1,\ldots,n_d\}$ to ∞. When that convergence in distribution only holds under the probability measure $\mathbb{P}$, we say that the central limit theorem is *annealed*.

Note that the existence of $\mathbb{P}^\omega$ is not a trivial question. Indeed, depending on the codomain of the random variables involved, it may or may not exist. In fact, Dieudonné [20] gave a counter example to the existence of a regular conditional probability. However, in the framework we are working with, the existence problem is solved as the codomain of the random variables is a Polish space.

4.3. **Quenched Central Limit Theorems over cubes**

Here we present the quenched functional central limit theorem over cubic regions of $\mathbb{Z}^d$. But first, we need to introduce a few more items of notation. For every $\boldsymbol{n}\in(\mathbb{N}^*)^d$, we define

$$\bar{S}_{\boldsymbol{n}} = S_{\boldsymbol{n}} - R_{\boldsymbol{n}} \quad \text{with} \quad R_{\boldsymbol{n}} = \sum_{i=1}^{d}(-1)^{i-1}\sum_{1\leq j_1<\cdots<j_i\leq d}\mathbb{E}\left[S_{\boldsymbol{n}}\middle|\,\mathcal{F}_{\boldsymbol{n}^{(j_1,\cdots,j_i)}}\right],$$

where $\boldsymbol{n}^{(j_1,\cdots,j_i)}$ is the multi-index obtained by replacing with 0 all the $j_1,\cdots,j_i$-th coordinates of the multi-index $\boldsymbol{n}$ and leaving the rest unchanged.

Under the framework mentioned above, we obtained some quenched functional central limit theorem over cubic regions.

Theorem 4.2. *Assume that* $(X_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ *is defined by* (30) *and that the filtration* $(\mathcal{F}_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ *given by* (31) *is commuting. Also assume that one of the transformations* $T_i, 1 \le i \le d$ *is ergodic and that in addition the condition* (32) *is satisfied. Then, for* $\mathbb{P}$*-almost all* $\omega \in \Omega$,

$$\left(\frac{1}{n^{d/2}}\bar{S}_{[n\boldsymbol{t}]}\right)_{\boldsymbol{t}\in[0,1]^d} \xrightarrow[n\to\infty]{\text{law}} (\sigma W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d} \quad \text{under } \mathbb{P}^{\omega},$$

where $\sigma^2 := \mathbb{E}\left[D_{\boldsymbol{0}}^2\right]$ *with* $D_{\boldsymbol{0}} = \sum_{\boldsymbol{i}\ge\boldsymbol{0}} \mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{i}})$, $(W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d}$ *is a standard Brownian sheet,* $[k\boldsymbol{t}] := ([kt_1], \dots, [kt_d])$ *for* $k \in \mathbb{Z}$ *and the convergence happens in the Skorokhod space* $D([0,1]^d)$ *endowed with the uniform topology. Moreover,* $\sigma^2 = \lim_{n\to\infty} \frac{\mathbb{E}[\bar{S}^2_{n,\dots,n}]}{n^d}$.

In Theorem 4.2, the random centering $R_{[n\boldsymbol{t}]}$ cannot be avoided without additional hypothesis. As a matter of fact, for $d = 1$, Volný and Woodroofe [76] constructed an example showing that the CLT for partial sums needs not be quenched. That being said, the following corollary gives a sufficient condition to get rid of the stochastic centering $R_{\boldsymbol{n}}$ in the previous theorem.

Corollary 4.3. *Assume that* $(X_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ *is defined by (30), that the filtration* $(\mathcal{F}_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ *defined by* (31) *is commuting and that one of the transformations* $T_i, 1 \le i \le d$ *is ergodic. If the following condition is satisfied:*

$$\sum_{\boldsymbol{u}\ge\boldsymbol{1}} \frac{\|\mathbb{E}\left[X_{\boldsymbol{u}}|\,\mathcal{F}_{\boldsymbol{1}}\right]\|_2}{|\boldsymbol{u}|^{\frac{1}{2}}} < \infty.$$

Then, for almost all $\omega \in \Omega$,

$$\left(\frac{1}{n^{d/2}}S_{[n\boldsymbol{t}]}\right)_{\boldsymbol{t}\in[0,1]^d} \xrightarrow[n\to\infty]{\text{law}} (\sigma W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d} \quad \text{under } \mathbb{P}^{\omega},$$

where σ^2 *is defined in Theorem 4.2,* $(W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d}$ *is a standard Brownian sheet and the convergence happens in the Skorokhod space* $D([0,1]^d)$ *endowed with the uniform topology.*

4.4. **Central Limit Theorems over rectangles**

In order to obtain a functional central limit theorem when we sum over rectangles, a stronger projective condition than (32) is necessary. Indeed, Peligrad and Volný [53] have given a counterexample to a quenched central limit theorem over rectangles for some stationary ortho-martingale under condition (32). This leads us to consider a projective condition in an Orlicz space associated to a specific Young function.

Following the work of Krasnosel'skii and Rutitskii [42], we define the Luxemburg norm associated to the Young function $\Phi_d : x \in [0,\infty) \mapsto x^2 \left(\log(1+x)\right)^{d-1} \in [0,\infty)$ as

$$\|f\|_{\Phi_d} = \inf\{t > 0 : \mathbb{E}\left[\Phi_d\left(|f|/t\right)\right] \le 1\}.$$

Using this new norm we can define a Hannan type condition which will be sufficient for the quenched central limit theorem to hold for sums computed over rectangular regions of $\mathbb{Z}^d$ instead of cubic ones.

Theorem 4.4. *Assume that* $(X_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ *is defined by* (30) *and that the filtration* $(\mathcal{F}_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ *given by* (31) *is commuting. Also assume that one of the transformations* $T_i, 1 \le i \le d$ *is ergodic and that in addition*

$$\sum_{\boldsymbol{u}\ge\boldsymbol{0}} \|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{u}})\|_{\Phi_d} < \infty. \tag{33}$$

Then, for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$\left(\frac{1}{\sqrt{|\boldsymbol{n}|}}\bar{S}_{[\boldsymbol{tn}]}\right)_{\boldsymbol{t}\in[0,1]^d} \xrightarrow[\boldsymbol{n}\to\infty]{\text{law}} (\sigma W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d} \quad \textit{under } \mathbb{P}^\omega,$$

where $[\boldsymbol{tn}] := ([t_1 n_1], \ldots, [t_d n_d])$, σ^2 is defined in Theorem 4.2, $(W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d}$ is a standard Brownian sheet and the convergence happens in the Skorokhod space $D([0,1]^d)$. In addition, $\sigma^2 = \lim_{\boldsymbol{n}\to\infty} \frac{\mathbb{E}[\bar{S}_{\boldsymbol{n}}^2]}{|\boldsymbol{n}|}$.

In the same manner as before, we give a sufficient condition to get rid of the random centering in Theorem 4.4.

Corollary 4.5. *Assume that $(X_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ is defined by (30), that the filtration $(\mathcal{F}_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ given by (31) is commuting and that one of the transformations $T_i, 1 \le i \le d$ is ergodic. If the following condition is satisfied:*

$$\sum_{\boldsymbol{u}\ge\boldsymbol{1}} \frac{\|\mathbb{E}\left[X_{\boldsymbol{u}} | \mathcal{F}_{\boldsymbol{1}}\right]\|_{\Phi_d}}{\Phi_d^{-1}(|\boldsymbol{u}|)} < \infty.$$

Then, for almost all $\omega \in \Omega$,

$$\left(\frac{1}{\sqrt{|\boldsymbol{n}|}}S_{[\boldsymbol{tn}]}\right)_{\boldsymbol{t}\in[0,1]^d} \xrightarrow[\boldsymbol{n}\to\infty]{\text{law}} (\sigma W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d} \quad \textit{under } \mathbb{P}^\omega,$$

where σ^2 is defined in Theorem 4.4, $(W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d}$ is a Brownian sheet and the convergence happens in the Skorokhod space $D([0,1]^d)$.

4.5. Examples

In this last section, we give two examples of applications of these results. The first one concerns Hölder functions of linear fields. Such processes have been studied in [16] for example and an annealed (i.e. non-quenched) central limit theorem under Hannan's condition has been derived. In the first example, we provide sufficient conditions for that functional central limit theorem to be quenched. Notice that, this example also covers the particular case of standard linear fields as well.

Example 4.6. *(Hölder function of a linear field) Let $(\xi_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ be a random field of independent, identically distributed, and centered random variables such that $\mathbb{E}\left[|\xi_{\boldsymbol{0}}|^2 \left(\log(1+|\xi_{\boldsymbol{0}}|)\right)^{d-1}\right] < \infty$. Consider an Hölder continuous function $f : \mathbb{R} \to \mathbb{R}$ of order $\alpha \in (0,1]$. For $\boldsymbol{k} \ge \boldsymbol{0}$, define*

$$X_{\boldsymbol{k}} = f\left(\sum_{\boldsymbol{j}\ge\boldsymbol{0}} a_{\boldsymbol{j}}\xi_{\boldsymbol{k}-\boldsymbol{j}}\right) - \mathbb{E}\left[f\left(\sum_{\boldsymbol{j}\ge\boldsymbol{0}} a_{\boldsymbol{j}}\xi_{\boldsymbol{k}-\boldsymbol{j}}\right)\right]$$

where $a_{\boldsymbol{u}}$ are real coefficients such that $\sum_{\boldsymbol{u}\ge\boldsymbol{0}} a_{\boldsymbol{u}}^2 < \infty$. If the coefficients $a_{\boldsymbol{u}}$ also satisfy one of the following conditions

$$\begin{cases} \sum_{\boldsymbol{k}\ge\boldsymbol{1}} \frac{1}{{\Phi_d}^{-1}(|\boldsymbol{k}|)} \sum_{\boldsymbol{j}\ge\boldsymbol{k}-\boldsymbol{1}} |a_{\boldsymbol{j}}|^\alpha < \infty \\ \sum_{\boldsymbol{k}\ge\boldsymbol{1}} \frac{1}{{\Phi_d}^{-1}(|\boldsymbol{k}|)} \left(\sum_{\boldsymbol{j}\ge\boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}^2\right)^{\frac{\alpha}{2}} < \infty \quad \textit{and} \quad \frac{1}{2} \le \alpha \le 1, \end{cases} \tag{34}$$

then the quenched functional central limit theorem in Corollary 4.5 holds.

Before moving on to the second example, we can give a more tractable sufficient condition for (34) to hold. Namely (34) will hold if either of the following conditions is satisfied:

$$\begin{cases} \sum_{\boldsymbol{k}\geq\boldsymbol{1}} \frac{(\log(|\boldsymbol{k}|))^{\frac{d-1}{2}}}{|\boldsymbol{k}|^{\frac{1}{2}}} \sum_{\boldsymbol{j}\geq\boldsymbol{k}-\boldsymbol{1}} |a_{\boldsymbol{j}}|^{\alpha} < \infty \\ \sum_{\boldsymbol{k}\geq\boldsymbol{1}} \frac{(\log(|\boldsymbol{k}|))^{\frac{d-1}{2}}}{|\boldsymbol{k}|^{\frac{1}{2}}} \left(\sum_{\boldsymbol{j}\geq\boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}^2\right)^{\frac{\alpha}{2}} < \infty \quad \text{and} \quad \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

In the second example, we focus on a well known example of a non-linear random field. Indeed, Volterra fields play an important role in the fields of financial mathematics as well as economics and in [79], the authors already established a quenched central limit theorems for this class of processes.

Example 4.7. *(Volterra field) Let $(\xi_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ be a random field of independent, identically distributed, and centered random variables satisfying $\mathbb{E}\left[|\xi_{\boldsymbol{0}}|^2 (\log(1+|\xi_{\boldsymbol{0}}|))^{d-1}\right] < \infty$. For $\boldsymbol{k} \geq \boldsymbol{0}$, define*

$$X_{\boldsymbol{k}} = \sum_{(\boldsymbol{u},\boldsymbol{v})\geq(\boldsymbol{0},\boldsymbol{0})} a_{\boldsymbol{u},\boldsymbol{v}} \xi_{\boldsymbol{k}-\boldsymbol{u}} \xi_{\boldsymbol{k}-\boldsymbol{v}}$$

where $a_{\boldsymbol{u},\boldsymbol{v}}$ are real coefficients with $a_{\boldsymbol{u},\boldsymbol{u}} = 0$ and $\sum_{\boldsymbol{u},\boldsymbol{v}\geq\boldsymbol{0}} a_{\boldsymbol{u},\boldsymbol{v}}^2 < \infty$. In addition, assume that

$$\sum_{\boldsymbol{k}\geq\boldsymbol{1}} \frac{1}{{\Phi_d}^{-1}(|\boldsymbol{k}|)} \Bigg(\sum_{\substack{(\boldsymbol{u},\boldsymbol{v})\geq(\boldsymbol{k}-\boldsymbol{1},\boldsymbol{k}-\boldsymbol{1}) \\ \boldsymbol{u}\neq\boldsymbol{v}}} a_{\boldsymbol{u},\boldsymbol{v}}^2 \Bigg)^{1/2} < \infty, \tag{35}$$

Then the quenched functional CLT in Theorem 4.5 holds.

In the same manner as before, we notice that (35) is satisfied whenever

$$\sum_{\boldsymbol{k}\geq\boldsymbol{1}} \frac{(\log(|\boldsymbol{k}|))^{\frac{d-1}{2}}}{|\boldsymbol{k}|^{\frac{1}{2}}} \Bigg(\sum_{\substack{(\boldsymbol{u},\boldsymbol{v})\geq(\boldsymbol{k}-\boldsymbol{1},\boldsymbol{k}-\boldsymbol{1}) \\ \boldsymbol{u}\neq\boldsymbol{v}}} a_{\boldsymbol{u},\boldsymbol{v}}^2 \Bigg)^{1/2} < \infty.$$

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