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# PRICING WITHOUT MARTINGALE MEASURE

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**Abstract.** For several decades, the martingale measures have played a major role in financial asset pricing theory. In this paper, we propose an approach based on the conditional support of the asset price increments that avoids the technical diffilcuties arising from the risk-neutral probability measures. This is illustrated by a numerical implementation on real data from the French CAC 40 index.

Keywords and phrases Financial market models; CAC 40; Super-hedging prices; AIP condition; Conditional support; Essential supremum.

## Introduction

The problem of giving a fair price to a financial asset $G$ is central in the economic and financial theory. A selling price should be an amount which is enough to initiate a hedging strategy for $G$, i.e., a strategy whose value at maturity is always above $G$. It seems also natural to ask for the infimum of such amount. This is the so called super-replication (or super-hedging) price and it has been introduced in the binomial setup for transaction costs by [3]. Characterizing and computing the super-replication price has become one of the central issue in the mathematical finance theory. The super-hedging theorem also called dual formulation shows that the super-replication price can be computed using martingale measures (see, [10], [8] and the references therein). We argue that, in a discrete time model without frictions, the super-replication price can be computed directly from the conditional support of the distribution of asset price increments. This is a well-known approach to characterize no-arbitrage conditions, see, [9]. Recently, in [5], [12] and [7], it has been shown that it is also fruitful for the computation of the super-replication prices even with transaction costs. In this paper, the super-replication prices are expressed using Fenchel conjugate and bi-conjugate without postulating any no-arbitrage condition. The finiteness of the super-hedging prices leads to a weak no-arbitrage condition called Absence of Instantaneous Profit (AIP).

This condition is equivalent to the fact that 0 belongs to the convex hull of the conditional support of the assets price increments. Under AIP condition, the one-step infimum super-hedging cost is the concave envelop of the payoff relatively to the convex envelop of the conditional support.

In this paper we propose, in the multiple-period framework, a recursive scheme for the computation of the super-hedging prices of an European option with convex payoff. Note that we obtain the same computation scheme as in [4] and [6], but under the weaker assumption of AIP and without using martingale measures.

So far, there has been little attention about the experimental implementation of this algorithm. We calibrate historical data of the French index CAC 40 and implement our super-hedging strategy for a call option. Our procedure is, somehow, model free and based only on statistical estimations. We actually

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observe that the same strategy, based on the implied volatility rather than the conditional support, provides worse results but at a lower price.

The paper is organized as follows. In Section 1, we present our algorithm and in Section 2, the numerical experiments. The proof of our algorithm is postponed to the appendix.

## 1. Theoritical algorithm

We present our framework and notations. Let $(\Omega, (\mathcal{F}_t)_{t\in\{0,\dots,T\}}\mathcal{F}_T, P)$ be a complete filtered probability space, where $T$ is the time horizon. For any $\sigma$-algebra $\mathcal{H}$, we denote by $L^0(\mathbb{R}, \mathcal{H})$ the set of $\mathcal{H}$-measurable and $\mathbb{R}$-valued random variables. We consider a non-negative process $S := \{S_t,\ t \in \{0,\dots,T\}\}$ such that $S_t \in L^0(\mathbb{R}, \mathcal{F}_t)$ for all $t \in \{0,\dots,T\}$. The vector $S_t$ represents the price at time $t$ of the risky asset in the financial market of consideration. Trading strategies are given by processes $\theta := \{\theta_t,\ t \in \{0,\dots,T-1\}\}$ such that $\theta_t \in L^0(\mathbb{R}, \mathcal{F}_t)$ for all $t \in \{0,\dots,T-1\}$. The vector $\theta_t$ represents the investor's holding in the risky asset between times $t$ and $t+1$. We assume that trading is self-financing and that the riskless asset's price is a constant equal to 1. The value at time $t$ of a portfolio $\theta$ starting from initial capital $x \in \mathbb{R}$ is then given by

$$V_t^{x,\theta} = x + \sum_{u=1}^{t} \theta_{u-1}\Delta S_u,$$

where $\Delta S_u = S_u - S_{u-1}$ for $u \geq 1$.

We propose an algorithm for the computation of the super-replication price of a European contingent claim $h(S_T)$, where $h$ is convex. This algorithm have already be obtained in Proposition 2.2 of [6], under the NA condition, using the dual representation of the superreplication price. This algorithm can be obtained under a weaker assumption, called Absence of Instantaneous Profit. This condition asserts that it is not possible to super-replicate the contingent claim 0 at a negative super-hedging price. This is also the minimal requirement for the super-hedging cost not to be equal to $-\infty$ which makes pricing computation feasible. The AIP condition is very weak : If the initial information is trivial, a one period instantaneous profit is a strategy starting from 0 and leading to a terminal wealth larger than some strictly positive constant.

We now define the super-hedging prices, the super-hedging cost and the AIP condition. For every $t \in \{0,\dots,T\}$, the set $\mathcal{R}_t^T$ of all claims that can be super-replicated from the zero initial endowment at time $t$ is defined by

$$\mathcal{R}_t^T := \left\{ \sum_{u=t+1}^{T} \theta_{u-1}\Delta S_u - \epsilon_T^+,\ \theta_{u-1} \in L^0(\mathbb{R}, \mathcal{F}_{u-1}),\ \epsilon_T^+ \in L^0(\mathbb{R}_+, \mathcal{F}_T) \right\}. \tag{1.1}$$

The set of super-hedging prices and the infimum super-hedging cost of contingent claim $g_T \in L^0(\mathbb{R}, \mathcal{F}_T)$ at time $t$ are given for all $t \in \{0,\dots,T\}$, by

$$\begin{aligned} \mathcal{P}_{T,T}(g_T) &= \{g_T\} + L^0(\mathbb{R}_+, \mathcal{F}_T), \text{ and } \pi_{T,T}(g_T) = g_T, \\ \mathcal{P}_{t,T}(g_T) &= \{x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists R \in \mathcal{R}_t^T, x_t + R = g_T \text{ a.s.}\}, \\ \pi_{t,T}(g_T) &= \operatorname{ess\,inf} \mathcal{P}_{t,T}(g_T). \end{aligned} \tag{1.2}$$

The infimum super-hedging cost is not necessarily a price if $\pi_{t,T}(g_T) \notin \mathcal{P}_{t,T}(g_T)$, i.e. when $\mathcal{P}_{t,T}(g_T)$ is not closed.

**Definition 1.1.** *The AIP condition holds true if for all, $t \in \{0,\dots,T\}$,*

$$\mathcal{P}_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}.$$

The AIP condition can be characterized using the notion of conditional support and of conditional essential supremum (and infimum). Recall that if $X_{t+1} \in L^0(\mathbb{R}, \mathcal{F}_{t+1})$, $\operatorname{supp}_{\mathcal{F}_t} X_{t+1}$, the conditional support of $X_{t+1}$ with respect to $\mathcal{F}_t$, is a random set $\Omega \twoheadrightarrow \mathbb{R}$ defined by

$$\operatorname{supp}_{\mathcal{F}_t} X_{t+1}(\omega) := \bigcap \left\{ A \subset \mathbb{R}^d, \text{ closed}, \ P(X_{t+1} \in A|\mathcal{F}_t)(\omega) = 1 \right\}, \tag{1.3}$$

when $P(X_{t+1} \in A|\mathcal{F}_t)(\omega)$ is a regular version of the conditional law of $X_{t+1}$ knowing $\mathcal{F}_t$. The conditional essential supremum allows to incorporate measurability in the definition of the essential supremum (see, [2] and [11]):

**Proposition 1.2.** *Let $\mathcal{F}_t$ and $\mathcal{F}_{t+1}$ be complete $\sigma$-algebras such that $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ and let $\Gamma = (\gamma_i)_{i \in I}$ be a family of real-valued $\mathcal{F}_{t+1}$-measurable random variables. There exists a unique $\mathcal{F}_t$-measurable random variable $\gamma_t \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{F}_t)$ denoted by $\operatorname{ess\,sup}_{\mathcal{F}_t} \Gamma$ which satisfies the following properties*

*(1) For every $i \in I$, $\gamma_t \geq \gamma_i$ a.s.*
*(2) If $\zeta_t \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{F}_t)$ satisfies $\zeta_t \geq \gamma_i$ a.s. $\forall i \in I$, then $\zeta_t \geq \gamma_t$ a.s.*

The following proposition gives a characterization of AIP, see Proposition 3.4 in [5].

**Proposition 1.3.** *The following assertions are equivalent (here $d = 1$).*

*(1) AIP holds true.*
*(2) $0 \in \operatorname{convsupp}_{\mathcal{F}_t} \Delta S_{t+1}$ a.s. or $S_t \in [\operatorname{ess\,inf}_{\mathcal{F}_t} S_{t+1}, \operatorname{ess\,sup}_{\mathcal{F}_t} S_{t+1}] \cap \mathbb{R}$ a.s. for all $t \leq T-1$.*
*(3) $\pi_{t,T}(0) = 0$ a.s. for all $t \in \{0, \dots, T-1\}$,*

*where $\operatorname{convsupp}_{\mathcal{F}_t} \Delta S_{t+1}$ is the convex envelop of $\operatorname{supp}_{\mathcal{F}_t} \Delta S_{t+1}$ i.e., the smallest convex set that contains $\operatorname{supp}_{\mathcal{F}_t} \Delta S_{t+1}$.*

We are now in position to present our pricing algorithm.

**Proposition 1.4.** *Suppose that $\operatorname{ess\,inf}_{\mathcal{F}_{t-1}} S_t = k^d_{t-1} S_{t-1}$ a.s. and $\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} S_t = k^u_{t-1} S_{t-1}$ a.s. where $(k^d_{t-1})_{t \in \{1,\dots,T\}}$, $(k^u_{t-1})_{t \in \{1,\dots,T\}}$ and $S_0$ are deterministic non-negative numbers. Then, the AIP condition is satisfied if and only if $k^d_{t-1} \in [0,1]$ and $k^u_{t-1} \in [1, +\infty]$ for all $t \in \{1, \dots, T\}$.*

*Suppose that the AIP condition holds. Let $h : \mathbb{R} \to \mathbb{R}$ be a non-negative convex function with $\operatorname{dom} h = \mathbb{R}$ such that $\lim_{z \to +\infty} \frac{h(z)}{z} \in [0, \infty)$. Then, the infimum super-hedging cost of the European contingent claim $h(S_T)$ is a price and it is given by $\pi_{t,T}(h) = h(t, S_t) \in \mathcal{P}_{t,T}(h(S_T))$a.s. where*

$$\begin{aligned} h(T, x) &= h(x) \\ h(t-1, x) &= \lambda_{t-1} h(t, k^d_{t-1} x) + (1 - \lambda_{t-1}) h\left(t, k^u_{t-1} x\right), \end{aligned} \tag{1.4}$$

*and $\lambda_{t-1} = \frac{k^u_{t-1} - 1}{k^u_{t-1} - k^d_{t-1}} \in [0,1]$ and $1 - \lambda_{t-1} = \frac{1 - k^d_{t-1}}{k^u_{t-1} - k^d_{t-1}} \in [0,1]$, with the following conventions. When $k^d_{t-1} = k^u_{t-1} = 1$ or $S_{t-1} = 0$, $\lambda_{t-1} = \frac{0}{0} = 0$ and $1 - \lambda_{t-1} = 1$ and when, $k^d_{t-1} < k^u_{t-1} = \infty$, $\lambda_{t-1} = \frac{\infty}{\infty} = 1$,*

$$(1 - \lambda_{t-1}) h(t, (+\infty) x) = (1 - k^d_{t-1}) x \frac{h(t, (+\infty x))}{(+\infty x)} = (1 - k^d_{t-1}) x \lim_{z \to +\infty} \frac{h(z)}{z}. \tag{1.5}$$

*Moreover, for every $t \in \{1, \dots, T\}$, $\lim_{z \to +\infty} \frac{h(z)}{z} = \lim_{z \to +\infty} \frac{h(t,z)}{z}$ and $h(\cdot, x)$ is non-increasing for all $x \geq 0$. The strategy associated to the infimum super-hedging price is given by:*

$$\theta^*_t = \frac{h(t+1, k^u_t S_t) - h(t+1, k^d_t S_t)}{(k^u_t - k^d_t) S_t}. \tag{1.6}$$

The infimum super-hedging cost of the European contingent claim $h(S_T)$ in our model is a price, precisely the same than the price we get in a binomial model where $S_t$ belongs to $\{k^d_{t-1} S_{t-1}, k^u_{t-1} S_{t-1}\}$ a.s., $t \in \{1, \dots, T\}$.

## 2. Numerical experiments

We suppose that the discrete dates are given by $t^n_i = \frac{iT}{n}$, $i \in \{0, \dots, n\}$ where $n \geq 1$ and that

$$\operatorname{ess\,inf}_{\mathcal{F}_{t^n_i}} S_{t^n_{i+1}} = k^d_{t^n_i} S_{t^n_i} \text{ a.s. and } \operatorname{ess\,sup}_{\mathcal{F}_{t^n_i}} S_{t^n_{i+1}} = k^u_{t^n_i} S_{t^n_i} \text{ a.s.}$$

Below we present different approach for the determination of $k^u_{t^n_i}$ and $k^d_{t^n_i}$. By Proposition 1.4, the infimum super-hedging cost of the European Call option $(S_T - K)^+$ is given by $h^n\left(t^n_i, S_{t^n_i}\right)$, where $h^n$ is defined by (1.4) with the terminal condition $h^n(T,x) = g(x) = (x-K)^+$. The associated super-hedging strategies $(\theta^*_{t^n_i})_{i\in\{0,\dots,n-1\}}$ are given by (1.6). We denote by $V_T$ the terminal value of our strategy starting from the super-hedging cost $V_0 = \pi_{0,T} = h(0,S_0)$, i.e., $V_T = V_0 + \sum_{i=0}^{n-1} \theta^*_{t^n_i} \Delta S_{t^n_{i+1}}$. Then, the super-hedging error is the difference between the terminal value of our strategy and the call option, i.e. $\varepsilon_T = V_T - (S_T - K)^+$. We expect that $\varepsilon_T \geq 0$.

## 2.1. **Implied volatility**

In this subsection, we assume that for $i \in \{0,\dots,n-1\}$

$$k^u_{t^n_i} = 1 + \sigma_{t^n_i}\sqrt{\Delta t^n_{i+1}} \text{ and } k^d_{t^n_i} = 1 - \sigma_{t^n_i}\sqrt{\Delta t^n_{i+1}} \geq 0, \tag{2.7}$$

where $t \mapsto \sigma_t$ is a positive Lipschitz-continuous function on $[0,T]$. Our aim is to calibrate $\sigma$ from the observed European call option prices. We extend the function $h^n$ on $[0,T]$ in such a way that $h^n$ is constant on each interval $[t^n_i, t^n_{i+1}[$, $i \in \{0,\dots,n\}$. Such a scheme is proposed by Milstein [14], where a convergence theorem is proved when the terminal condition, i.e., the payoff function is smooth. Precisely, the sequence of functions $(h^n(t,x))_{n\geq 1}$ converges uniformly to $h(t,x)$, solution of the Black and Scholes formula with time-dependent volatility:

$$\partial_t h(t,x) + \sigma_t^2 \frac{x^2}{2}\partial_{xx} h(t,x) = 0, \quad h(T,x) = g(x). \tag{2.8}$$

In [14], it is supposed that the successive derivatives of the solution of the P.D.E. solution $h$ are uniformly bounded. This is not the case for the Call payoff function $g$. On the contrary, the successive derivatives of the solution of the P.D.E. explode at the horizon date, see [13]. In [1], it is proven that the uniform convergence still holds when the payoff function is not smooth provided that the successive derivatives of the solution of the P.D.E. do not explode *too much*.

Supposing that $\Delta t^n_i$ is closed to 0, we make the observed prices of the Call option coincided to the theoretical limit prices $h(t, S_t)$ at any instant $t$ given by (2.8). For several strikes, matching the observed prices to the theoretical ones allows to deduce the associated *implied* volatility $t \mapsto \sigma_t$.

The data set is composed of historical values of the french index CAC 40 and European call option prices of maturity 3 months from the 23rd of October 2017 to the 19th of January 2018. The values of $S$ are distributed as in Figure 1.

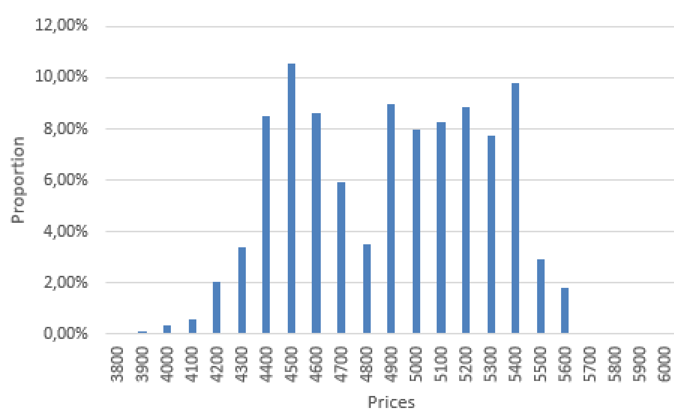



FIGURE 1. Distribution of the observed prices.

Now, we want to evaluate the proportion of observations satisfying (2.9), which is the direct consequence of the assumptions made on the multipliers $k^u_{t^n_{i-1}}$ and $k^d_{t^n_{i-1}}$ in (2.7):

$$\left|\frac{S_{t^n_{i+1}}}{S^n_{t_i}} - 1\right| \le \sigma_{t^n_i}\sqrt{\Delta t^n_{i+1}}, \text{a.s.} \tag{2.9}$$

The results are satisfactory for strikes lower that 5000, see Fig. 2. Note that, when the strikes are too large with respect to the current price $S$, price observations are less available for the calibration, see Figure 1. This could explain the degradation of our results.

| Strike | 4800 | 4900 | 5000 | 5100 | 5200 | 5300 |
|---|---|---|---|---|---|---|
| Ratio | 96,7% | 95,1% | 95,1% | 88,5% | 86,9% | 80,3% |
| Strike | 5400 | 5500 | 5600 | 5700 | 5800 | 5900 |
| Ratio | 70,5% | 78,7% | 75,4% | 77,0% | 73,8% | 75,4% |

FIGURE 2

## 2.2. **Direct calibration**

We propose another approach where the coefficients $k^u$ and $k^d$ of Proposition 1.4 are calibrated directly on the value of the CAC 40 and the quality of the model is measured through the super-hedging error. This allows to obtain coefficients $k^u$ and $k^d$ which are not strike dependent as before, i.e. are model free.

The data set is composed of historical daily closing values of the French index CAC 40 from the 5th of January 2015 to the 12th of March 2018. The chosen interval $[0, T]$ corresponds to one week of 5 working days and $n = 4$. We propose two estimators for $k^d_{t^4_{i-1}}$ and $k^u_{t^4_{i-1}}$. The first approach is called symmetric : $\sigma_{t^4_i}$ is estimated as an upper bound in (2.9) and then the $k^d_{t^4_i}$ and $k^u_{t^4_i}$ are given as in (2.7).

$$\begin{aligned} \sigma_{t^4_i} &= \overline{\max}\left(\left|\frac{S_{t^4_{i+1}}}{S^4_{t_i}} - 1\right| / \sqrt{\Delta t^4_{i+1}},\right) \\ k^u_{t^4_i} &= 1 + \sigma_{t^4_i}\sqrt{\Delta t^4_{i+1}} \text{ and } k^d_{t^4_i} = 1 - \sigma_{t^4_i}\sqrt{\Delta t^4_{i+1}}. \end{aligned} \tag{2.10}$$

The asymmetric approach is more intuitive as the empirical minimum and maximum are considered.

$$k^d_{t^4_i} = \overline{\min}\frac{S_{t^4_{i+1}}}{S_{t^4_i}} \text{ and } k^u_{t^4_i} = \overline{\max}\frac{S_{t^4_{i+1}}}{S_{t^4_i}}.$$

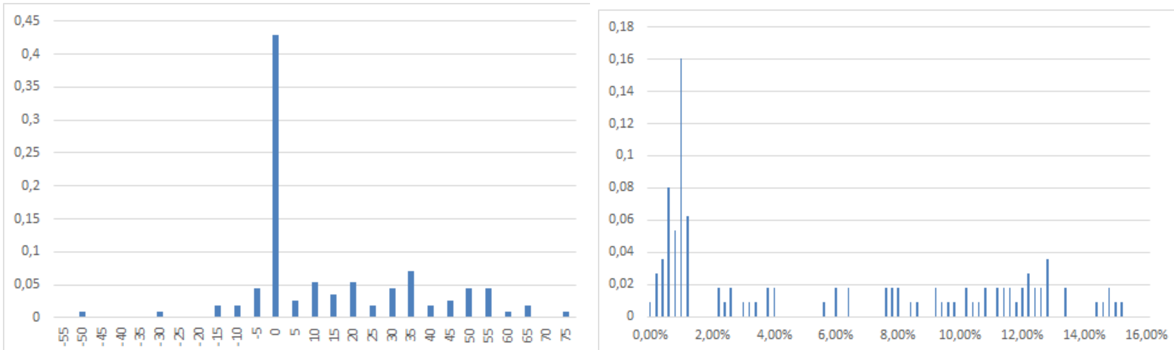
Note that $\overline{\max}$ (resp. $\overline{\min}$), the empirical maximum (resp. min), is taken over a one year sliding sample window of 52 weeks. We estimate the parameters on 52 weeks and we implement our hedging strategy on the fifty third one. We then repeat the procedure by sliding the window of one week.

We study below the super-hedging error $\varepsilon_T = V_T - (S_T - K)^+$ for different strikes. In the symmetric case, we present in Figures 3a and 3b the distribution of the super-hedging error $\varepsilon_T$ and of $V_0/S_0$ for $K = 4700$.

The asymmetric case is presented in Figures 4a and 4b still for $K = 4700$.

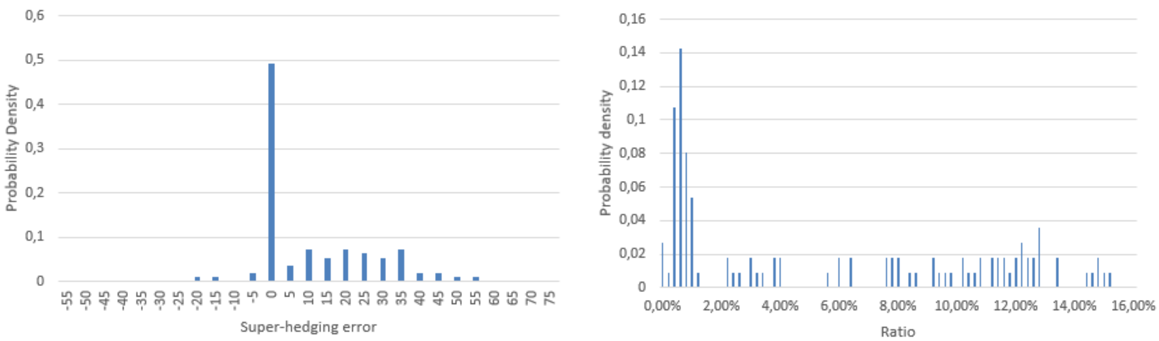
The graphs in the Symmetric and Asymmetric cases look similar. So, we now compare both methods in the table below.

In the symmetric (resp. asymmetric) case the empirical average of the error $\varepsilon_T$ is 12.76 (resp. 9.47) and its standard deviation is 21.65 (resp. 14.20). This result is rather satisfactory in comparison to the large value of the empirical mean of $S_0$ equal to 4844.93. We observe that $E(S_T - K)^+ \simeq 278.73$. This empirically confirms the efficiency of our suggested method. The empirical probability of $\{\varepsilon_T < 0\}$ is equal to 14.29% (resp. 8.04%) but the Value at Risk at 95 % is $-10.33$ (resp. -1.81) which shows that our strategy is conservative.




(A) Distribution of $\varepsilon_T = V_T - (S_T - K)^+$.   (B) Distribution of the ratio $V_0/S_0$ .

FIGURE 3. Symmetric case for $K = 4700$.




(A) Distribution of $\varepsilon_T = V_T - (S_T - K)^+$.   (B) Distribution of the ratio $V_0/S_0$.

FIGURE 4. Asymmetric case with $K = 4700$.

| | Mean of $V_0/S_0$ | Variance of $V_0/S_0$ | Mean of $\varepsilon_T$ | Variance of $\varepsilon_T$ | $P(\varepsilon_T < 0)$ | VaR 95 % |
|---|---|---|---|---|---|---|
| Symmetric | 5.61% | 5.14 % | 12.76 | 21.65 | 14.29% | -10.33 |
| Asymmetric | 5.52% | 5.22% | 9.47 | 14.20 | 8.04% | -1.81 |

FIGURE 5. Comparison of the two methods of estimation for $K = 4700$. The mean of $S_0$ is 4844,93 and the mean of $(S_0 - K)^+$ is 278.73.

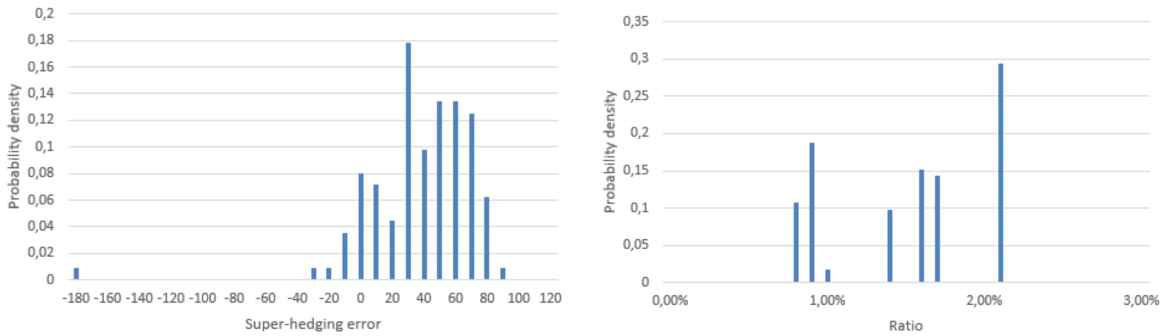
Now, we compare the cost of our strategy to $S_0$, which is the theoretical super-hedging price in some incomplete markets (this is for example the case when $k^d = 0$ and $k^u = \infty$, in particular when the dynamics of $S$ is modeled by a (discrete) geometric Brownian motion, see [4]). The empirical average of $V_0/S_0$ is 5.61% (resp. 5.52%) and its standard deviation is 5.14% (resp. 5.22%). This means that $V_0$ is much smaller than $S_0$.

Note that the huge loss (50 in the symmetric case) is explained by the so-called *black friday* week that occurs the 24th of June 2016. Large falls of risky assets were observed in European markets, mainly explained by the Brexit vote. In particular, the CAC 40 felt from 4340 to 4106, with a loss of $-8\%$ on Friday. If we want to prevent such a risk which is not observable in the data, it suffices to make the model more robust by reducing the coefficients $k^d$ and increasing the coefficients $k^u$. The choice of the new coefficients is clearly subjective and increases the super-hedging price.

We also present the “at the money” case $K = S_0$, see Fig. 6.

We also provide the comparison between the symmetric and asymmetric cases.

We see that the results are better than for $K = 4700$: $V_0$ is small with respect to $S_0$ and the probability of loss is small as well.




(A) Distribution of $\varepsilon_T = V_T - (S_T - K)^+$.

(B) Distribution of the ratio $V_0/S_0$.

FIGURE 6. Asymmetric case with $K = S_0$ .

| | Mean of $V_0/S_0$ | Variance of $V_0/S_0$ | Mean of $\varepsilon_T$ | Variance of $\varepsilon_T$ | $P(\varepsilon_T < 0)$ | VaR 95 % |
|---|---|---|---|---|---|---|
| Symmetric | 1.51% | 0.47 % | 35.69 | 34.11 | 9.82 % | -11.41% |
| Asymmetric | 1.47% | 0.49% | 33.37 | 32.78 | 12.50% | -9.29 |

FIGURE 7. Comparison of the two methods of estimation for $K = S_0$.

## 2.3. **Comparison between implied volatility and direct calibration**

We finish with the comparison of the results of Section 2.2 in the symmetric case, i.e., when $\sigma_{t_i}$ is given by (2.10), to the ones when $\sigma_{t_i}$ is the implied volatility. The comparison is made for $K = 5400$. The implementation of the classical approach through implied volatility gives worse results as we may expect. Indeed, as observed in Section 2.1, the implied volatility does not capture the coefficients as the empirical observations do. Therefore, from the implied volatility, we get larger coefficients $k^d$ and smaller coefficients $k^u$. Thus, the minimal super-hedging prices are smaller, as observed in Figure 8, but the super-hedging error is negative in most of the cases, see Figure 9. Our approach is definitively more conservative.

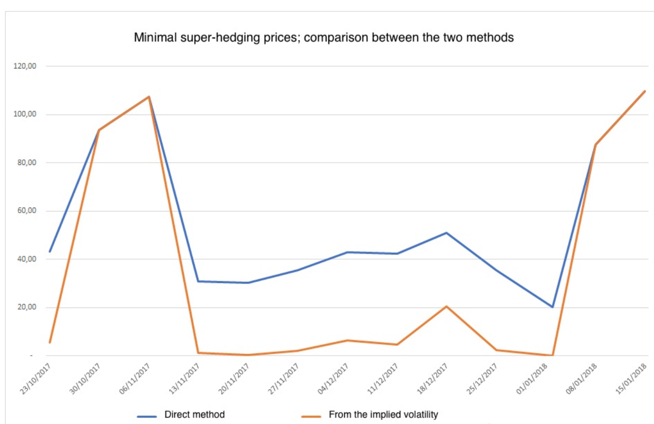



FIGURE 8. Minimum Super-hedging prices $K = 5400$.

# 3. APPENDIX

*Proof of Proposition 1.4* The conditions $k^d_{t-1} \in [0, 1]$ and $k^u_{t-1} \in [1, +\infty]$ for all $t \in \{1, \ldots, T\}$ are equivalent to the AIP condition (see Proposition 1.3). Let $M = \frac{h(\infty)}{\infty}$ and $M_t = \lim_{z\to+\infty} \frac{h(t,z)}{z}$. We prove the second statement. Assume that AIP holds true. We establish (i) the recursive formulation $\pi_{t,T}(h(S_T)) = h(t, S_t)$

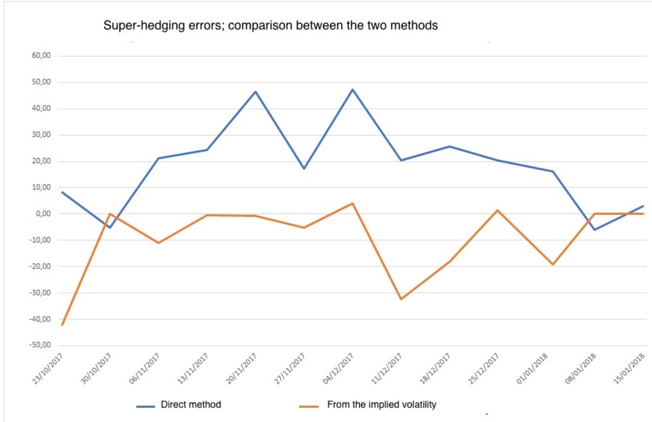



FIGURE 9. Super-heding errors for $K = 5400$ expected to be non negative.

given by (1.4), (ii) $h(t,\cdot) \geq h(t+1,\cdot)$ and (iii) $M_t = M_{t+1}$. The case $t = T$ is immediate. As $h : \mathbb{R} \to \mathbb{R}$ is a convex function with $\mathrm{dom}\,\mathrm{h} = \mathbb{R}$, $h$ is clearly a $\mathcal{F}_{T-1}$-normal integrand, we can apply Corollary 2.23 in [5] and we get that a.s.

$$
\begin{aligned}
\pi_{T-1,T}(h(S_T)) &= \inf\{\theta S_{T-1} + \beta,\ \theta \in \mathbb{R},\ \beta \in \mathbb{R},\ \theta x + \beta \geq h(x),\ \forall x \in \mathrm{supp}_{\mathcal{F}_{T-1}} S_T\} \\
&= \theta^*_{T-1} S_{T-1} + \beta^* = h(\operatorname{ess\,inf}_{\mathcal{F}_{T-1}} S_T) + \theta^*_{T-1}\left(S_{T-1} - \operatorname{ess\,inf}_{\mathcal{F}_{T-1}} S_T\right) \\
&= h(k^d_{T-1} S_{T-1}) + \theta^*_{T-1}\left(S_{T-1} - k^d_{T-1} S_{T-1}\right), \\
\theta^*_{T-1} &= \frac{h(k^u_{T-1} S_{T-1}) - h(k^d_{T-1} S_{T-1})}{k^u_{T-1} S_{T-1} - k^d_{T-1} S_{T-1}},
\end{aligned}
$$

with the conventions $\theta^*_{T-1} = \frac{0}{0} = 0$ if either $S_{T-1} = 0$ or $k^u_{T-1} = k^d_{T-1} = 1$ and $\theta^*_{T-1} = \frac{h(\infty)}{\infty} = M$ if $k^d_{T-1} < k^u_{T-1} = +\infty$. Moreover, we obtain that

$$\pi_{T-1,T}(h(S_T)) + \theta^*_{T-1}\Delta S_T \geq h(S_T) \text{ a.s., i.e., } \pi_{T-1,T}(h(S_T)) \in \mathcal{P}_{T-1,T}(h(S_T)).$$

For any $t \in \{0,\ldots,T-1\}$ and any $g_{t+1} \in L^0(\mathbb{R},\mathcal{F}_{t+1})$, we introduce the one-step super-hedging prices and cost :

$$
\begin{aligned}
\mathcal{P}_{t,t+1}(g_{t+1}) &= \left\{x_t \in L^0(\mathbb{R},\mathcal{F}_t), \exists\, \theta_t \in L^0(\mathbb{R},\mathcal{F}_t),\ x_t + \theta_t \Delta S_{t+1} \geq g_{t+1} \text{ a.s.}\right\} \\
\pi_{t,t+1}(g_{t+1}) &= \operatorname{ess\,inf} \mathcal{P}_{t,t+1}(g_{t+1})
\end{aligned}
$$

Then, using Lemma 3.1 of [5], we get that $\mathcal{P}_{T-2,T}(h(S_T)) = \mathcal{P}_{T-2,T-1}(\pi_{T-1,T}(h(S_T)))$,

$$\pi_{T-2,T}(h(S_T)) = \pi_{T-2,T-1}(\pi_{T-1,T}(h(S_T)))$$

and we may continue the recursion as soon as $\pi_{T-1,T}(h(S_T)) = h(T-1, S_{T-1})$ where $h(T-1,\cdot)$ satisfies (1.4), is convex with domain equal to $\mathbb{R}$, is such that $h(T-1,z) \geq 0$ for all $z \geq 0$ and $M_{T-1} = M \in [0,\infty)$. To see that, we distinguish three cases. If either $S_{T-1} = 0$ or $k^u_{T-1} = k^d_{T-1} = 1$, $\pi_{T-1,T}(h(S_T)) = h(S_{T-1})$ and $h(T-1,z) = h(z) = h(T,z)$ satisfies all the required conditions. If $k^d_{T-1} < k^u_{T-1} = +\infty$, $\theta^*_{T-1} = M$ and $\pi_{T-1,T}(h(S_T)) = h(T-1,S_{T-1})$, where

$$
\begin{aligned}
h(T-1,z) &= h(k^d_{T-1} z) + Mz\left(1 - k^d_{T-1}\right) \\
&= \lim_{k^u \to +\infty}\left(\frac{k^u - 1}{k^u - k^d_{T-1}} h(k^d_{T-1} z) + \frac{1 - k^d_{T-1}}{k^u - k^d_{T-1}} h(k^u z)\right),
\end{aligned}
$$

using (1.5). The term in the r.h.s. above is larger than $h(z) = h(T, z)$ by convexity. As $k^d_{T-1} \in [0,1]$ and $M \in [0,\infty)$, $h(T-1,z) \geq 0$ for all $z \geq 0$, we get that $h(T-1,\cdot)$ is convex function with domain equal to $\mathbb{R}$ since $h$ is so. The function $h(T-1,\cdot)$ also satisfies (1.4) (see (1.5)). Finally

$$M_{T-1} = \lim_{z\to+\infty} k^d_{T-1} \frac{h(k^d_{T-1}z)}{k^d_{T-1}z} + M\left(1 - k^d_{T-1}\right) = M.$$

The last case is when $S_{T-1} \neq 0$ and $k^u_{T-1} \neq k^d_{T-1}$ and $k^u_{T-1} < +\infty$. It is clear that (1.6) implies (1.4). Moreover, as $k^d_{T-1} \in [0,1]$ and $k^u_{T-1} \in [1,+\infty)$, $\lambda_{T-1} \in [0,1]$, $1-\lambda_{T-1} \in [0,1]$ and (1.4) implies that $h(T-1,z) \geq 0$ for all $z \geq 0$, $h(T-1,\cdot)$ is convex with domain equal to $\mathbb{R}$ since $h$ is so. Finally, by convexity

$$M_{T-1} \quad = \quad \lambda_{T-1}k^d_{T-1} \lim_{z\to+\infty} \frac{h(k^d_{T-1}z)}{k^d_{T-1}z} + (1-\lambda_{T-1})k^u_{T-1} \lim_{z\to+\infty} \frac{h(k^u_{T-1}z)}{k^u_{T-1}z} = M.$$

□

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